SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS: VARIATIONAL ITERATION APPROACH

by

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Dedication

To my parents, brother and sisters

for their endless love, patience and support.

Abstract

This thesis discusses a newly introduced numerical scheme for fractional differential equations. The strategy is an extension of the Variational Iteration Method (VIM) and is suitable for boundary value problems (BVPs). The presented Generalized Variational Iteration Method (GVIM) is particularly used for tackling BVPs. The scheme yields accurate solutions for which the errors are uniformly distributed across the domain. The numerical results confirm that this approach is reliable, and has a high convergence rate compared to other existing methods.

Keywords: *Generalized Variational Iteration Method, GVIM, Variational Iteration Approach, VIM, fractional differential equations*.

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Chapter 1: Introduction

Differential equations are used to model dynamical systems in various fields in sciences and engineering. These models, in general, are nonlinear and as such exact solutions are difficult to obtain. Therefore numerical methods are applied to obtain approximate solutions. Of the most used numerical methods in recent years, we mention the Variational Iteration Method (VIM), which was developed in 1999 by He [1] has been extensively employed by numerous authors for several applications. For example Abukhaled used the VIM to solve nonlinear singular two-point boundary value problem [2], Khuri and Wazwaz employed the VIM to solve BVP arising in electric conducting solids, elliptic BVPs, and integro-differential forms of Lane-Emden problems, [3–7]. Wu et al. [8] used the VIM to solve reaction-diffusion equtaion. Analytical solutions for boundary value problems were also derived by using coupled Green's Function with Fixed Point Iteration Methods (GFIM) [9]. For example Khuri et al. used the GFIM to efficiently solve BVPs, Toresh's problem and Bratu-like equations [10–12]. Abukhaled et al. used the GFIM to find highly accurate semi-analytical solution for the one dimensional curvature equation, a class of strongly nonlinear oscillators, amperometric enzymatic reactions, and a class of BVPs arising in heat transfer [13–16]. Homotopy perturbation method [17] and homotopy analysis method [18] have also been effective in deriving analytical solutions for many applied problems in sciences and engineering.

Although less desired than analytical or semi-analytical solutions, many numerical collocation methods lead to highly discrete approximate solutions. For example Taylor collocation method [19], Chebyshev collocation method [20], spline collocation methods [21], and wavelet-based methods [22].

Fