ON SOME APPLICATIONS OF THE MAXIMUM PRINCIPLES TO A VARIETY OF ELLIPTIC AND PARABOLIC PROBLEMS

by

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Dedication

To my family.

Abstract

One of the most important and useful tools used in the study of partial differential equations is the maximum principle. This principle is a natural extension to higher dimensions of an elementary fact of calculus: any function, which satisfies the inequality $f'' > 0$ on an interval [a,b], achieves its maximum at one of the endpoints of the interval. In this context, we say that the solution to the differential inequality $f'' > 0$ satisfies a maximum principle. In this thesis we will discuss the maximum principles for partial differential equations and their applications. More precisely, we will show how one may employ the maximum principles to obtain information about uniqueness, approximation, boundedness, convexity, symmetry or asymptotic behavior of solutions, without any explicit knowledge of the solutions themselves. The thesis will be organized in two main parts. The purpose of the first part is to briefly introduce in Chapter 1 the terminology and the main tools to be used throughout this thesis. We will start by introducing the second order linear differential operators of elliptic and parabolic type. Then, we will develop the first and second maximum principles of E. Hopf for elliptic equations, respectively the maximum principles of L. Nirenberg and A. Friedman for parabolic equations. Next, in the second part, namely in Chapter 2 and 3, we will introduce various P-functions, which are nothing else than appropriate functional combinations of the solutions and their derivatives, and derive new maximum principles for such functionals. Moreover, we will show how to employ these new maximum principles to get isoperimetric inequalities, symmetry results and convexity results in the elliptic case (Chapter 2), respectively spatial and temporal asymptotic behavior of solutions, in the parabolic case (Chapter 3).

Search Terms Maximum principles, isoperimetric inequalities, overdetermined problems, symmetry, convexity, time decay estimates, spatial decay estimates.

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Chapter 1: Maximum Principles

One of the most important and useful tools used in the study of partial differential equations is the maximum principle. This principle is a natural extension to higher dimensions of an elementary fact of calculus: any function, which satisfies the inequality $f'' > 0$ on an interval $[a, b]$, achieves its maximum at one of the endpoints of the interval, unless it is identically constant. In this context, we say that the solution to the differential inequality $f'' > 0$ satisfy a maximum principle. In the more general context of the partial differential equations of elliptic and parabolic type, a similar idea applies and the aim of this chapter is to present such extensions to some general classes of second order elliptic and parabolic problems.

1.1 Second Order Linear Differential Operators

Let Ω be a non-empty open bounded set of $\mathbb{R}^N$, $N \geq 2$.

Definition 1.1. We say that the operator L, defined as

$$Lu(\mathbf{x}) := a_{ij}(\mathbf{x})u_{,ij} + b_i(\mathbf{x})u_{,i} + c(\mathbf{x})u,\ u \in C^2(\Omega),\ \mathbf{x} \in \Omega, \tag{1.1}$$

is a *linear partial differential operator of order 2.* In (1.1), the coefficients

$$a = (a_{ij}) : \Omega \to \mathbb{R}^{N^2}, \qquad b = (b_i) : \Omega \to \mathbb{R}^N, \qquad c : \Omega \to \mathbb{R}^N, \tag{1.2}$$

are given measurable functions. Moreover, without loss of generality, we may assume that $a = (a_{ij})$ is a symmetric matrix, since $u_{,ij} = u_{,ji}$, so that we can write $a_{ij}u_{,ij} = \frac{1}{2}\left(a_{ij} + a_{ji}\right)u_{,ij}$, if necessary. Finally, in (1.1) and throughout this thesis, we will make use of the following notations

$$u_{,i} := \frac{\partial u}{\partial x_i}, \qquad u_{,ij} := \frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{1.3}$$

and summation from 1 to N is understood on repeated indices. Using these notations we have, for instance,

$$u_{,ij}u_{,i}u_{,j} = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}. \tag{1.4}$$

Definition 1.2. We say that the operator L, defined in (1.1), is:

(i) *elliptic at* $\mathbf{x} \in \Omega$, if there exists a number $\lambda(\mathbf{x}) > 0$ such that

$$a_{ij}(\mathbf{x})\xi_i\xi_j \geq \lambda(\mathbf{x}) \left|\xi\right|^2, \text{ for all } \xi \in \mathbb{R}^N; \tag{1.5}$$

(ii) *elliptic in* Ω, if L is elliptic at every $\mathbf{x} \in \Omega$;

(iii) *uniformly elliptic in* Ω, if L is elliptic in Ω and there exists a constant $\lambda_0 > 0$, such that $\lambda(\mathbf{x}) \geq \lambda_0$, for all $\mathbf{x} \in \Omega$. The largest such λ_0 is called *the uniform modulus of ellipticity of* L.

Now, let $u(\mathbf{x}, t) \in C^2(\Omega_T)$, with $\Omega_T := \Omega \times (0, T]$, where T is a positive constant. Let us define the second order partial differential operator

$$\overline{L}u(\mathbf{x},t) := \alpha_{ij}(\mathbf{x},t)u_{,ij} + \beta_i(\mathbf{x},t)u_{,i} + \gamma(\mathbf{x},t)u, \ (\mathbf{x}, t) \in \Omega_T, \tag{1.6}$$

where the coefficients

$$\alpha = (\alpha_{ij}) : \Omega_T \to \mathbb{R}^{N^2}, \qquad \beta = (\beta_i) : \Omega_T \to \mathbb{R}^N, \qquad \gamma : \Omega_T \to \mathbb{R}^N, \tag{1.7}$$

are given measurable functions. As before, without any loss of generality, we assume that the matrix $\alpha = (\alpha_{ij})$ is symmetric. Next, let us define the operator

$$\pounds u(\mathbf{x}, \mathbf{t}) := \left(\overline{L} - \frac{\partial}{\partial t}\right) u(\mathbf{x}, \mathbf{t}) = \overline{L}u(\mathbf{x}, \mathbf{t}) - u_{,t}(\mathbf{x}, \mathbf{t}). \tag{1.8}$$

Definition 1.3. We say that the operator $\pounds$, defined in (1.8), is:

(i) *parabolic at* $(\mathbf{x},t) \in \Omega_T$, if there exists a number $\mu(\mathbf{x},t) > 0$ such that

$$\alpha_{ij}(\mathbf{x},t)\xi_i\xi_j \geq \mu(\mathbf{x},t) \left|\xi\right|^2, \text{ for all } \xi \in \mathbb{R}^N; \tag{1.9}$$

(ii) *parabolic in* Ω_T, if $\pounds$ is parabolic at every $(\mathbf{x},t) \in \Omega_T$;

(iii) *uniformly parabolic in* Ω_T, if $\pounds$ is parabolic in Ω_T and there exists a constant $\mu_0 > 0$ such that $\mu(\mathbf{x},t) \geq \mu_0$, for all $(\mathbf{x},t) \in \Omega_T$. The largest such μ_0 is called *the uniform modulus of parabolicity of* $\pounds$.

Example 1.4.

(i) *The Laplace operator* $Lu(\mathbf{x}) := \Delta u(\mathbf{x})$ is *uniformly elliptic* in each $\Omega \subseteq \mathbb{R}^N$, with uniform modulus of ellipticity $\lambda_0 = 1$.

(ii) *The heat operator* $\pounds u(\mathbf{x}, t) := \Delta u - u_{,t}$ is *uniformly parabolic* in each $\Omega_T \subseteq \mathbb{R}^N \times (0, T]$, with uniform modulus of parabolicity $\mu_0 = 1$.

1.2 Maximum Principles for Elliptic Operators

In this section we consider the operator L, defined in (1.1), with $c(\mathbf{x}) \leq 0,$ for all $\mathbf{x} \in \Omega$. Moreover, we assume that the coefficients $a_{ij}(\mathbf{x}),\, b_j(\mathbf{x})$ and $c(\mathbf{x})$ are bounded measurable functions and L is uniformly elliptic, with modulus of ellipticity $\lambda_0 > 0$.

Definition 1.5. We say that $u \in C^2(\Omega)$ is:

(i) a *sub-solution relative to* L *and* Ω, if $Lu(\mathbf{x}) \geq 0$ in Ω;

(ii) a *super-solution relative to* L *and* Ω, if $Lu(\mathbf{x}) \leq 0$ in Ω;

(iii) a *solution relative to* L *and* Ω, if $Lu(\mathbf{x}) = 0$ in Ω.

In most of the applications it is sufficient to apply the following weak form of the maximum principle, known in the litterature as the *weak maximum principle*.

Theorem 1.6.

Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ *be a sub-solution relative to* L *and* Ω.

(i) If $c(\mathbf{x}) \equiv 0$, then

$$\max_{\overline{\Omega}} u(\mathbf{x}) = \max_{\partial\Omega} u(\mathbf{x}). \tag{1.10}$$

(ii) If $c(\mathbf{x}) \leq 0$, then

$$\max_{\overline{\Omega}} u(\mathbf{x}) \leq \max_{\partial\Omega} u^+(\mathbf{x}), \tag{1.11}$$

where $u^+(\mathbf{x}) = \max\{u(\mathbf{x}), 0\}$.

Proof.

(i) For an arbitrary $\varepsilon > 0$ and a constant α to be chosen later, we define

$$v(\mathbf{x}) := u(\mathbf{x}) + \varepsilon e^{\alpha x_1}, \text{ for all } \mathbf{x} = (x_1, ..., x_N) \in \overline{\Omega}, \tag{1.12}$$

Then

$$\begin{aligned} Lv(\mathbf{x}) &= L\left(u(\mathbf{x}) + \varepsilon e^{\alpha x_1}\right) = Lu(\mathbf{x}) + \varepsilon L\left(e^{\alpha x_1}\right) \\ &\geq \alpha\varepsilon\left\{a_{11}(\mathbf{x})\alpha + b_1(\mathbf{x})\right\} e^{\alpha x_1} \\ &\geq \alpha\varepsilon\left(\lambda_0\alpha - \sup_{\overline{\Omega}} |b_1|\right) e^{\alpha x_1} > 0, \text{ for all } \mathbf{x} \in \Omega, \end{aligned} \tag{1.13}$$

if we chose $\alpha > \dfrac{1}{\lambda_0} \sup_{\overline{\Omega}} |b_1|$. Therefore, with such a choice for α, we have

$$Lv(\mathbf{x}) > 0, \text{ for all } \mathbf{x} \in \Omega. \tag{1.14}$$

On the other hand, since $v \in C\left(\overline{\Omega}\right)$ and $\overline{\Omega}$ is a compact set in $\mathbb{R}^N$, then v takes its maximum at some point $\mathbf{x}_0 \in \overline{\Omega}$. In what follows we will show that, in fact, $\mathbf{x}_0 \notin \Omega$, so that $\mathbf{x}_0 \in \partial\Omega$. To this end, we assume contrariwise that $\mathbf{x}_0 \in \Omega$. Then, at this point of maximum we have

$$v_{,i}(\mathbf{x}_0) = 0,\ i = 1, ..., N, \tag{1.15}$$

and the hessian matrix $H(\mathbf{x}_0) := (v_{,ij}(\mathbf{x}_0)) \leq 0$ (that is, $H(\mathbf{x}_0)$ is a negative semi-definite matrix).

Now, since $A(\mathbf{x}_0) = (a_{ij}(\mathbf{x}_0))$ is symmetric and positive semi-definite, then there exists an orthogonal matrix $R = (r_{ij})$, such that

$$RA(\mathbf{x}_0)R^t = diag\,(d_1, ..., d_N)\,, \qquad RR^t = I. \tag{1.16}$$

By the hypothesis of ellipticity, we also have $d_j \geq 0,\ j = 1, ..., N$. Let us denote $\mathbf{y} = R\mathbf{x}$ and compute

$$v_{,i}\left(\mathbf{x}\right) = v_{,k}\left(\mathbf{y}\right)\mathbf{y}_{k,i}\left(\mathbf{x}\right) = v_{,k}\left(\mathbf{y}\right)r_{ki}, \tag{1.17}$$

$$v_{,ij}\left(\mathbf{x}\right) = v_{,kl}\left(\mathbf{y}\right)r_{ki}r_{lj}. \tag{1.18}$$

Then

$$\begin{aligned} a_{,ij}(\mathbf{x}_0)v_{,ij}(\mathbf{x}_0) &= a_{,ij}(\mathbf{x}_0)v_{,kl}\left(\mathbf{y}_0\right)r_{ki}r_{lj} = \left(RA\left(\mathbf{x}_0\right)R^t\right)_{kl}v_{kl}\left(\mathbf{y}_0\right) \\ &= \sum_{j=1}^{N} d_j v_{,jj}\left(\mathbf{y}_0\right), \end{aligned} \tag{1.19}$$

so that

$$Lv(\mathbf{x}_0) = d_j v_{,jj}\left(\mathbf{y}_0\right) + b_j\left(\mathbf{x}_0\right)v_{,j}\left(\mathbf{x}_0\right) \leq 0, \tag{1.20}$$

contradicting thus (1.14). Therefore, $v(\mathbf{x})$ takes its maximum on $\partial\Omega$, so

$$\max_{\overline{\Omega}} v(\mathbf{x}) = \max_{\partial\Omega} v(\mathbf{x}). \tag{1.21}$$

Letting $\varepsilon \to 0$, we obtain

$$\max_{\overline{\Omega}} u(\mathbf{x}) = \max_{\partial\Omega} u(\mathbf{x}), \tag{1.22}$$

and the proof of (i) is thus achieved.

(ii) Let

$$\Omega^+ := \left\{\mathbf{x} \in \Omega : u(\mathbf{x}) > 0\right\}. \tag{1.23}$$

Clearly, Ω^+ is an open set (since $u(\mathbf{x})$ is continuous). We will analyze separately the following two possible cases:

1) $\Omega^+ = \emptyset$: In this case $\max_{\overline{\Omega}} u(\mathbf{x}) \leq 0$ and the theorem is true.

2) $\Omega^+ \neq \emptyset$: In this case, we define a new operator $\widetilde{L}$, as follows

$$\widetilde{L}u := Lu - cu. \tag{1.24}$$

Since L is uniformly elliptic in Ω, then $\widetilde{L}$ is also uniformly elliptic in Ω. Moreover, $\widetilde{L}u \geq 0$, in Ω^+. Using now (i), we obtain

$$\max_{\overline{\Omega^+}} u(\mathbf{x}) = \max_{\partial\Omega^+} u(\mathbf{x}). \tag{1.25}$$

Therefore, there exists a point $\mathbf{x}_0 \in \partial\Omega^+$ such that

$$u(\mathbf{x}_0) = \max_{\overline{\Omega^+}} u(\mathbf{x}) > 0. \tag{1.26}$$

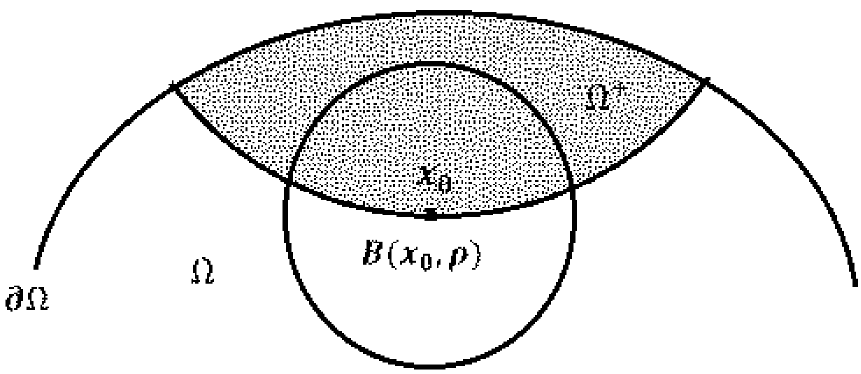

Fig. 1.1: The weak maximum principle

If $\mathbf{x}_0 \in \Omega$ (see Fig. 1.1) we get a contradiction. Indeed, if $\mathbf{x}_0 \in \Omega$, the continuity of u implies that $u(\mathbf{x}) > 0$ in $B(\mathbf{x}_0, \rho)$, for a $\rho > 0$. On the other hand, $B(\mathbf{x}_0, \rho)$ contains points in $\Omega \diagdown \Omega^+$, because $\mathbf{x}_0 \in \partial\Omega^+$, and $u(\mathbf{x}) \leq 0$ at such points. Therefore, $\mathbf{x}_0 \in \partial\Omega$ and the proof is thus achieved. $\square$

Remark 1.7.

(i) If $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a super-solution relative to L and Ω, then $-u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a sub-solution relative to L and Ω. Applying Theorem 1.6 (ii) to $-u$, we obtain

$$\max_{\overline{\Omega}} \left(-u(\mathbf{x})\right) \leq \max_{\partial\Omega} \left(-u(\mathbf{x})\right)^{+}, \tag{1.27}$$

where

$$\left(-u\right)^{+}(\mathbf{x}) = \max\left\{-u(\mathbf{x}), 0\right\} = -\min\left\{u(\mathbf{x}), 0\right\} =: -u^{-}(\mathbf{x}). \tag{1.28}$$

Therefore

$$\max_{\overline{\Omega}}(-u(\mathbf{x})) \leq \max_{\partial\Omega}(-u^{-}(\mathbf{x})), \tag{1.29}$$

which is equivalent to

$$\min_{\overline{\Omega}} u(\mathbf{x}) \geq \min_{\partial\Omega} u^{-}(\mathbf{x}). \tag{1.30}$$

(ii) If $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a solution relative to L and Ω, then

$$\min_{\partial\Omega} u^{-}(\mathbf{x}) \leq u(\mathbf{x}) \leq \max_{\partial\Omega} u^{+}(\mathbf{x}), \text{ for all } \mathbf{x} \in \overline{\Omega}. \tag{1.31}$$

(iii) *The Dirichlet problem for* L in a bounded domain Ω consists in finding $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{cases} Lu(\mathbf{x}) = f(\mathbf{x}) \text{ in } \Omega, \\ u(\mathbf{x}) = g(\mathbf{x}) \text{ on } \partial\Omega. \end{cases} \tag{1.32}$$

where f and g are given functions. This problem can have at most one solution. Indeed, if we apply the weak maximum principle to the difference $u := u_1 - u_2$ (where we assume that u_1, u_2 are two possible solution), then we obtain that $u(\mathbf{x}) \equiv 0$, in Ω.

Despite the fact that the use of the weak maximum principle is sufficient in some applications, sometimes it is necessary to have a stronger form, which eliminates the possibility of having a non-trivial maximum in an interior point of the domain. In what follows we will obtain such a stronger result making use of a *boundary point lemma*, obtained by E. Hopf [12], which is equally useful in some applications.

Lemma 1.8. (Hopf's lemma for balls)

Assume that $B \subset \Omega$ *is a ball and* $u \in C^1(\overline{B}) \cap C^2(B)$ *is a sub-solution relative to* L *and* B*. Assume also that there exists a point* $\mathbf{p} \in \partial B$ *such that*

$$u(\mathbf{x}) < u(\mathbf{p}), \text{ for all } \mathbf{x} \in B, \tag{1.33}$$

with $u(\mathbf{p}) \geq 0$*, if* $c(\mathbf{x}) \neq 0$*. Then,*

$$\frac{\partial u}{\partial \mathbf{m}}(\mathbf{p}) > 0, \tag{1.34}$$

whenever this directional derivative exists. In (1.34), $\mathbf{m}$ *is an outward unit vector at* $\mathbf{p}$ *(*$\mathbf{m} \cdot \mathbf{n} > 0$ *and* $|\mathbf{m}| = 1$*, where* $\mathbf{n}$ *denotes the outward unit normal at* $\mathbf{p} \in \partial B$*).*

Proof.

As in Fig. 1.2., let us define

$$B := B(\mathbf{q}, \rho), \tag{1.35}$$

$$A := B(\mathbf{q}, \rho) \diagdown \overline{B(\mathbf{q}, \frac{1}{2}\rho)}, \tag{1.36}$$

and

$$M := u(\mathbf{p}) = \sup_{\overline{B}} u(\mathbf{x}). \tag{1.37}$$

Next, we define on $\overline{A}$ the following function

$$v(\mathbf{x}) := \delta \left(e^{-\alpha r^2} - e^{-\alpha \rho^2} \right), \ r := |\mathbf{x} - \mathbf{q}|, \tag{1.38}$$

where δ and α are two positive constants to be chosen later. We thus notice that $v \in C^2(\overline{A})$ and

$$v(\mathbf{p}) = 0, \tag{1.39}$$

$$\frac{\partial v}{\partial \mathbf{m}}(\mathbf{p}) = (\mathbf{m} \cdot \mathbf{n}) \frac{dv}{dr}\bigg|_{r=\rho} < 0. \tag{1.40}$$

On the other hand, with

$$\left(e^{-\alpha r^2} \right)_{,i} = e^{-\alpha r^2} (-\alpha) 2 (x_i - q_i), \tag{1.41}$$

$$\left(e^{-\alpha r^2} \right)_{,ij} = e^{-\alpha r^2} \left\{ 4\alpha^2 (x_i - q_i)(x_j - q_j) - 2\alpha \delta_{ij} \right\}, \tag{1.42}$$

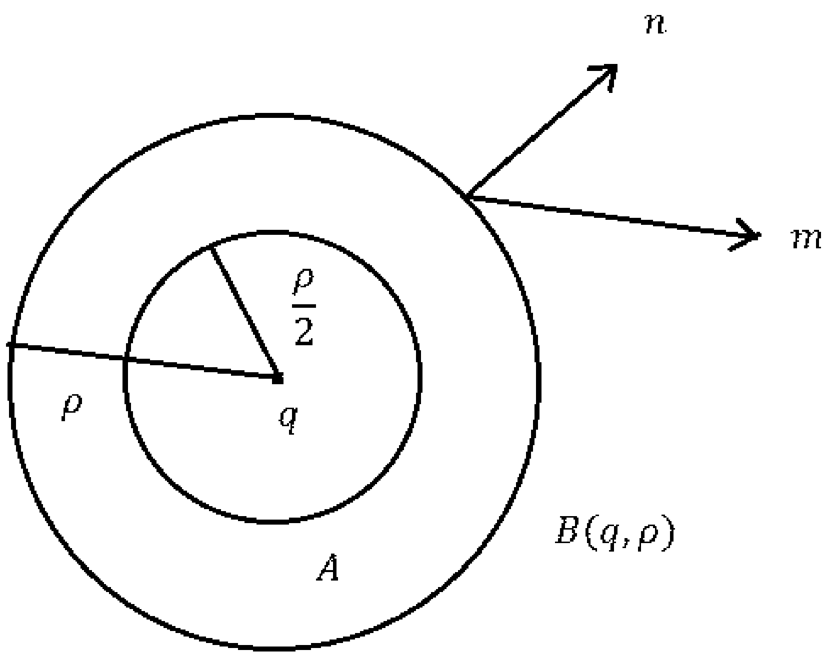

Fig. 1.2: Hopf's lemma for balls

we obtain

$$\begin{aligned} \frac{1}{\delta} Lv(\mathbf{x}) &= e^{-\alpha r^2} \{4\alpha^2 a_{ij}(\mathbf{x})(x_i - q_i)(x_j - q_j) - 2\alpha a_{jj}(\mathbf{x}) \\ &\quad -2\alpha b_j(\mathbf{x})(x_j - q_j)\} + c(\mathbf{x}) \left\{ e^{-\alpha r^2} - e^{-\alpha\rho^2} \right\}. \end{aligned} \tag{1.43}$$

Since L is uniformly elliptic in Ω, with modulus of ellipticity λ_0, we have

$$\begin{aligned} \frac{1}{\delta} Lv(\mathbf{x}) &\geq e^{-\alpha r^2} \left\{ 4\alpha^2 \lambda_0 r^2 - 2\alpha \sup_A (|a_{jj}| + |b|\,\rho) - \sup_A |c| \right\} \\ &> 0, \text{ for all } \mathbf{x} \in \overline{A}, \end{aligned} \tag{1.44}$$

if we choose α sufficiently large, because $r^2 \geq \left(\frac{1}{2}\rho\right)^2$. Let's fix such a value of α and define

$$w(\mathbf{x}) := u(\mathbf{x}) + v(\mathbf{x}), \text{ for all } \mathbf{x} \in \overline{A}. \tag{1.45}$$

Then

$$Lw(\mathbf{x}) = Lu(\mathbf{x}) + Lv(\mathbf{x}) > 0, \text{ for all } \mathbf{x} \in \overline{A}, \tag{1.46}$$

so that the weak maximum principle implies

$$\max_{\overline{A}} w = \max_{\partial A} w. \tag{1.47}$$

Next, we will analyze the following two cases:

1) $r = \rho$: In this case, we have $u(\mathbf{x}) \leq M$ and $v(\mathbf{x}) = 0$. Therefore, $u(\mathbf{x}) + v(\mathbf{x}) \leq M$, with equality at $\mathbf{x} = \mathbf{p}$.

2) $r = \frac{1}{2}\rho$: In this case, we have $u(\mathbf{x}) < M$, by (1.33). If we denote by $M - \beta$ the maximum of $u(\mathbf{x})$ for $r = \frac{1}{2}\rho$ (since the supremum of a continuous function on a compact set is always attained), then $\beta > 0$. On the other hand, choosing $\delta = \beta$, we have $v(\mathbf{x}) < \beta$. Therefore, $u + v < M$ in this case.

In conclusion,

$$\max_{\overline{A}} (u + v)(\mathbf{x}) = \max_{\partial A} (u + v)(\mathbf{x}) = M. \tag{1.48}$$

Finally, for $0 < t < \frac{1}{2}\rho$ and for an unit outward vector $\mathbf{m}$, we have

$$\frac{w(\mathbf{p}) - w(\mathbf{p} - t\mathbf{m})}{t} = \frac{M - w(\mathbf{p} - t\mathbf{m})}{t} \geq 0, \tag{1.49}$$

by (1.39) and (1.49). Therefore,

$$\begin{aligned} \liminf_{t \searrow 0} \frac{u(\mathbf{p}) - u(\mathbf{p} - t\mathbf{m})}{t} &= \liminf_{t \searrow 0} \left\{ \frac{w(\mathbf{p}) - w(\mathbf{p} - t\mathbf{m})}{t} - -\frac{v(\mathbf{p}) - v(\mathbf{p} - t\mathbf{m})}{t} \right\} \\ &\geq \liminf_{t \searrow 0} \frac{-v(\mathbf{p}) + v(\mathbf{p} - t\mathbf{m})}{t} \\ &= -\frac{\partial v}{\partial \mathbf{m}}(\mathbf{p}) > 0, \end{aligned} \tag{1.50}$$

by (1.18), and the proof is thus achieved. □

The above Hopf's lemma is the main tool in the proof of the following *strong maximum principle*, also known in the literature as *Hopf's first maximum principle*:

Theorem 1.9. (E. Hopf [11], 1927)

Assume that Ω is connected (possibly unbounded!) and $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a sub-solution relative to L and Ω. Assume also that $\sup_{\overline{\Omega}} u(\mathbf{x}) \geq 0$, if $c(\mathbf{x}) \neq 0$. If the supremum of $u(\mathbf{x})$ is attained at some interior point of Ω, then $u(\mathbf{x}) \equiv$ const. in Ω.

Proof.

Let $M := \sup_{\overline{\Omega}} u(\mathbf{x})$ and assume that the supremum is attained at $\mathbf{x}_1 \in \Omega$. Let us define

$$F := \{\mathbf{x} \in \Omega : u(\mathbf{x}) = M\}, \tag{1.51}$$

$$G := \{\mathbf{x} \in \Omega : u(\mathbf{x}) < M\}. \tag{1.52}$$

Then, clearly F is closed (because $u(\mathbf{x})$ is continuous) and non-empty (because $\mathbf{x}_1 \in F$), and G is open (because $u(\mathbf{x})$ is continuous). We will analyse the following two possible cases:

1) $G = \varnothing$: In this case, the theorem is true.

2) $G \neq \varnothing$: In this case, G contains at least one point, let's say $\mathbf{x}_0 \in G$. Using Theorem 1.6 we will show that this is impossible.

First of all, since $\Omega \subset \mathbb{R}^N$ is a domain (open and connected set), then G is connected by arcs. Therefore, there exists a continuous arc

$$\gamma := \{\xi(t) : 0 \leq t \leq 1\} \subset \Omega,\ \xi(0) = \mathbf{x}_0,\ \xi(1) = \mathbf{x}_1, \tag{1.53}$$

as in Fig. 1.3. Here, $\xi \in C([0,1], \mathbb{R}^n)$, so γ is compact. If Ω has a boundary, then

$$dist(\gamma, \partial\Omega) > 0, \tag{1.54}$$

since $\partial\Omega$ is closed in $\mathbb{R}^N$ and $\partial\Omega \cap \gamma = \emptyset$.

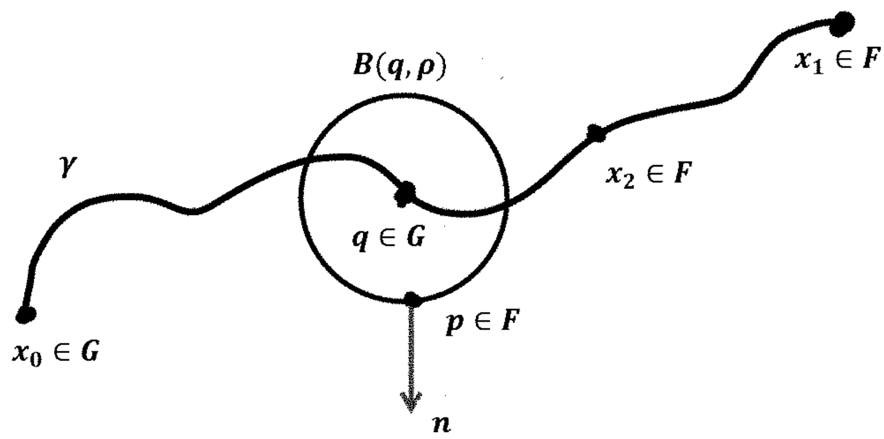

Fig. 1.3: Hopf's first maximum principle

Let $\mathbf{x}_2$ be the first point of γ, in F, while the curve γ is traversed from $\mathbf{x}_0$ to $\mathbf{x}_1$, the point where $u(\mathbf{x})$ attains its maximum M. Possibly $\mathbf{x}_2 = \mathbf{x}_1$. Let $\mathbf{q}$ be a point of γ located between $\mathbf{x}_0$ and $\mathbf{x}_2$, such that $|\mathbf{q} - \mathbf{x}_2| < dist(\gamma, \partial\Omega)$, if Ω has a boundary. We consider the ball $B := B(\mathbf{q}, \rho)$, with $\rho := dist(\mathbf{q}, F)$. Then

$$\rho \leq |\mathbf{q} - \mathbf{x}_2| < dist(\gamma, \partial\Omega), \tag{1.55}$$

so $B \subset \Omega$. Also, by construction, $B \subset G$. Since F is closed, then there exists a point $\mathbf{p} \in F \cap \partial B$ (possibly $\mathbf{p} = \mathbf{x}_2$). Applying now Hopf's lemma, we have

$$\frac{\partial u}{\partial \mathbf{n}}(\mathbf{p}) = \mathbf{n} \cdot (\nabla u)(\mathbf{p}) > 0. \tag{1.56}$$

On the other hand, since $\mathbf{p} \in F$, then $\mathbf{p}$ is an interior point of maximum for $u \in C^1(\Omega)$. Therefore, $\nabla u(\mathbf{p}) = 0$, so that we get a contradiction and the proof of the theorem is thus achieved. $\square$

To state *Hopf's lemma*, also known in the literature as *Hopf's second maximum principle*, we need the following definition:

Definition 1.10. A set Ω *has the interior ball property at some point* $\mathbf{p} \in \partial\Omega$, if there exists a ball $B_0 \subset \Omega$ such that $\mathbf{p} \in \partial B_0$ (see Fig. 1.4).

Theorem 1.11. (E. Hopf [12], 1952)

Assume that $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ is a sub-solution relative to L and Ω. Assume also that there exists a point $\mathbf{p} \in \partial\Omega$, such that

$$u(\mathbf{x}) \leq u(\mathbf{p}), \textit{ for all } \mathbf{x} \in \Omega, \tag{1.57}$$

with $u(\mathbf{p}) \geq 0$, if $c(\mathbf{x}) \neq 0$, and Ω has the interior ball property at $\mathbf{p}$. Let $\mathbf{m}$ an unit outward vector at $\mathbf{p}$. Then, either

$$\frac{\partial u}{\partial \mathbf{m}}(\mathbf{p}) > 0, \tag{1.58}$$

whenever this directional derivative exists, or $u(\mathbf{x}) \equiv$ *const. in* Ω.

Proof.

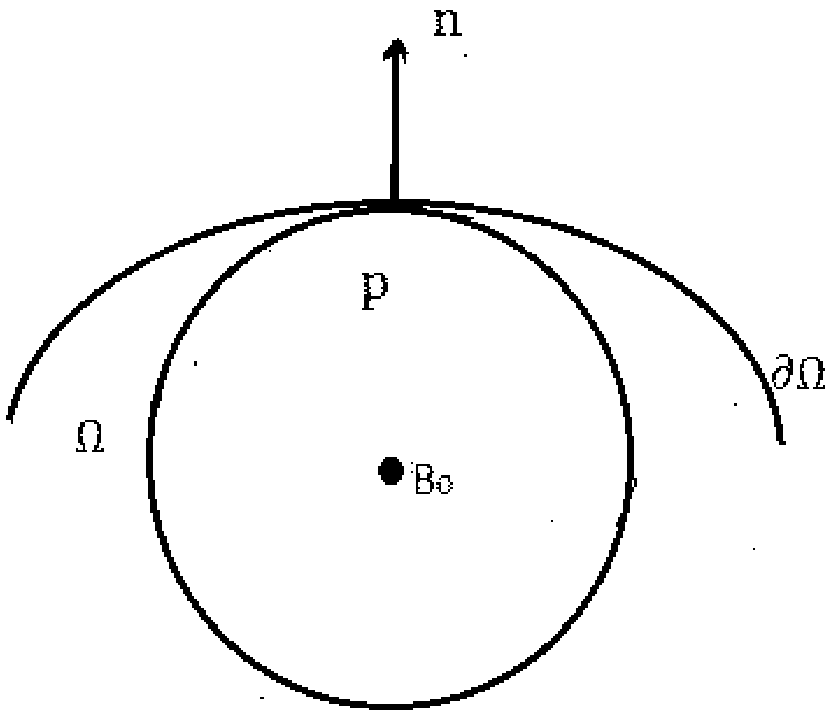

Fig. 1.4: Hopf's lemma

Let B_0 be the ball in Ω such that $\mathbf{p} \in \partial\Omega$, $\mathbf{x}_0$ is the center of the ball B_0 and $\rho_0 := |\mathbf{p} - \mathbf{x}_0|$ is the radius of the ball, that is $B_0 = B(\mathbf{x}_0, \rho_0)$. Let us consider another ball, smaller than B_0 and located inside B_0, defined as

$$B := B(\mathbf{q}, \frac{1}{2}\rho_0), \text{ with } \mathbf{q} := \frac{1}{2}(\mathbf{p} + \mathbf{x}_0). \tag{1.59}$$

Since $\overline{B} \subset B_0 \cup \{\mathbf{p}\}$, we have $\overline{B} \subset \Omega \cup \{\mathbf{p}\}$, so $u \in C(\overline{B})$. If $u(\mathbf{x}) < u(\mathbf{p})$, for all $\mathbf{x} \in B$, Hopf's lemma for balls implies (1.58). If $u(\mathbf{x}_1) = u(\mathbf{p})$ for a $\mathbf{x}_1 \in B$, then $u(\mathbf{x}_1) = \sup\limits_{\Omega} u(\mathbf{x})$ and the strong maximum principles implies that $u(\mathbf{x}) \equiv const.$ in Ω. The proof is thus achieved. □

1.3 Maximum Principles for Parabolic Operators

In this section we will consider the operator $\mathcal{L}$ defined in (1.8), with $\gamma(\mathbf{x}, t) \leq 0$ in Ω_T. We also assume that the coefficients $\alpha_{ij}(\mathbf{x}, t)$, $\beta_j(\mathbf{x}, t)$ and $\gamma(\mathbf{x}, t)$ are bounded measurable functions and $\mathcal{L}$ is an uniformly parabolic operator, with constant of parabolicity $\mu_0 > 0$. Moreover, let us denote by $\Gamma_T := \overline{\Omega}_T \diagdown \Omega_T$ the *parabolic boundary of* Ω_T.

Definition 1.12. We say that $u \in C^2(\Omega_T)$ is:

(i) a *sub-solution relative to* $\pounds$ *and* Ω_T, if $\pounds u(\mathbf{x},t) \geq 0$ in Ω_T;

(ii) a *super-solution relative to* $\pounds$ *and* Ω_T, if $\pounds u(\mathbf{x},t) \leq 0$ in Ω_T;

(iii) a *solution relative to* $\pounds$ *and* Ω_T, if $\pounds u(\mathbf{x},t) = 0$ in Ω_T.

Similarly to the elliptic case, in some applications it is sufficient to use the following weak form of the maximum principle. This result is known in the literature as *the weak maximum principe* for parabolic operators.

Theorem 1.13.

Let $u \in C(\overline{\Omega}_T) \cap C_1^2(\Omega_T)$ *a sub-solution relative to* $\pounds$ *and* Ω_T.

(i) If $\gamma(\mathbf{x},t) \equiv 0$ *in* Ω_T*, then*

$$\max_{\overline{\Omega}_T} u(\mathbf{x},t) = \max_{\Gamma_T} u(\mathbf{x},t). \tag{1.60}$$

(ii) If $\gamma(\mathbf{x},t) \leq 0$ *in* Ω_T*, then*

$$\max_{\overline{\Omega}_T} u(\mathbf{x},t) = \max_{\Gamma_T} u^+(\mathbf{x},t), \tag{1.61}$$

with $u^+(\mathbf{x},t) := \max\{u(\mathbf{x},t), 0\}$.

Proof

(i) Let $\varepsilon > 0$ be an arbitrary constant. We define the following function

$$v(\mathbf{x},t) := u(\mathbf{x},t) + \varepsilon t, \text{ for all } (\mathbf{x},t) \in \overline{\Omega}_T, \tag{1.62}$$

Then

$$\pounds v(\mathbf{x},t) = \pounds u(\mathbf{x},t) + \varepsilon > 0, \text{ for all } (\mathbf{x},t) \in \Omega_T. \tag{1.63}$$

We assume that the maximum of $v(\mathbf{x},t)$ on $\overline{\Omega}_T$ is attained at some point $(\mathbf{x}_0, t_0) \in \Omega_T$. We will show that this fact leads us to a contradiction, so that, in fact, the maximum of $v(\mathbf{x},t)$ is attained on Γ_T. We analyse the following two possible cases:

1) $t_0 \in (0,T)$: In this case, $(\mathbf{x}_0, t_0) \in \Omega_T$ is an interior point of maximum. Therefore,

$$v_{,t}(\mathbf{x}_0, t_0) = 0. \tag{1.64}$$

On the other hand, if we proceed exactly as in the proof of Theorem 1.6, then we can show that $\overline{L}v(\mathbf{x}_0, t_0) \leq 0$, so

$$\pounds v(\mathbf{x}_0, t_0) = \overline{L}v(\mathbf{x}_0, t_0) - v_{,t}(\mathbf{x}_0, t_0) \leq 0, \tag{1.65}$$

which contradicts (1.63).

2) $t_0 = T$: In this case, as the maximum of $v(\mathbf{x}, t)$ over Ω_T is attained at $(\mathbf{x}_0, t_0)$ then

$$v_{,t}\left(\mathbf{x}_0, t_0\right) \geq 0. \tag{1.66}$$

On the other hand, the inequality $\overline{L}v(\mathbf{x}_0, t_0) \leq 0$ remains true in this case. Therefore, we obtain a contradiction as in the previous case.

In conclusion

$$\max_{\overline{\Omega}_T} v(\mathbf{x}, t) = \max_{\Gamma_T} v(\mathbf{x}, t). \tag{1.67}$$

Letting $\varepsilon \to 0$, we obtain

$$\max_{\overline{\Omega}_T} u(\mathbf{x}, t) = \max_{\Gamma_T} u(\mathbf{x}, t), \tag{1.68}$$

which achieves the proof of (i).

(ii) Assume that the positive maximum of $u(\mathbf{x}, t)$ over $\overline{\Omega}_T$ is attained at some point in Ω_T. Then the function $v(\mathbf{x}, t)$, defined in (1.62), also attains a positive maximum at some point in Ω_T, let's say $(\mathbf{x}_0, t_0)$, if $\varepsilon > 0$ is sufficiently small. Since $v(\mathbf{x}_0, t_0) > 0$ and $\gamma(\mathbf{x}_0, t_0) \geq 0$, then

$$\pounds v(\mathbf{x}_0, t_0) \geq 0 \tag{1.69}$$

which contradicts (1.63). In conclusion,

$$\max_{\overline{\Omega}_T} u(\mathbf{x}, t) = \max_{\Gamma_T} u^+(\mathbf{x}, t), \tag{1.70}$$

and this achieves the proof of the theorem. □

To prove a strong maximum principle we need a result known in the literature as *Harnack's inequality*, which states that if $u(\mathbf{x}, t)$ is a non-negative solution relative to $\pounds$ and Ω_T, then the maximum of $u(\mathbf{x}, t)$ at an interior point of a region can be estimated by its minimum in the same region, but at a later time.

Theorem 1.14. (Harnack's inequality)

Let $u \in C_1^2\left(\Omega_T\right)$ be a solution relative to $\pounds$ and Ω_T, $u \geq 0$ in Ω_T, and let $U \subset\subset \Omega$ be a connected set. Then, for all $0 < t_1 < t_2 \leq T$, there exists a constant C such that

$$\sup_U u\left(\cdot, t_1\right) \leq C \inf_U u(\cdot, t_2). \tag{1.71}$$

The constant C depends only on U, t_1, t_2 and the coefficients of $\overline{L}$.

Proof. See L.C. Evans [6], pp. 370-375. □

Harnack's inequality is the main tool in the proof of the strong maximum principle which, as in the elliptic case, eliminates the possibility of having a point of non-trivial maximum at an interior point of Ω_T. This result is known in the literature as the *Nirenberg's maximum principle*:

Theorem 1.18. (L. Nirenberg [14], 1953)

Assume that Ω is connected and $u \in C_1^2(\Omega_T) \cap C\left(\overline{\Omega}_T\right)$ is a sub-solution relative to $\pounds$ and Ω_T.

(i) If $\gamma(\mathbf{x}) \equiv 0$ in Ω_T and u attains its maximum at a point $(\mathbf{x}_0, t_0) \in \Omega_T$, then $u \equiv const.$ in Ω_{t_0}.

(i) If $\gamma(\mathbf{x}) \leq 0$ in Ω_T and u attains its non-negative maximum at some point $(\mathbf{x}_0, t_0) \in \Omega_T$, then $u \equiv const.$ in Ω_{t_0}.

Proof.

(i) Assume that u attains its maximum at some point $(\mathbf{x}_0, t_0) \in \Omega_T$. We consider an open set $U \subset\subset \Omega$, with $x_0 \in U$. Let v be the solution of:

$$\begin{cases} \pounds v(\mathbf{x}, t) = 0, & \text{for all } (\mathbf{x}, t) \in U_T, \\ v(\mathbf{x}, t) = u(\mathbf{x}, t), & \text{for all } (\mathbf{x}, t) \in \Delta_T, \end{cases} \tag{1.72}$$

where $\Delta_T := \overline{U}_T - U_T$ denotes the parabolic boundary of U_T (see L.C. Evans [6] for the existence of the solution). Then, the weak maximum principle implies the following inequality

$$u(\mathbf{x}, t) \leq v(\mathbf{x}, t), \text{ for all } (\mathbf{x}, t) \in U_T. \tag{1.73}$$

As $u \leq v \leq M$, with $M := \max\limits_{\overline{\Omega}_T} u(\mathbf{x}, t)$, we deduce that $v(\mathbf{x}_0, t_0) = M$.

Next, we denote

$$w(\mathbf{x}, t) := M - v(\mathbf{x}, t), \text{ for all } (\mathbf{x}, t) \in U_T. \tag{1.74}$$

Since $\gamma(\mathbf{x}, t) \equiv 0$, then we have

$$\pounds w(\mathbf{x}, t) = \overline{L} w(\mathbf{x}, t) - w_t(\mathbf{x}, t) = 0, \qquad w(\mathbf{x}, t) \geq 0, \text{ for all } (\mathbf{x}, t) \in U_T. \tag{1.75}$$

We chose $V \subset\subset U$, with $x_0 \in V$, V connected. Let $0 < t < t_0$. Then, Harnack's inequality implies

$$\max_V w(\cdot, t) \leq C \inf_V w(\cdot, t_0). \tag{1.76}$$

But

$$\inf_V w(\cdot, t_0) \leq w(\mathbf{x}_0, t_0) = 0. \tag{1.77}$$

Since $w(\mathbf{x}, t) \geq 0$ for all $(\mathbf{x}, t) \in U_T$, then

$$w(\mathbf{x}, t) \equiv 0, \text{ for all } (\mathbf{x}, t) \in V_{t_0}. \tag{1.78}$$

But this is true for any set V chosen as above. Therefore,

$$w(\mathbf{x}, t) \equiv 0, \text{ for all } (\mathbf{x}, t) \in U_{t_0}, \tag{1.79}$$

which is equivalent to

$$v(\mathbf{x}, t) \equiv M, \text{ for all } (\mathbf{x}, t) \in U_{t_0}. \tag{1.80}$$

On the other hand, since $v(\mathbf{x}, t) = u(\mathbf{x}, t)$, for all $(\mathbf{x}, t) \in \Delta_T$, we obtain

$$u(\mathbf{x}, t) \equiv M, \text{ for all } (\mathbf{x}, t) \in \partial U \times [0, t_0]. \tag{1.81}$$

But this conclusion is true for any set U chosen as above. Therefore,

$$u(\mathbf{x}, t) \equiv M, \text{ for all } (\mathbf{x}, t) \in U_{t_0}. \tag{1.82}$$

(ii) Let $M := \sup_{\overline{\Omega}_T} u(\mathbf{x}, t)$. Assume that $M \geq 0$ and $u(\mathbf{x}, t)$ attains this maximum at $(\mathbf{x}_0, t_0) \in \Omega_T$. We will analyse the following two possible cases:

1) If $M = 0$: In this case, we obtain as before

$$\pounds w(\mathbf{x}, t) = 0, \qquad w(\mathbf{x}, t) \geq 0, \text{ for all } (\mathbf{x}, t) \in U_T. \tag{1.83}$$

2) If $M > 0$: In this case, we consider as before a set $U \subset\subset \Omega$, with $\mathbf{x}_0 \in U$. Let $v(\mathbf{x}, t)$ the solution to the problem

$$\begin{cases} Kv(\mathbf{x}, t) - v_{,t} = 0, & \text{for all } (\mathbf{x}, t) \in U_T, \\ v(\mathbf{x}, t) = u^+(\mathbf{x}, t), & \text{for all } (\mathbf{x}, t) \in \Delta_T, \end{cases} \tag{1.84}$$

where $\Delta_T := \overline{U}_T - U_T$ denotes the parabolic boundary of U_T and

$$Kv := \overline{L}v - \gamma v. \tag{1.85}$$

We note that $0 \leq v \leq M$. Since

$$Ku(\mathbf{x},t) - u_t(\mathbf{x},t) = -\gamma u(\mathbf{x},t) \leq 0, \text{ for all } (\mathbf{x},t) \in U_T^+, \tag{1.86}$$

where $U_T^+ := \{(\mathbf{x},t) : u(\mathbf{x},t) \geq 0\}$, the weak maximum principle implies

$$u \leq v. \tag{1.87}$$

The same approach as in (i) gives

$$v(x_0, t_0) = M. \tag{1.88}$$

Now, we denote

$$w(\mathbf{x},t) := M - v(\mathbf{x},t), \text{ for all } (\mathbf{x},t) \in \Omega_T. \tag{1.89}$$

Since the operator K doesn't have zero order terms, then we have

$$Kw(\mathbf{x},t) - w_{,t}(\mathbf{x},t) = 0, \qquad w(\mathbf{x},t) \geq 0, \text{ for all } (\mathbf{x},t) \in U_T. \tag{1.90}$$

Next, let $V \subset\subset U$, with $\mathbf{x}_0 \in V$, V connected, and $0 < t < t_0$. Harnack's inequality then implies

$$v(\mathbf{x},t) \equiv u^+(\mathbf{x},t) \equiv M, \text{ for all } \partial U \times [0, t_0]. \tag{1.91}$$

Since $M > 0$, we deduce that

$$u(\mathbf{x},t) \equiv M, \text{ for all } (\mathbf{x},t) \in \partial V \times [0, t_0]. \tag{1.92}$$

But this conclusion is true for any set U given as above. Therefore,

$$u(\mathbf{x},t) \equiv M, \text{ for all } (\mathbf{x},t) \in U_{t_0}, \tag{1.93}$$

and the proof of the theorem is thus achieved. □

In what follows, we will formulate the counterpart in the parabolic case of the Hopf's second maximum principle, known in the literature as *Friedman's maximum principle*.

Theorem 1.19. (A. Friedman [9], 1958)

Assume that $u \in C^1(\overline{\Omega}_T) \cap C_1^2(\Omega_T)$ *is a sub-solution relative to* $\pounds$ *and* Ω_T. *Assume also that there exists a point* $\mathbf{p} := (\mathbf{x}_0, t_0) \in \Gamma_T$ *such that*

$$u(\mathbf{x}, t) \leq u(\mathbf{x}_0, t_0) =: M, \text{ for all } (\mathbf{x}, t) \in \Omega_T, \tag{1.94}$$

with $u(\mathbf{x}_0, t_0) \geq 0$, *if* $\gamma(\mathbf{x}, t) \neq 0$. *Moreover, assume that* Ω *has the interior ball property at* $\mathbf{x}_0$. *Let* $\mathbf{m}$ *be an unit outward vector to* Ω_{t_0} *at* $(\mathbf{x}_0, t_0)$. *Then, either*

$$\frac{\partial u}{\partial \mathbf{m}}(\mathbf{x}_0, t_0) > 0, \tag{1.95}$$

whenever this directional derivative exists, or $u(\mathbf{x}, t) \equiv$ *const. in* Ω_T.

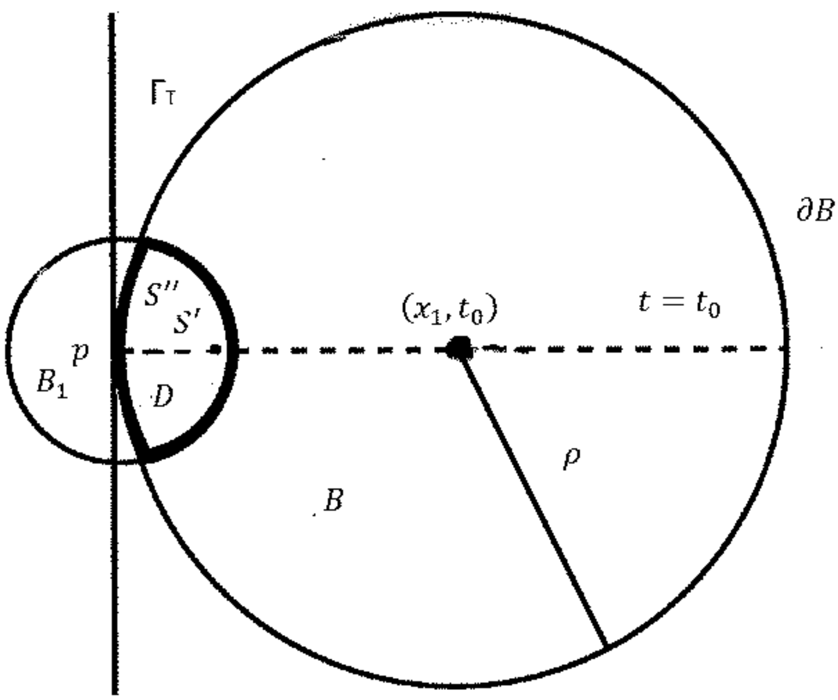

Fig. 1.5: Friedman's maximum principle

Proof.

As in Fig. 1.5, let B be a ball of center (x_1, t_0) and radius

$$\rho = |\mathbf{x_1} - \mathbf{x_0}|\,, \tag{1.96}$$

which is tangent to Ω_T at (x_0, t_0). We will build a ball B_1 centered at (x_0, t_0) and of radius smaller than ρ. Let us denote

$$S' = \partial B_1 \cap B_{t_0}, \tag{1.97}$$

$$S'' = \partial B \cap B_1 \cap \Omega_{t_0}, \tag{1.98}$$

and note that the surfaces S', S'' and $\{(\mathbf{x}, t) \in \Omega_T : t = t_0\}$ form the boundary of a region D. Choosing the disk B to be small enough, if necessary, we can make $u < M$ on $S'' \diagdown \{\mathbf{p}\}$.

Since $u < M$ on S', then we can establish the following three facts:

(i) $u < M$ on $S'' \diagdown \{\mathbf{p}\}\,,$

(ii) $u = M$ at $(x_0, t_0)\,,$

(iii) there exists η sufficiently small such that $u \leq M - \eta$ on S'.

Let us now define on $\overline{D}$ the following function

$$v(\mathbf{x}) := e^{-\alpha r^2} - e^{-\alpha \rho^2}, r := |(\mathbf{x}, t) - (\mathbf{x}_1, t_0)|\,, \tag{1.99}$$

where the positive constant α will be chosen later. We thus note that $v \in C^2(\overline{D})$ and

$$v(\mathbf{p}) = 0, \tag{1.100}$$

$$\frac{\partial v}{\partial \mathbf{m}}(\mathbf{p}) = (\mathbf{m} \cdot \mathbf{n}) \frac{dv}{dr}\bigg|_{r=\rho} < 0. \tag{1.101}$$

Making the computations as in the proof of Lemma 1.8, we obtain

$$Lv(\mathbf{x}) \geq e^{-\alpha r^2} \left\{ 4\alpha^2 \mu_0 r^2 - 2\alpha \sup_D \left(|a_{jj}| + |b_j|\,\rho\right) + (t - t_0) \right\}. \tag{1.102}$$

Therefore,

$$Lv(\mathbf{x}) > 0, \text{ on } D \cup \partial D, \tag{1.103}$$

if we choose α to be sufficiently large. Let us now fix such a value of α and define

$$w(\mathbf{x}) := u(\mathbf{x}) + \varepsilon v(\mathbf{x}), \text{ for all } (\mathbf{x}, t) \in \overline{D}. \tag{1.104}$$

Then

$$Lw(\mathbf{x}) = Lu(\mathbf{x}) + \varepsilon Lv(\mathbf{x}) > 0, \text{ for all } (\mathbf{x}, t) \in \overline{D}. \tag{1.105}$$

On the other hand, by (iii) we can chose ε sufficiently small such that

$$w < M \text{ on } S'. \tag{1.106}$$

Since

$$v = 0 \text{ on } \partial B, \tag{1.107}$$

we obtain, using (i), that

$$w < M \text{ on } S'' \diagdown \{\mathbf{p}\} \tag{1.108}$$

and

$$w(\mathbf{p}) = M. \tag{1.109}$$

Now, we restrict our attention to the region D and we apply the weak maximum principle to conclude that the maximum of w on $\overline{D}$ is attained at one single point $\mathbf{p}$. Therefore

$$\frac{\partial w}{\partial \mathbf{m}}(\mathbf{p}) = \frac{\partial u}{\partial \mathbf{m}}(\mathbf{p}) + \varepsilon \frac{\partial v}{\partial \mathbf{m}}(\mathbf{p}) \geq 0. \tag{1.110}$$

One the other hand,

$$\frac{\partial v}{\partial \mathbf{m}}(\mathbf{p}) = m \cdot n \frac{\partial v}{\partial r} = -2\mathbf{m} \cdot \mathbf{n} \alpha \rho e^{-\alpha \rho^2} < 0. \tag{1.111}$$

Therefore, we deduce that

$$\frac{\partial u}{\partial \mathbf{m}}(\mathbf{p}) > 0, \tag{1.112}$$

and the proof of the theorem is achieved. □

Remark 1.20.

The material in this chapter was inspired from the books of L.E. Frankel [8], L.C. Evans [6] and M.H. Protter-H.F. Weinberger [17]. We refer the reader to these books for more detail on this topic and extension of these results to more general nonlinear operators.

Chapter 2: Applications to the Elliptic Case

In the following two chapters we will develop some maximum principles for some appropriate functionals involving the solution of some problems and their derivatives. Such functionals are usually called in the literature as *P-functions*, due to L.E. Payne, who had a lot of contributions in this direction in the '70s (please see the book of R. Sperb [20] on this topic and the references therein).

For the sake of simplicity, we will start this chapter with a standard problem, to better understand the main ideas.

2.1 A Standard Problem: The Torsion Problem

Let $\Omega \subseteq R^N$ be a bounded domain, with $\partial\Omega \in C^{2,\varepsilon}$ and $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be the solution of the *torsion problem:*

$$\begin{cases} \Delta u = -1 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.1}$$

From Hopf's maximum principles, we immediately notice that $u > 0$ in Ω and $\frac{\partial u}{\partial n} = -|\nabla u| < 0$ on $\partial\Omega$. However, our goal here is to rather build some new maximum principles for P-functions and use these new maximum principles to get interesting results in applications.

2.1.1 P-functions with maximum on $\partial\Omega$. Let us first consider the following P-function

$$P^{(1)} := |\nabla u|^2 + \frac{2}{N}u. \tag{2.2}$$

If we differentiate successively (2.2), we get

$$P^{(1)}_{,k} = 2u_{,ik}\, u_{,i} + \frac{2}{N}u_{,k}\,, \tag{2.3}$$

$$\begin{aligned} \Delta P^{(1)} &= 2u_{,ik}\, u_{,ik} + 2(\Delta u)_{,i}\, u_{,i} + \tfrac{2}{N}\Delta u \\ &= 2u_{,ik}\, u_{,ik} - \tfrac{2}{N}. \end{aligned} \tag{2.4}$$

On the other hand, making use of Cauchy-Schwarz inequality, we have

$$u_{,ik}\, u_{,ik} \geq u_{,ii}u_{,ii} \geq \frac{1}{N}\left(\Delta u\right)^2 = \frac{1}{N}. \tag{2.5}$$

Inserting now (2.5) into (2.4), we get

$$\Delta P^{(1)} \geq 0 \text{ in } \Omega. \tag{2.6}$$

Hopf's first maximum principle then implies:

Theorem 2.1.

$P^{(1)}$ takes its maximum on $\partial\Omega$, unless it is identically constant.

Next, we apply this result to solve an overdetermined problem. More precisely, we have:

Theorem 2.2.

If a solution to problem (2.1) also satisfies $|\nabla u| = c$ on $\partial\Omega$, then Ω is a ball of radius Nc, and

$$u = \frac{N^2c^2 - r^2}{2N}. \tag{2.7}$$

Proof. For the proof, let us first notice that $P^{(1)} \equiv c^2$ on $\partial\Omega$, so either

$$P^{(1)} < c^2 \text{ in } \Omega, \tag{2.8}$$

or

$$P^{(1)} \equiv c^2 \text{ in } \Omega. \tag{2.9}$$

We will show that (2.8) cannot holds. To this end, we first note that

$$\Delta(r\frac{\partial u}{\partial r}) = r\frac{\partial}{\partial r}(\Delta u) + 2\Delta u = -2. \tag{2.10}$$

Next, we make use of Green's theorem, to compute

$$\begin{aligned} \int_\Omega [2u - r\tfrac{\partial u}{\partial r}]dx &= \int_\Omega [-u\,\Delta(r\tfrac{\partial u}{\partial r}) + r\tfrac{\partial u}{\partial r}\Delta u]dx \\ &= \int_{\partial\Omega} [-u\tfrac{\partial}{\partial n}(r\tfrac{\partial u}{\partial r}) + r\tfrac{\partial u}{\partial r}\tfrac{\partial u}{\partial n}]ds \\ &= \int_{\partial\Omega} r\tfrac{\partial r}{\partial n}(\tfrac{\partial u}{\partial n})^2 ds \\ &= c^2\int_{\partial\Omega} r\tfrac{\partial r}{\partial n}ds = Nc^2V, \end{aligned} \tag{2.11}$$

where $V = |\Omega|$ denotes the volume of Ω. Using again Green's theorem, we have

$$\int\limits_{\Omega} r\frac{\partial r}{\partial r}dx = \int\limits_{\Omega} \nabla(\frac{r^2}{2})\nabla u dx = -N\int\limits_{\Omega} u dx. \quad (2.12)$$

Therefore,

$$(N+2)\int\limits_{\Omega} u dx = Nc^2V. \quad (2.13)$$

Now, if we assume that (2.8) holds, then

$$\int\limits_{\Omega} P^{(1)}dx < \int\limits_{\Omega} c^2 dx = c^2V. \quad (2.14)$$

But

$$\int\limits_{\Omega} P^{(1)}dx = \int\limits_{\Omega} (|\nabla u|^2 + \frac{2}{N}u)dx = (1+\frac{N}{2})\int\limits_{\Omega} u dx = \frac{N}{2}c^2V, \quad (2.15)$$

where (2.13) was used to get the last identity. Comparing now (2.14) and (2.15), one can easily notice that we get a contradiction. In conclusion,

$$P^{(1)} = |\nabla u|^2 + \frac{2}{N}u \equiv c^2 \text{ in } \Omega. \quad (2.16)$$

Therefore

$$\Delta P^{(1)} = 0 \text{ in } \Omega, \quad (2.17)$$

so the equality must hold in inequality (2.5), that is $u_{ij} = -\delta_{ij}/N$, so

$$u = \frac{1}{2N}(A - r^2), \quad (2.18)$$

where A is a constant. Finally, since $u = 0$ and $|\nabla u|^2 = c^2$ on $\partial\Omega$, we obtain that Ω is a ball of radius $A^{1/2} = Nc$. The proof is thus achieved. $\square$

Remark 2.3.

i) The above proof was given by H.F. Weinberger [21] but there are few other nice methods to get similar results: moving plane method (see J. Serrin [19], shape optimization technique (see M. Choulli and A. Henrot [3]), Newton inequalities (see B. Brandolini, C. Nitsch, P. Salani and C. Trombetti [2]) etc.

ii) The method of H.F. Weinberger was used by N. Garofalo and J. Lewis in [10] for a larger class of overdetermined problems.

iii) For another alternative method to get the result from (2.16) we refer the reader to a paper of A. Farina and B. Kawohl [7].

Next, let us consider the following P-function:

$$P^{(2)} := x \cdot \nabla u - 2u = x_i u,_i - 2u. \tag{2.19}$$

If we differentiate successively (2.19), we get

$$P^{(2)}_{,k} = \delta_{ik} + x_i u,_{ik} - 2u,_k = x_i u,_{ik} - u,_k \,, \tag{2.20}$$

$$\Delta P^{(2)} = \delta_{ik} u,_{ik} + x_i (\Delta u)_i - \Delta u = \Delta u - \Delta u = 0. \tag{2.21}$$

Therefore, Hopf's first maximum principle implies:

Theorem 2.4.

$P^{(2)}$ takes its maximum and minimum on $\partial\Omega$, unless it is constant.

Next, we give a application of this result to some overdetermined problem.

Theorem 2.5.

If the solution of (2.1) also satisfies

$$x \cdot \nabla u \equiv c\text{=const. on } \partial\Omega, \tag{2.22}$$

then Ω is the interior of an ellipsoid.

For the proof of Theorem 2.5 we will make use of the following result:

Theorem 2.6.

Let u be the solution of problem (2.1), with the origin $O \in \Omega$. Suppose that there exist $\lambda \in (0,1)$, such that

$$\Omega_\lambda := \{\lambda x \in R^N : x \in \Omega\} \subset \Omega, \tag{2.23}$$

and $\alpha > 0$, such that

$$u(x) = \alpha \text{ on } \partial\Omega_\lambda. \tag{2.24}$$

Then Ω must be the interior of an ellipsoid.

Remark 2.7.

i) If Ω is starshaped with respect to the origin, then (2.23) is satisfied for all $\lambda \in (0,1)$.

ii) If we choose $\lambda > 0$ sufficiently small, then (2.23) holds for any domain Ω .

iii) Theorem 1.5 and 1.6 still hold if in problem (2.1) we replace the Laplacian with a general fully nonlinear elliptic operator $F(D^2u)$ (see C. Enache-S. Sakaguchi [5]).

Proof of Theorem 2.6. Let us introduce the following functions

$$v_1(x) = u(x) - \alpha, \qquad v_2(x) = \lambda^2 u(\lambda^{-1}x). \tag{2.25}$$

Then, clearly v_1 and v_2 are solutions to the following problem

$$\begin{cases} \Delta v = -1 \text{ in } \Omega_\lambda, \\ v = 0 \text{ on } \partial\Omega_\lambda. \end{cases} \tag{2.26}$$

Therefore, by the uniqueness theorem, we have

$$v_1(x) = v_2(x), \text{ for all } x \in \Omega_\lambda. \tag{2.27}$$

Differentiating twice (2.27), with respect to x_i and x_j, we obtain

$$u_{,ij}(x) = u_{,ij}(\lambda^{-1}x), \text{ for all } x \in \Omega_\lambda. \tag{2.28}$$

On the other hand, since $\Omega_\lambda \subset \Omega$, we have

$$x \in \Omega_\lambda, \text{ for all } x \in \Omega_\lambda, n \in \mathbb{N}. \tag{2.29}$$

Therefore,

$$u_{,ij}(x) = u_{,ij}(\lambda^n x), \text{ for all } x \in \Omega_\lambda, n \in \mathbb{N}. \tag{2.30}$$

Now, if we let $n \longrightarrow \infty$ in (2.30), we obtain

$$u_{,ij}(x) = u_{,ij}(0), \text{ for all } x \in \Omega_\lambda, \tag{2.31}$$

which means that $u(x)$ must be a quadratic function in Ω_λ. Since $u(x) = \alpha$ on $\partial\Omega_\lambda$, then it follows that Ω_λ must be the interior of an ellipsoid and the proof is thus achieved. $\square$

Proof of Theorem 2.5. We first notice that

$$P^{(2)} = x \cdot \nabla u - 2u \equiv c \text{ on } \partial\Omega. \tag{2.32}$$

Therefore, (2.21) and Hopf's first maximum principle implies that

$$P^{(2)} \equiv c \text{ on } \overline{\Omega}. \tag{2.33}$$

Now, let us define

$$U(x) := u(x) - \min_{\overline{\Omega}} u. \tag{2.34}$$

We then have

$$x_k \cdot U_{,k} - 2U = 0 \text{ in } \Omega, \tag{2.35}$$

so U is homogeneous of degree 2 in Ω. Therefore, the level sets of U (so, also the level sets of u) are homothetic with Ω. Theorem 2.6 then implies that Ω is the interior of an ellipsoid. □

2.1.2 P-functions with minimum on $\partial\Omega$.

Assume that $\Omega \subseteq R^2$. Let us introduce the following P -function

$$P^{(3)} := \begin{vmatrix} u_{,11} & u_{,12} & u_{,1} \\ u_{,21} & u_{,22} & u_{,2} \\ u_{,1} & u_{,2} & 2u \end{vmatrix} = u_{,ij}\, u_{,i}\, u_{,j} - |\nabla u|^2 \Delta u + u[(\Delta u)^2 - u_{,ij}\, u_{,ij}]. \tag{2.36}$$

Differentiating successively (2.36), we obtain

$$P^{(3)}_{,k} = u_{,ijk}\, u_{,i}\, u_{,j} - 2u.u_{,ijk}\,.u_{,ij}\,, \tag{2.37}$$

$$\begin{aligned} \Delta P^{(3)} &= \Delta u_{,ij}\, u_{,i}\, u_{,j} + 2u_{,ijk}\,.u_{,ik}\, u_{,j} - 2u_{,k}\, u_{,ijk}\,.u_{,j} - 2u\Delta u_{,ij}\, u_{,j} - 2u_{,ijk}\, u_{,ijk} \\ &= -2uu_{,ijk}\, u_{,ijk} \leq 0. \end{aligned} \tag{2.38}$$

Therefore, Hopf's first maximum principle implies:

Theorem 2.8.

$P^{(3)}$ takes its minimum on $\partial\Omega$, unless it is identically constant.

As an application of this result we have the following convexity result, due to L. Makar-Limanov [13]:

Theorem 2.9.

If u is the solution of problem (2.1), with $\Omega \subseteq R^2$ convex, then $v = \sqrt{u}$ is strictly concave in Ω.

For the proof we make use of of the following:

Lemma 2.10.

Let $\Omega \subseteq R^2$ be convex bounded domain and $w \in C^2(\overline{\Omega})$ be strictly super harmonic function, i.e $\Delta w < 0$ in Ω. Then w is strictly concave in Ω if and only if $\det(D^2 u) > 0$.

Proof of Theorem 2.9. Since $v = \sqrt{u}$, then $u = v^2$ and we get successively, by differentiation, the following relations

$$v_{,i} = \frac{1}{2} u^{-1/2} u_{,i} \tag{2.39}$$

$$v_{,ij} = \frac{1}{2} u^{-1/2} u_{,ij} - \frac{1}{4} u^{-3/2} u_{,i}\, u_{,j}\,. \tag{2.40}$$

Therefore

$$\Delta v = -\frac{1}{v}(|\nabla v|^2 + \frac{1}{2}) < 0. \tag{2.41}$$

Also, since

$$v_{,ij} = \frac{1}{4u\sqrt{u}} [2uu_{ij} - u_{,i}\, u_{,j}\,], \tag{2.42}$$

one can easily note that

$$\det(\Delta^2 v) = \frac{1}{16u^3} \begin{vmatrix} 2uu_{,11} - u_{,1}^2 & 2uu_{,12} - u_{,1}u_{,2} \\ 2uu_{,12} - u_{,1}u_{,2} & 2uu_{,22} - u_{,2}^2 \end{vmatrix}. \tag{2.43}$$

On the other hand, we also notice that

$$P^{(3)} = \frac{1}{2u} \begin{vmatrix} 2uu_{,11} - u_{,1}^2 & 2uu_{,12} - u_{,1}u_{,2} \\ 2uu_{,12} - u_{,1}u_{,2} & 2uu_{,22} - u_{,2}^2 \end{vmatrix}. \tag{2.44}$$

Therefore, to show that $\det(D^2 v) > 0$, it is sufficient to show that $P^{(3)} > 0$.

From Theorem 2.8 we know that $P^{(3)}$ attains its maximum on $\partial\Omega$. But on $\partial\Omega$ we have

$$P^{(3)} = u_{,ij}\, u_{,i}\, u_{,j} - |\nabla u|^2 \Delta u = K\, |\nabla u|^3\,, \tag{2.45}$$

where K is the curvature of $\partial\Omega$. Therefore, since Ω is convex, the strong minimum principle implies that $P^{(3)} > 0$ in Ω and the proof is thus achieved. □

2.1.3 P-functions with maximum at a critical point of u.

Let us consider the following P-function

$$P^{(4)} := |\nabla u|^2 + 2u. \tag{2.46}$$

If we differentiate successively (2.46), we get

$$P^{(4)}_{,k} = 2u_{,ik}\, u_{,i} + 2u_{,k}\,, \tag{2.47}$$

$$\Delta P^{(4)} = 2u_{,ik}\, u_{,ik} + 2\left(\Delta u\right)_{,i} u_{,i} + 2\Delta u = 2u_{,ik}\;\, u_{,ik} - 2. \tag{2.48}$$

Next, making use of Cauchy-Schwarz inequality, we have

$$u_{,ik}\, u_{,ik}\, u_{,j}\, u_{,j} \geq u_{,ik}\, u_{,i}\, u_{,jk}\, u_{,j}\,. \tag{2.49}$$

Now, using (2.47) in (2.49), we obtain

$$u_{,ik}\, u_{,ik} \geq 1 + \ldots \text{ in } \Omega \backslash \omega, \tag{2.50}$$

where, here and in all that follows in this thesis, $\omega := \{x : \nabla u(x) = 0\}$ and the dots stand for linear terms containing the first derivatives of the P-function ($P^{(4)}_{,k}$ in this case). Therefore

$$\Delta P^{(4)} + \ldots \geq 0, \text{ in } \Omega \backslash \omega, \tag{2.51}$$

and Hopf's first maximum principles implies:

Theorem 2.12.

$P^{(4)}$ takes its maximum either on $\partial\Omega$ or at a point critical point of u, unless it is identically constant.

Remark 2.13.

i) Theorem 2.12 holds independently of the boundary condition for $u(x)$. However, we will see that $P^{(4)}$ cannot take its maximum on $\partial\Omega$, if Ω is a convex domain and $u = 0$ on $\partial\Omega$.

ii) In the case $N = 2$ we can replace the Cauchy-Schwarz inequality (2.49), used to get a bound for $u_{,ik}\, u_{,ik}$, by the following identity

$$u_{,ik}\, u_{,ik}\, |\nabla u|^2 = |\nabla u|^2 (\Delta u)^2 + 2u_{,i}\, u_{,ij}\, u_{,k}\, u_{,kj} - 2\Delta u u_{,ij}\, u_{,i}\, u_{,j}\,, \tag{2.52}$$

and obtain

$$\Delta P^{(4)} + \ldots = 0 \text{ in } \Omega \backslash \omega. \tag{2.53}$$

Therefore, when $N = 2$, $P^{(4)}$ takes its minimum either on $\partial\Omega$ or at a point critical point of u, unless it is identically constant.

Next, let's assume that Ω is convex and $P^{(4)}$ takes its maximum value at a point

$Q \in \partial\Omega$. Then, Hopf's second maximum principle implies that either $P^{(4)} \equiv const.$ in Ω or $\partial P^{(4)}/\partial n > 0$ at Q. Using now the normal coordinates with respect to the boundary and the fact that $u = 0$ on $\partial\Omega$, we have

$$\frac{\partial P^{(4)}}{\partial n} = 2u_n u_{nn} + 2u_n, \tag{2.54}$$

where, here and in the remainder of this thesis, u_n and u_{nn} represent the first and the second normal derivative of u, respectively.

On the other hand, since $\partial\Omega$ is smooth, equation (2.1) is also satisfied on $\partial\Omega$, so

$$\Delta u = -1 \text{ on } \partial\Omega \Longleftrightarrow u_{nn} + (N-1)Ku_n = -1 \text{ on } \partial\Omega, \tag{2.55}$$

where K is the mean curvature of $\partial\Omega$. Therefore, inserting (2.55) into (2.54), we get

$$\frac{\partial P^{(4)}}{\partial n}(Q) = 2u_n[-1-(N-1)ku_n] + 2u_n = -2(N-1)Ku_n^2 \leq 0. \tag{2.56}$$

Since, from Hopf's second max principle we have $\frac{\partial P^{(4)}}{\partial n} > 0$, unless $P^{(4)}$ is identically constant, then from (2.56) we get:

Theorem 2.14.

If Ω is convex, $P^{(4)}$ takes its maximum value at critical point of u, unless it is identically constant.

In what follows we give an application of this result:

Theorem 2.15.

If in problem (2.1) Ω is also convex, then

$$u(x) \leq \frac{d^2}{2}, \tag{2.57}$$

where d is the radius of the largest ball inscribed in Ω.

Proof. From Theorem 2.14 we know that $P^{(4)} := |\nabla u|^2 + 2u$ takes its maximum value at a critical point of u. Therefore,

$$|\nabla u|^2 \leq 2(u_m - u), \tag{2.58}$$

where $u_m = \max_{\Omega} u(x)$. Next, let $A \in \Omega$ be a point where $u = u_m$, and $B \in \partial\Omega$ be a point nearest to A. Let r measure the distance from A along the ray connecting A and

B. Then

$$\frac{du}{dr} \leq |\nabla u|. \tag{2.59}$$

Integrating from A to B relation (2.59) and making use of (2.58), we get

$$-2\int_0^{u_m} \frac{du}{2\sqrt{u_m - u}} \leq \sqrt{2}\int_A^B dr = \sqrt{2}\delta \leq \sqrt{2}d, \tag{2.60}$$

and the result follows from (2.60). $\square$

2.2 Some Extensions to Nonlinear Problems

Let $\Omega \subseteq R^N$ be bounded domain, with $\partial\Omega \in C^{2.\varepsilon}$ and $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be a solution of the following nonlinear problem in divergence form:

$$\begin{cases} div(g(|\nabla u|^2)\nabla u) + \rho(|\nabla u|^2)f(u) = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.61}$$

where $f \in C^1$, $g \in C^2$ and $\rho \in C^1$ satisfy the following conditions

$$g(s) > 0, \qquad \rho(s) > 0, \qquad G(s) := g(s) + 2sg'(s) > 0, \qquad \text{for all } s > 0. \tag{2.62}$$

2.2.1 P-functions with maximum on $\partial\Omega$.

Let us consider the following P-function

$$P^{(5)} := \int_0^{|\nabla u|^2} \frac{G(s)}{\rho(s)}ds + \frac{2}{N}\int_0^u f(s)ds. \tag{2.63}$$

We have:

Theorem 2.16.

If u is a solution of equation (2.61) and

$$\frac{2}{N}\rho' f^2 - Gf' \geq 0, \tag{2.64}$$

then $P^{(5)}$ takes its maximum value on $\partial\Omega$, unless it is identically constant.

Proof. For simplicity, we consider only the case $g \equiv 1$. We differentiate $P^{(5)}$ successively to get

$$P^{(5)}_{,k} = \frac{1}{\rho}2u_{,ik}\,u_i + \frac{2}{N}fu_{,k}\,, \tag{2.65}$$

$$\Delta P^{(5)} = -\frac{\rho'}{\rho^2} 4u_{,ik}\, u_i u_{lk} u_l + \frac{2}{\rho} u_{,i}\,(\Delta u)_{,i} + \frac{2}{\rho} u_{,ik}\, u_{,ik} + \frac{2}{N} f'(|\nabla u|^2) + \frac{2}{N} f \Delta u. \quad (2.66)$$

To estimate $u_{,i}\,(\Delta u)_{,i}$ we differentiate equation (2.61), and obtain

$$2u_{,i}\,(\Delta u)_{,i} = 2u_{,i}\,[-f(u)\rho(|\nabla u|^2)_{,i} = -2f'\rho|\nabla u|^2 - 4f\rho' u_{,ik}\, u_{,k}\, u_{,i}\,. \quad (2.67)$$

Therefore

$$\begin{aligned} \Delta P^{(5)} &= -4\tfrac{\rho'}{\rho^2} u_{,ik}\, u_{,i} u_{,lk}\, u_{,l} - \tfrac{2}{\rho} f \rho |\nabla u|^2 - \tfrac{4}{\rho} f \rho' u_{,ik}\, u_{,k}\, u_{,i} \\ &\quad + \tfrac{2}{\rho} u_{,ik}\, u_{,ik} + \tfrac{2}{N} f' |\nabla u|^2 - \tfrac{2}{N} f^2 \rho \end{aligned} \quad (2.68)$$

Next, we first remind that

$$u_{,ik}\, u_{,ik} \geq \frac{1}{N}(\Delta u)^2 = \frac{1}{N} f^{\,2} \rho^2. \quad (2.69)$$

On the other hand, using the expression of $P_{,k}^{5}$ we obtain

$$u_{,ik}\, u_{,k}\, u_{,i} = -\frac{1}{N} f\rho|\nabla u|^2 + ..., \quad (2.70)$$

$$u_{,ik}\, u_{,i}\, u_{,lk}\, u_{,l} = \frac{1}{N^2} f^{\,2} \rho^2 |\nabla u|^2 + ... \,. \quad (2.71)$$

Replacing (2.69), (2.70) and (2.71) in (2.68), after some reductions we obtain

$$\Delta P^{(5)} \geq (2 - \frac{2}{N})|\nabla u|^2 \left[\frac{2}{N}\rho' f^{\,2} - f'\right] \geq 0. \quad (2.72)$$

Then the conclusion follows now from Hopf's first maximum principle. □

Remark 2.17.

i) Theorem 2.16 held independently of the convexity of Ω and of the boundary condition for $u(x)$.

ii) If the solution of equation (2.61) also satisfies

$$u \equiv a = \text{const. on } \partial\Omega, \quad (2.73)$$

and $f(u)$ does not change sign, then using some computations in normal coordinates one can also obtain the following bound

$$\frac{g(|\nabla u|^2_{\max})|\nabla u|_{\max}}{\rho(|\nabla u)|^2_{\max})} \leq \frac{f(a)}{N K_{\min}}, \quad (2.74)$$

where $|\nabla u|_{\max} = \max\limits_{\overline{\Omega}} |\nabla u|$, $K_{\min} = \min\limits_{\partial\Omega} K(s)$ and $K(s)$ represents the mean curvature of $\partial\Omega$ (see L.E. Payne-G.A. Philippin [15]).

In what follows we'll give some applications of Theorem 2.16 to surfaces of constant mean curvature. The mathematical model describing 2-dimensional surfaces of constant mean curvature Λ, with planar boundary, is given by the following problem:

$$\begin{cases} div\left(\frac{\nabla u}{\sqrt{2+|\nabla u|^2}}\right) = -2\Lambda \ \text{ in } \Omega \subset \mathbb{R}^2, \Lambda = \text{const} > 0, \\ u = 0 \text{ on } \ \partial\Omega. \end{cases} \tag{2.75}$$

In this case Serrin's existence criterion [18] states that the following condition should be satisfied

$$k(s) \geq 2\Lambda \text{ on } \partial\Omega. \tag{2.76}$$

Now, with $g(s) = \frac{1}{\sqrt{1+s}}$, $\rho(s) = 1$ and $f(s) = 2\Lambda$, Theorem 2.16 implies that

$$P^{(5)} = 2\left(1 - \frac{2}{\sqrt{1+|\nabla u|^2}} + \Lambda u\right), \tag{2.77}$$

takes its maximum value at some point $Q \in \partial\Omega$, unless it is identically constant. This means that

$$\Lambda u \leq \frac{1}{\sqrt{1+|\nabla u|^2}} - \frac{1}{\sqrt{1+|\nabla u|^2_{\max}}} \text{ in } \Omega, \tag{2.78}$$

where $|\nabla u|_{\max} = \max\limits_{\overline{\Omega}}|\nabla u|$. Evaluating this inequality at a point where u takes its maximum, we get

$$\Lambda u_{\max} \leq 1 - \frac{1}{\sqrt{1+|\nabla u|^2_{\max}}}. \tag{2.79}$$

Next, using (2.74), we have

$$K_{\min} \leq k(Q) \leq \frac{2\sqrt{1+|\nabla u|^2_{\max}}}{2|\nabla u|_{\max}}, \tag{2.80}$$

so that we get the following inequality

$$|\nabla u|^2_{\max} \leq \frac{\Lambda^2}{K^2_{\min} - \Lambda^2}. \tag{2.81}$$

Inserting now (2.81) into (2.79), we obtain

Theorem 1.18.

The solution of mean curvature problem (2.75) *satisfies*

$$u_{\max} \leq \frac{1}{\Lambda}\left(1-\frac{\sqrt{K_{\min}-\Lambda^2}}{K_{\min}}\right). \tag{2.82}$$

The equality is obtained in (2.82) *when* Ω *is a disk.*

We note that other interesting isoperimetric inequalities can also be derived. For instance, if we denote by A the area of the surface

$$S := \{(x_{1,}x_2, u(x_1,x_2)),\ x_3 = u(x_1,x_2), (x_1,x_2) \in \Omega\}, \tag{2.83}$$

and by V the volume between Ω and S, then we have

$$V = \int_{\Omega} u dx, \qquad A = \int_{\Omega} \sqrt{1+|\nabla u|^2} dx. \tag{2.84}$$

Therefore, integrating the inequality

$$\Lambda u \leq \frac{1}{\sqrt{1+|\nabla u|^2}} = \sqrt{1+|\nabla u|^2} - \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}}, \tag{2.85}$$

we obtain

$$\Lambda \int_{\Omega} u dx \leq \int_{\Omega} \sqrt{1+|\nabla u|^2} dx - \int_{\Omega} \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} dx. \tag{2.86}$$

On the other hand,

$$2\Lambda \int_{\Omega} u dx = -\int_{\Omega} u div \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) dx = \int_{\Omega} \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} dx. \tag{2.87}$$

Therefore, inserting (2.87) into (2.86), we get

Theorem 2.19.

With the notations given above, we have

$$3\Lambda V \leq A. \tag{2.88}$$

The equality holds in (2.88) *when* Ω *is a disk of radius* $1/\Lambda$. *This mean that the volume* V, *bounded by* Ω *and* S, *is greatest when* Ω *is a disk and* S *a hemisphere.*

2.2.2 P-functions with maximum at a critical point of u.

Let us consider the following P-function

$$P^{(6)} := \int_0^{|\nabla u|^2} \frac{G(s)}{\rho(s)} ds + 2\int_0^u f(s)ds, \tag{2.89}$$

where u is a solution to problem (2.61). We then have:

Theorem 1.20.

$P^{(6)}$ takes maximum either on $\partial\Omega$ or at a point critical point of u, unless it is identically constant.

Proof. For simplicity, we consider again only the case $g \equiv 1$. Differentiating successively $P^{(6)}$, we get

$$P^{(6)}_{,k} = \frac{1}{\rho} 2u_{,ik}\, u_{,i} + 2f u_{,k}\,, \tag{2.90}$$

$$\Delta P^{(6)} = -\frac{\rho'}{\rho^2} 4u_{,ik}\, u_{,i}\, u_{,lk}\, u_{,l} + \frac{2}{\rho} u_{,ik}\, u_{,ik} + \frac{2}{\rho} u_{,i}\, (\Delta u)_{,i} + 2f'|\nabla u|^2 + 2f\Delta u. \tag{2.91}$$

Now, to estimate $u_{,i}\,(\Delta u)_{,i}$ we use (2.67). On the other hand, from Cauchy-Schwarz inequality we have

$$u_{,ik}\, u_{,ik}\;\; u_{,j}\, u_{,j} \geq u_{,ik}\, u_{,i}\, u_{,jk}\, u_{,j}\,. \tag{2.92}$$

Moreover, using the expression of $P^{(6)}_{,k}$, we get

$$u_{,ik}\;\, u_{,i}\;\, u_{,k} = \rho f|\nabla u|^2 + ... \text{ in } \Omega, \tag{2.93}$$

$$u_{,ik}\;\, u_{,i}\, u_{,lk}\;\, u_{,l} = \rho^2 f^2 |\nabla u|^2 + ... \text{ in } \Omega, \tag{2.94}$$

respectively, using (2.92),

$$u_{,ik} u_{,ik} \geq \rho^2 f^2 + ... \text{in } \Omega\backslash\omega, \tag{2.95}$$

where $\omega := \{x \in \Omega : \nabla u(x) = 0\}$. Replacing (2.67), (2.93), (2.94) and (2.95) in the expression of $\Delta P^{(6)}$, we get

$$\begin{aligned} \Delta P^{(6)} &\geq -4\tfrac{\rho'}{\rho^2}\rho^2 f^2 |\nabla u|^2 + \tfrac{2}{\rho} f^2 \rho^2 + \tfrac{4}{\rho} f\rho'\rho f(|\nabla u|^2) - 2f^2\rho \\ &= 0 \text{ in } \Omega\backslash\omega, \end{aligned} \tag{2.96}$$

and the result follows from Hopf's first maximum principle. □

Let's now assume that Ω is convex and u satisfies the Dirichlet boundary condition from problem (2.61), namely $u = 0$ on $\partial\Omega$. We will show that, in such a case, $P^{(6)}$ cannot take its maximum on $\partial\Omega$, so that the maximum of $P^{(6)}$ is attained at a critical point of u, unless $P^{(6)}$ is identically constant.

Assume that $P^{(6)}$ takes its maximum value at a point $Q \in \partial\Omega$. Then Hopf's second maximum principle implies that either $P^{(6)} \equiv const.$ or $\frac{\partial P^{(6)}}{\partial n} > 0$ at Q. Now, using the normal coordinates with respect to the boundary and the fact that $u = 0$ on $\partial\Omega$, we have

$$\frac{\partial P^{(6)}}{\partial n} = 2\frac{G}{\rho} u_{nn} u_n + 2 f u_n. \tag{2.97}$$

On the other hand, since $\partial\Omega$ is smooth, equation (2.61) is also satisfied on $\partial\Omega$, so that we have

$$G u_{nn} + (N-1) K g u_n + f\rho = 0 \text{ on } \partial\Omega. \tag{2.98}$$

Inserting now (2.98) into (2.97), we get

$$\begin{aligned} \tfrac{\partial P^{(6)}}{\partial n} &= 2\tfrac{G}{\rho} u_n \left(-\tfrac{f\rho}{G} - \tfrac{(N-1)Kgu_n}{G} \right) + 2 f u_n \\ &= -2(N-1)Kgu_n^2 \leq 0, \end{aligned} \tag{2.99}$$

since Ω is convex. Hopf's second maximum principles then implies that:

Theorem 2.21.

If Ω is convex, then $P^{(6)}$ takes its maximum value at critical point of u, unless it is identically constant.

As an application of this result, we will find a bound for the first eigenvalue of the Dirichlet-Laplacian. More precisley, let u be the first eigenfunction of the Dirichlet-Laplacian on a bounded convex domain $\Omega \subseteq R^N$, i.e. u satisfies

$$\begin{cases} \Delta u + \lambda_1 u = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.100}$$

In this case, $g \equiv 1$, $\rho \equiv 1$, $f(s) = \lambda_1 s$, so the P-function becomes

$$P^{(6)} = |\nabla u|^2 + \lambda_1 u^2, \tag{2.101}$$

and Theorem 2.21 implies

$$|\nabla u|^2 \leq \lambda_1 (u_{\max}^2 - u^2). \tag{2.102}$$

Next we proceed as the proof of Theorem 2.15. Let A be a point where $u = u_{\max}$ and B a point on $\partial\Omega$, nearest to A. Let r measure the distance from A along the ray connecting A and B. We then have

$$-\frac{du}{dr} \leq |\nabla u| \leq \sqrt{\lambda_1}\sqrt{u_{\max}^2 - u^2}. \tag{2.103}$$

Integrating (2.103) from A to B, we have

$$\int_0^{u_{\max}} \frac{du}{\sqrt{u_{\max}^2 - u^2}} \leq \sqrt{\lambda_1}\int_A^B dr = \sqrt{\lambda_1}|AB| \leq \sqrt{\lambda_1}d, \tag{2.104}$$

where d is the radius of the largest ball inscribed in Ω. Therefore

$$\left[\sin^{-1}\frac{u}{u_{\max}}\right]_u^{u_{\max}} = \frac{\pi}{2}, \tag{2.105}$$

which leads to the following:

Theorem 1.22.

With the notations given above, we have

$$\lambda_1 \geq \frac{\pi^2}{4d^2}, \tag{2.106}$$

with equality when Ω is a strip.

Chapter 3: Applications to the Parabolic Case

This chapter deals with two semi-linear heat diffusion problems, whose solutions, without appropriate restrictions on the data, might blow up in time or space. Our aim is to present some conditions to insure that the solutions remain bounded, as well as some conditions which allow us to derive explicit exponential decay bounds for the solutions and their derivatives.

3.1 P-functions with Maximum at $t = 0$

In this section we deal with a heat emission process in a medium with a nonnegative source and no heat emission in a cold medium. In other words, we will consider the following initial-boundary value problem

$$\begin{cases} \Delta u - u_{,t} = -f(u) & , \mathbf{x} \in \Omega,\ t > 0, \\ u(\mathbf{x}, t) = 0 & , \mathbf{x} \in \partial\Omega,\ t > 0, \\ u(\mathbf{x}, 0) = h(\mathbf{x}) & , \mathbf{x} \in \Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded convex domain in $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega \in C^{2,\varepsilon}$, while $f \in C^1$ and $h \in C^2$ are given functions assumed to satisfy the following conditions:

$$f(0) = 0, \qquad sf'(s) \geq f(s) > 0, \qquad s > 0, \tag{3.2}$$

$$h \geq 0, \qquad h(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial\Omega. \tag{3.3}$$

Under these assumptions, it then follows from Nirenberg's maximum principle that $u(\mathbf{x}, t)$ is nonnegative. We also note that (3.2) implies in particular that $f(s)/s$ is a nondecreasing function of s.

It is well known that the solution of problem (3.1) may not exist for all time and that the only way that the solution can fail to exist is by becoming unbounded at some finite time t^* (see J.M. Ball [1]). In this section we first determine conditions on the data sufficient to guarantee global boundedness of solution. Thereafter, making use of the maximum principles, we will derive some explicit exponential decay estimates in time for the solution and its derivatives. These results have been obtained in Payne-Philippin [16].

As the solution $u(\mathbf{x}, t)$ of problem (3.1) might blow up in a finite time t^*, it follows, in this case, that the solution exists in an interval $(0, \tau)$ with $\tau < t^*$. Let us denote

$$u_m := \max_{\Omega \times (0,\tau)} u(\mathbf{x}, t)\,(< \infty)\,. \tag{3.4}$$

In what follows, we will derive the conditions on the data which will guarantee that $u(\mathbf{x},t)$ remains bounded for all time $t>0$, i.e. such that the solution of the problem (3.1) doesn't blow up. In establishing this condition we make use of the first eigenvalue λ_1 of the Dirichlet-Laplacian and of the corresponding eigenfunction $\Phi_1(\mathbf{x})$, for a region $\widetilde{\Omega}\supseteq\Omega$:

$$\begin{cases} \Delta\Phi_1(\mathbf{x})+\lambda_1\Phi_1(\mathbf{x})=0,\ \Phi_1(\mathbf{x})>0 & ,\ \mathbf{x}\in\ \widetilde{\Omega}, \\ \Phi_1(\mathbf{x})=0 & ,\ \mathbf{x}\in\ \partial\widetilde{\Omega}. \end{cases} \tag{3.5}$$

Moreover, since $\Phi_1(\mathbf{x})$ is determined up to an arbitrary multiplicative constant, we normalize $\Phi_1(\mathbf{x})$ by the condition

$$\max_{\widetilde{\Omega}}\Phi_1(\mathbf{x})=1. \tag{3.6}$$

The reason for replacing Ω by $\widetilde{\Omega}\supseteq\Omega$ in our investigation is merely to allow an explicit computation of Φ_1 and λ_1 by considering, for instance, that $\widetilde{\Omega}$ is a ball or a rectangle.

Lemma 4.1.

The classical solution of problem (3.1) satisfies the following inequality

$$0\le u(\mathbf{x},t)\le\Gamma_1\exp\left(-\left(\lambda_1-\frac{f(u_m)}{u_m}\right)t\right),t\in[0,\tau], \tag{3.7}$$

where

$$\Gamma_1:=\max_{\Omega}\frac{h(\mathbf{x})}{\Phi_1(\mathbf{x})}<\infty. \tag{3.8}$$

Proof. We consider the following auxiliary function

$$v(\mathbf{x},t):=u(\mathbf{x},t)\exp\left(-\frac{f(u_m)}{u_m}t\right). \tag{3.9}$$

We compute

$$\begin{aligned} \{\Delta v-v_{,t}\}\exp\left(\frac{f(u_m)}{u_m}t\right) &= \Delta u-u_{,t}+\frac{f(u_m)}{u_m}u \\ &\ge \Delta u-u_{,t}+f(u)=0, \end{aligned} \tag{3.10}$$

where we have used the fact that $\dfrac{f(s)}{s}$ is nondecreasing. We thus have

$$\begin{cases} \Delta v - v_{,t} \geq 0 & , \mathbf{x} \in \Omega,\ t \in (0,\tau), \\ v(\mathbf{x},0) = h(\mathbf{x}) & , \mathbf{x} \in \Omega, \\ v(\mathbf{x},t) = 0 & , \mathbf{x} \in \partial\Omega,\ t \in (0,\tau). \end{cases} \tag{3.11}$$

The comparison principle implies

$$v(\mathbf{x},t) \leq \Gamma_1 \Phi_1 e^{-\lambda_1 t} =: w(\mathbf{x},t), \tag{3.12}$$

because we have

$$\begin{cases} \Delta w - w_{,t} = 0 & , \mathbf{x} \in \Omega,\ t \in (0,\tau), \\ w(\mathbf{x},0) = \Gamma_1 \Phi_1(\mathbf{x}) \geq h(\mathbf{x}) & , \mathbf{x} \in \Omega, \\ w(\mathbf{x},t) \geq 0 & , \mathbf{x} \in \partial\Omega,\ t \in (0,\tau). \end{cases} \tag{3.13}$$

Now, combining (3.9) and (3.12) we get the desired inequality (3.7). $\square$

Theorem 3.2.
If Γ_1 *satisfies the condition*

$$\frac{f(\Gamma_1)}{\Gamma_1} < \lambda_1, \tag{3.14}$$

then $t^* = \infty$ *and we have*

$$\max_{\Omega} \frac{f(u(\mathbf{x},t))}{u(\mathbf{x},t)} < \lambda_1,\ 0 \leq t < \infty. \tag{3.15}$$

Proof. We suppose that (3.15) is violated and establish a contradiction. By continuity, there exists a first time $\widetilde{t}$ for which $\dfrac{f(u)}{u}$ reaches the value λ_1, in the sense that

$$\max_{\Omega} \frac{f\left(u\left(\mathbf{x},\widetilde{t}\right)\right)}{u\left(\mathbf{x},\widetilde{t}\right)} = \lambda_1. \tag{3.16}$$

Since $\dfrac{f(s)}{s}$ is a nondecreasing function of $s > 0$, Lemma 3.1. implies

$$u(\mathbf{x},t) \leq \Gamma_1,\ 0 \leq t \leq \widetilde{t}, \tag{3.17}$$

which lead to the following chain of inequalities

$$\frac{f\left(u\left(\mathbf{x},t\right)\right)}{u\left(\mathbf{x},t\right)} \leq \frac{f\left(\Gamma_1\right)}{\Gamma_1} < \lambda_1,\ \mathbf{x}\in\Omega,\ 0\leq t\leq \widetilde{t}, \tag{3.18}$$

in view of (3.14). In particular, we have

$$\max_{\Omega}\frac{f\left(u\left(\mathbf{x},\widetilde{t}\right)\right)}{u\left(\mathbf{x},\widetilde{t}\right)} < \lambda_1, \tag{3.19}$$

which is in contradiction with the definition of $\widetilde{t}$. We then conclude that $\widetilde{t}=\infty$ and the proof of the theorem is complete. □

In what follows we will establish sufficient conditions on the data to derive exponential decay bounds in time for the solution $u\left(\mathbf{x},t\right)$, its derivatives and some data of problem (3.1). For this aim, we shall derive some maximum principles for the following P-function

$$\Phi\left(\mathbf{x},t\right) := \left\{\left|\nabla u\right|^2 + 2\int_0^u f(s)ds + au^2\right\} e^{2\beta t}, \tag{3.20}$$

where a and β are some real positive parameters to be appropriately chosen.

The main result of this section is formulated in the following theorem:

Theorem 3.3.

Let $u\left(\mathbf{x},t\right)$ be the classical solution of problem (3.1). Assume that the domain Ω and the initial data $h\left(\mathbf{x}\right)$ are small enough in the following sense

$$a \leq \frac{\pi^2}{4d^2} - \frac{f\left(\Gamma_1\right)}{\Gamma_1}, \tag{3.21}$$

where d is the inradius of Ω, and $a\in(0,1]$ is a constant. Then, the auxiliary function $\Phi\left(\mathbf{x},t\right)$ defined in (3.20) takes its maximum value at $t=0$, i.e.

$$\left|\nabla u\right|^2 + 2\int_0^u f(s)ds + au^2 \leq H^2 e^{-2at},\ \mathbf{x}\in\Omega,\ t>0, \tag{3.22}$$

with

$$H^2 := \max_{\Omega}\left\{\left|\nabla h\right|^2 + 2\int_0^h f(s)ds + ah^2\right\}. \tag{3.23}$$

Proof of theorem 3.3. The proof will be given in several steps.

Step 1.

Differentiating (3.20), we obtain successively

$$\Phi_{,k} = 2\left\{u_{,ik}u_{,i} + fu_{,k} + auu_{,k}\right\}e^{2at}, \tag{3.24}$$

$$\begin{aligned} \Delta\Phi &= 2\left\{u_{,ik}u_{,ik} + u_{,i}\Delta u_{,i} + a\left|\nabla u\right|^2 + au\Delta u + f'\left|\nabla u\right|^2 + f\Delta u\right\} \\ &= 2\left\{u_{,ik}u_{,ik} + u_{,i}u_{,it} + a\left|\nabla u\right|^2 + (au+f)(u_t - f)\right\} \end{aligned} \tag{3.25}$$

$$\Phi_{,t} = 2\left\{u_{,it}u_{,i} + fu_{,t} + auu_{,t} + a\left|\nabla u\right|^2 + 2a\int_0^u f(s)ds + a^2u^2\right\}e^{2at}, \tag{3.26}$$

Next, we differentiate the equation (3.1), to obtain

$$\Delta u_i = u_{,ti} - f'u_{,i}, \tag{3.27}$$

from which we get

$$\Delta u_{,i}u_{,i} = u_{,ti}u_{,i} - f'\left|\nabla u\right|^2. \tag{3.28}$$

Making use of the Cauchy-Schwarz inequality, in the following form,

$$\left|\nabla u\right|^2 u_{,ik}u_{,ik} \geq u_{,ik}u_{,k}u_{,ij}u_{,j}, \tag{3.29}$$

and of (3.24), we obtain

$$u_{,ik}u_{,ik} \geq (f+au)^2 e^{2at} + ..., \text{ in } \Omega\diagdown\omega. \tag{3.30}$$

Next, using the differential equation (3.1) in the equivalent form

$$\Delta u = -f + u_{,t}, \tag{3.31}$$

and inserting (3.28), (3.30) and (3.31) into (3.25), we obtain after some reductions

$$L\Phi := \Delta\Phi - \Phi_{,t} + ... \geq 2e^{2at}\left\{uf(u) - 2\int_0^u f(s)\,ds\right\} \geq 0 \text{ in } \Omega\diagdown\omega. \tag{3.32}$$

It follows from Nirenberg's maximum principle that Φ takes its maximum value either

(i) at a point **P** on $\partial\Omega$ for some $t > 0$, or

(ii) at a critical point of $u(\mathbf{x}, t)$ for some $t > 0$, or

(iii) at a point **P** in Ω at time $t = 0$.

Step 2.

Using Friedman's maximum principle, we will see that $\Phi(\mathbf{x}, t)$ cannot take its maximum value on $\partial\Omega$, that is the first possibility, namely (i), is eliminated.

Indeed, suppose that $\Phi(\mathbf{x},t)$ takes its maximum value at $\widehat{P} = \left(\widehat{\mathbf{x}},\widehat{t}\right)$ on $\partial\Omega$. We will compute the the outward normal derivative $\dfrac{\partial\Phi}{\partial n}$ at an arbitrary point of $\partial\Omega$. Since $u=0$ on $\partial\Omega$, we obtain

$$\frac{\partial\Phi}{\partial n} = 2u_{nn}u_n e^{2at}. \tag{3.33}$$

From equation (3.1), evaluated on $\partial\Omega \in C^{2,\varepsilon}$, in normal coordinates, we have

$$u_{nn} + (N-1)Ku_n = 0. \tag{3.34}$$

Inserting (3.34) into (3.33), we get

$$\frac{\partial\Phi}{\partial n} = -2(N-1)Kg^2u_n^2 \leq 0, \text{ on } \partial\Omega. \tag{3.35}$$

Therefore, Friedman's maximum principle implies that $\Phi(\mathbf{x},t)$ cannot take its maximum value on $\partial\Omega$. We also note that $\nabla u \neq 0$ on $\partial\Omega$ in view of Friedman's maximum principle.

Step 3.

Assume that the second possibility (ii) holds, i.e. $\Phi(\mathbf{x},t)$ takes its maximum value at a critical point $\overline{P} := \left(\overline{\mathbf{x}},\overline{t}\right)$. Then we would have

$$\Phi(\mathbf{x},t) \leq \Phi\left(\overline{\mathbf{x}},\overline{t}\right),\ \mathbf{x}\in\Omega, t>0. \tag{3.36}$$

Evaluating (3.36) in $t=\overline{t}$, we obtain

$$|\nabla u|^2 \leq 2\int_u^{u_m} f(s)\,ds + 2(u_m^2 - u^2), \mathbf{x}\in\Omega, \tag{3.37}$$

with $u_m := \max_\Omega u\left(\mathbf{x},\overline{t}\right)$. Using Cauchy's mean value theorem we can write

$$\begin{aligned} 2\textstyle\int_u^{u_m} f(s)\,ds &= 2\left[\textstyle\int_0^{u_m} f(s)\,ds - \int_0^u f(s)\,ds\right] = \frac{f(\xi)}{\xi}\left[u_m^2 - u^2\left(\mathbf{x},\overline{t}\right)\right] \\ &\leq \frac{f(u_m)}{u_m}\left[u_m^2 - u^2\left(\mathbf{x},\overline{t}\right)\right] \end{aligned} \tag{3.38}$$

where ξ is some intermediate value between u and u_m. Replacing (3.38) in (3.37) we obtain

$$\left|\nabla u\left(\mathbf{x},\overline{t}\right)\right|^2 \leq \left(\frac{f(u_m)}{u_m} + a\right)\left[u_m^2 - u^2\left(\mathbf{x},\overline{t}\right)\right], \mathbf{x}\in\Omega, \tag{3.39}$$

or

$$\frac{du}{\sqrt{u_m^2 - u^2\left(\mathbf{x},\overline{t}\right)}} \leq \sqrt{\left(\frac{f(u_m)}{u_m} + a\right)}d\tau. \tag{3.40}$$

Integrating (3.40) on a straight line from $\overline{\mathbf{x}}$ to the nearest point $\mathbf{x}_0 \in \partial\Omega$, we obtain

$$\frac{\pi}{2} \leq \sqrt{\left(\frac{f\left(u_m\right)}{u_m} + a\right)} \left|\overline{\mathbf{x}}\mathbf{x}_0\right| \leq \sqrt{\left(\frac{f\left(u_m\right)}{u_m} + a\right)} d. \tag{3.41}$$

Consequently,

$$\frac{f\left(u_m\right)}{u_m} + a \geq \frac{\pi^2}{4d^2}. \tag{3.42}$$

The inequality (3.42) is a necessary condition in order that $\Phi\left(\mathbf{x}, t\right)$ takes its maximum at a critical point of $u\left(\mathbf{x}, t\right)$. On the other hand, using the fact that $\dfrac{f(s)}{s}$ is a nondecreasing function we obtain the following chain of inequalities

$$\frac{f\left(u_m\right)}{u_m} \leq \frac{f\left(\Gamma_1\right)}{\Gamma_1} < \frac{\pi^2}{4d^2} g\left(\Gamma_1\right) - a \leq \frac{\pi^2}{4d^2} - a, \tag{3.43}$$

which is in contradiction with (3.42).

This achieves the proof of the theorem. □

3.2 P-functions with Maximum on $\partial\Omega$

In this section we will study a semilinear heat equation in a long cylindrical region for which the far end and the lateral surface are held at zero temperature and a nonzero temperature is applied at the near end. More precisely, the specific domain we consider is a finite cylinder $\Omega := D \times [0, L]$, where D is a bounded convex domain in the (x_1, x_2)-plane, with smooth boundary $\partial D \in C^{2,\varepsilon}$, the generators of the cylinder are parallel to the x_3-axis and its length is L. The heat diffusion problem we consider is the following:

$$\begin{cases} \Delta u - u_{,t} = -f(u) & , \mathbf{x} \in \Omega,\ t \in (0, T), \\ u(\mathbf{x}, t) = 0 & , \mathbf{x} \in \partial\Omega_L \cup \partial\Omega_{lat},\ t \in (0, T), \\ u(\mathbf{x}, t) = h(x_1, x_2, t) & , \mathbf{x} \in \partial\Omega_0,\ t \in (0, T), \\ u(\mathbf{x}, 0) = 0 & , \mathbf{x} \in \Omega, \end{cases} \tag{3.44}$$

where $\partial\Omega_0 := D \times \{0\}$, $\partial\Omega_L := D \times \{L\}$, $\partial\Omega_{lat} := \partial D \times (0, L)$ and T is assumed to be any time prior to blow-up time. We also suppose that $h(x_1, x_2, t)$ is a prescribed nonnegative function, with $h(x_1, x_2, 0) = 0$, and f is a nonnegative function satisfying the following conditions

$$\lim_{s \to 0} \frac{f(s)}{s} \text{ exists}, \qquad f'(s) \leq p(s), \qquad f''(s) \leq q(s), \qquad s \geq 0, \tag{3.45}$$

where $p(\sigma)$ and $q(\sigma)$ are nondecreasing functions of s.

We are interested in the spatial decay bounds for the solution to (3.44). Since the solution $u(\mathbf{x},t)$ can blow up at some point in space time, our first goal is to derive sufficient conditions on the data which will guarantee that the solution remains bounded. Moreover, under such conditions, we will obtain some explicit spatial decay bounds for the solution and its derivatives. The results presented here have been obtained by C. Enache [4]. The method we will use is similar to the one from the previous section, in the sense that the main idea is to construct a maximum principle for an appropriate P-function.

The P-function that we consider is

$$\Phi(\mathbf{x},t) := \left\{u_{,\alpha}u_{,\alpha} + u^2 + u_{,t}^2\right\} e^{2[\beta x_3 - \gamma t]}, \tag{3.46}$$

where $u(\mathbf{x},t)$ is the solution to (3.44), while β and γ are positive constants to be appropriately chosen. As in the previous section, to derive a maximum principle for the P-function defined in (3.46), we have to derive a parabolic inequality for Φ and apply the maximum principles of Nirenberg and Friedman.

Differentiating successively (3.46), we get

$$\Phi_{,k} = 2\left\{u_{,\alpha}u_{,\alpha k} + uu_{,k} + u_{,t}u_{,tk}\right\} e^{2[\beta x_3 - \gamma t]} + 2\beta\Phi\delta_{3k}, \tag{3.47}$$

$$\begin{aligned} \Delta\Phi &= 2\Big\{u_{,\alpha k}u_{,\alpha k} + |\nabla u|^2 + u_{tk}u_{,tk} + u_{,\alpha}\left(\Delta u\right)_{,\alpha} + u\Delta u \\ &\quad + u_{,t}\left(\Delta u\right)_{,t}\Big\} e^{2[\beta x_3 - \gamma t]} + 4\beta\Phi_{,3} - 4\beta^2\Phi, \end{aligned} \tag{3.48}$$

with $\delta_{3k} = 0$, if $k \neq 3$, $\delta_{33} = 1$, and

$$\frac{\partial\Phi}{\partial t} = 2\left\{u_{,\alpha t}u_{,\alpha t} + u_{,t} + u_{,t}u_{,tt}\right\} e^{2[\beta x_3 - \gamma t]} - 2\gamma\Phi. \tag{3.49}$$

Combining (3.48) and (3.49), using equation (3.44) and conditions (3.45), regrouping appropriately the various terms and dropping the nonnegative quantities $u_{,\alpha k}u_{,\alpha k}$ and $u_{,tk}u_{,tk}$, we obtain

$$\begin{aligned} L\Phi &:= \Delta\Phi - 4\beta\Phi_{,3} - \Phi_{,t} \geq 2\left\{u_{,\alpha}u_{,\alpha}\left[1 + \gamma - 2\beta^2 - p(u)\right] + u_{,3}^2\right. \\ &\quad \left. + u^2\left[\gamma - 2\beta^2 - p(\xi)\right] + u_{,t}^2\left[\gamma - 2\beta^2 - p(u)\right] e^{2[\beta x_3 - \gamma t]}\right., \end{aligned} \tag{3.50}$$

where ξ is an intermediate value between 0 and u. Now, it is clear that if $p(u)$ is bounded from above, then we can choose β and γ such that $L\Phi \geq 0$, in $\Omega \times [0,T)$.

Next, we will derive a condition on $h(x_1,x_2,t)$ which guarantees that $u(\mathbf{x},t)$ remains bounded for all time, that is the solution to problem (3.44) does not blow up. In establishing this condition we make use of the first eigenfunction $\varphi_1(x_1, x_2)$ of the Dirichlet-Laplacian and the corresponding eigenvalue λ_1, for a region $\widetilde{D} \supseteq D$:

$$\begin{cases} \Delta\varphi_1 + \lambda_1\varphi_1 = 0,\ \varphi_1 > 0 & ,\ \mathbf{x} \in\ \widetilde{D}, \\ \varphi_1 = 0 & ,\ \mathbf{x} \in\ \partial\widetilde{D}. \end{cases} \tag{3.51}$$

Moreover, since φ_1 is determined up to an arbitrary multiplicative constant, we normalize φ_1 by the condition

$$\max_{\widetilde{D}} \varphi_1 = 1. \tag{3.52}$$

We have :

Lemma 3.4.

Let x_0 and M be positive constants such that

$$h(x_1, x_2, t) \leq M \frac{\varphi_1}{\sqrt{t}} \exp\left(-\frac{x_0^2}{4t}\right), \tag{3.53}$$

and let $\widehat{h}$ be defined as

$$\widehat{h} := M \max_{t>0} \left\{ \frac{1}{\sqrt{t}} \exp\left(-\frac{x_0^2}{4t}\right) \right\} = \frac{M}{x_0}\sqrt{\frac{2}{e}}, \tag{3.54}$$

with $e = 2.718281...$ We also assume that $\hat{h}$ is small enough in the following sense:

$$p(\widehat{h}) < \lambda_1. \tag{3.55}$$

Then the solution $u(\mathbf{x}, t)$ of the problem (3.44) exists for all time. Moreover, the function $p(u)$ remains bounded away from λ_1 for all time, i.e.

$$p(u(\mathbf{x}, t)) < \lambda_1,\ \mathbf{x} \in \Omega,\ \ t > 0, \tag{3.56}$$

and we have the following estimate

$$u(\mathbf{x}, t) \leq U(\mathbf{x}, t) := \frac{M\varphi_1}{\sqrt{t}} \exp\left(-\frac{(x_0 + x_3)^2}{4t}\right),\ \mathbf{x} \in \Omega,\ \ t > 0. \tag{3.57}$$

Proof.

The function $U(\mathbf{x},t)$, defined in (3.57), satisfies the following properties

$$\begin{cases} \Delta U - U_{,t} + \lambda_1 U = 0 & , \ \mathbf{x} \in \Omega, \ t > 0, \\ U(\mathbf{x},t) = 0 & , \ \mathbf{x} \in \partial\Omega_{lat}, \ t > 0, \\ U(\mathbf{x},t) \geq 0 & , \ \mathbf{x} \in \partial\Omega_L, \ t > 0, \\ U(\mathbf{x},t) \geq h(x_1,x_2,t) & , \ \mathbf{x} \in \partial\Omega_0, \ t > 0, \\ U(\mathbf{x},t) = 0 & , \ \mathbf{x} \in \Omega, \ t \to 0. \end{cases} \tag{3.58}$$

Suppose that (3.56) is violated. Then there exists, by continuity, a first time $\widetilde{t}$ for which $p(u)$ reaches the value λ_1 in the sense that

$$\sup_{\mathbf{x}\in\Omega} p(u(\mathbf{x},\widetilde{t})) = \lambda_1. \tag{3.59}$$

Combining (3.58) and (3.59), we have

$$\Delta U - U_{,t} \leq -U \max_{\mathbf{x}\in\Omega} p(u(\mathbf{x},t)), \ t \in \left[0,\widetilde{t}\right]. \tag{3.60}$$

Setting $z = U - u$ and making use of the mean value theorem we obtain

$$\Delta z - z_{,t} < -z \max_{\mathbf{x}\in\Omega} p(u(\mathbf{x},t)), \ t \in \left[0,\widetilde{t}\right]. \tag{3.61}$$

It follows, from Nirenberg's maximum principle, that $z(\mathbf{x},t)$ is a positive function in $\Omega \times \left[0,\widetilde{t}\right]$. Thus, we obtain the inequality (3.57) in $\left[0,\widetilde{t}\right]$. Moreover, making use of (3.53), we obtain

$$u(\mathbf{x},t) \leq \widehat{h}, \ \ t \in \left[0,\widetilde{t}\right]. \tag{3.62}$$

Since p is a nondecreasing function, we are led to the following chain of inequalities

$$p\left(u\left(\mathbf{x},t\right)\right) \leq p(\widehat{h}) < \lambda_1, \ \mathbf{x} \in \Omega, \ \ t \in \left[0,\widetilde{t}\right], \tag{3.63}$$

in view of (3.55). In particular, we have

$$\max_{\mathbf{x}\in\Omega} p(u(\mathbf{x},\widetilde{t})) < \lambda_1. \tag{3.64}$$

which is in contradiction to the definition of $\widetilde{t}$. We then conclude that $\widetilde{t} = \infty$, and the proof of the Lemma 3.4 is thus achieved. □

Now, under the condition (3.55) of Lemma 3.4, it is clear that choosing the positive parameters β and γ to satisfy the following condition

$$\gamma - 2\beta^2 \geq \lambda_1, \tag{3.65}$$

we obtain the inequality $L\Phi \geq 0,$ in $\Omega \times (0, T).$ Nirenberg's maximum principle then implies that $\Phi(\mathbf{x}, t)$ ($\neq$ $const.$) attains its maximum value either at $t = 0$ (which is excluded, since $\Phi = 0$ at $t = 0$) or on $\partial\Omega.$

Next, using Friedman's maximum principle, we will see that $\Phi(\mathbf{x}, t)$ ($\neq$ $const.$) cannot take its maximum value on $\partial\Omega_L \cup \partial\Omega_{lat}$. Indeed,

$$\frac{\partial\Phi}{\partial n} = \frac{\partial\Phi}{\partial x_3} = 0, \text{ on } \partial\Omega_L, \tag{3.66}$$

and

$$\frac{\partial\Phi}{\partial n} = 2u_n u_{nn} e^{2[\beta x_3 - \gamma t]} = -2u_n^2 K e^{2[\beta x_3 - \gamma t]} \leq 0, \text{ on } \partial\Omega_{lat}, \tag{3.67}$$

where K is the mean curvature of ∂D (which is nonnegative since D is a convex domain) and where used equation (3.44) in normal coordinates with respect to the boundary. Thus $\Phi(\mathbf{x}, t)$ ($\neq$ $const.$) cannot take its maximum value on $\partial\Omega_L \cup \partial\Omega_{lat}$. Consequently, the maximum value of Φ occurs on $\partial\Omega_0$ and we have:

Theorem 3.5.

Let $u(\mathbf{x}, t)$ be the classical solution of (3.44). Suppose that the initial data h on $\partial\Omega_0$ is small enough, in the sense that h satisfies the condition (3.55) of Lemma 3.44, and that the positive parameters β and γ are chosen to satisfy the inequality (3.65). Then, the auxiliary function Φ, defined in (3.46), takes its maximum value on $\partial\Omega_0 \times (0, T)$, i.e. we have the inequality

$$u_{,\alpha}u_{,\alpha} + u^2 + u_{,t}^2 \leq \Gamma^2 e^{2[\gamma t - \beta x_3]}, \ \mathbf{x} \in \Omega, \ \ t \in [0, T), \tag{3.68}$$

with

$$\Gamma^2 = \max_{D \times [0,T]} \left\{ h_{,\alpha}h_{,\alpha} + h^2 + h_{,t}^2 \right\} e^{-2\gamma t}, \tag{3.69}$$

valid for arbitrary $T > 0$. Clearly, (3.68) holds for the quantities $u^2, u_{,\alpha}u_{,\alpha}$ and $u_{,t}^2$ separately.

Remark 4.6. Using the same idea, one may also prove that the solution to problem (3.44) depends continuously on the data $h(x_1, x_2, t)$ at the near end of the cylinder (see C. Enache [4] for more details).

Chapter 4: Conclusion

The first aim of this thesis was to give a complete and rigourous presentation of the classical maximum principles known for general classes of second order linear elliptic and parabolic operators. The second aim was to show how one may apply these maximum principles to obtain various information about the solutions of some important partial differential equations of elliptic and parabolic type, which appear as model for real life problems. To this end, in Chapter 1 we have introduced the terminology and the maximum principles of E. Hopf, in the elliptic case, respectively the maximum principles of L. Nirenberg and A. Friedman, in the parabolic case. Thereafter, in Chapters 2 and 3 we have developed some maximum principles for auxiliary functions involving the solutions of some problems and their derivatives. More precisely, using these new maximum principles, we found several optimal a priori bounds for quantities of interest in problems from physics and geometry, whose solutions are not known explicitly. Moreover, in Chapter 3 we found explicit time and spatial decay estimates for two different heat diffusion problems, whose solutions are not usually known in an explicit form. Our next aim is to adapt these techniques to fully nonlinear elliptic equations, respectively to nonlinear parabolic equations in divergence form and publish some papers in this directions of research.

Bibliography

[1] J.M. Ball (1977), *Remarks on blow-up nonexistence theorems for nonlinear evolution equations*, Quart. J. Math., 28, 473-486.

[2] B. Brandolini, C. Nitsch, P. Salani, C. Trombetti (2008), *Serrin-type overdetermined problems: an alternative proof*, Arch. Ration. Mech. Anal., 190(2), 267-280.

[3] M. Choulli, A. Henrot (1998), *Use of the domain derivative to prove symmetry results in partial differential equations*, Math. Nachr., 192(1), 91-103.

[4] C. Enache (2006), *Spatial decay bounds and continuous dependence on the data for a class of parabolic initial-boundary value problems*, J. Math. Anal. Appl., 323(2), 993-1000.

[5] C. Enache, S. Sakaguchi (2011), *Some fully nonlinear elliptic boundary value problems with ellipsoidal free boundaries*, Math. Nachr., 284(14-15), 1872-1879.

[6] L.C. Evans (1998), *Partial Differential Equations*, American Mathematical Society.

[7] A. Farina, B. Kawohl (2008), *Remarks on an overdetermined boundary value problem*, Calc.Var. PDEs, 31(3), 351-357.

[8] L.E. Fraenkel (2000), *An introduction to maximum principles and symmetry in elliptic problems*, Cambridge University Press.

[9] A. Friedman (1958), *Remarks on the maximum principle for parabolic equations and its applications*, Pacific J. Math., **8**, 201-211.

[10] N. Garofalo, J.L. Lewis (1989), *A symmetry result related to some overdetermined boundary value problems*, Amer. J. Math., 111(1), 9-33.

[11] E. Hopf (1927), *Elementare Bemerkung über die Lösung partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Berlin Sber. Preuss. Akad. Wiss, 19, 147-152.

[12] E. Hopf (1952), *A remark on linear elliptic differential equations of the second order*, Proc. Amer. Math. Soc., **3**, 791-793.

[13] L.G. Makar-Limanov (1971), *Solution of Dirichlet's problem for the equation* $\Delta u = 1$ *in a convex region*, Mat. Zametki, 9, 89-92.

[14] L. Nirenberg (1953), *A strong maximum principle for parabolic equations*, Comm. Pure Appl. Math., 6, 167-177.

[15] L.E. Payne, G.A. Philippin (1979), *Some maximum principles for nonlinear elliptic equations in divergence form with applications to capillary surface and to surfaces of constant mean curvature*, Nonlin. Anal, 3, 193-211.

[16] L.E. Payne, G.A. Philippin (1995), *Decay bounds for solutions of second order parabolic problems and their derivatives*, Math. Models Methods Appl. Sci., 5, 95-110.

[17] M.H. Protter, H.F. Weinberger (1975), *Maximum principles in differential equations,* Springer Verlag.

[18] J.B. Serrin (1969), *The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables*, Phil. Trans. R. Soc. London, 264, 413-496.

[19] J. Serrin (1971), *A symmetry problem in potential theory,* Arch. Rat. Mech. Anal., 43, 304-318.

[20] R.P. Sperb (1981), *Maximum Principles and Their Applications*, Academic Press.

[21] H.F. Weinberger (1971), *Remark on the preceding paper of Serrin*, Arch. Rat. Mech. Anal., 43, 319-320.

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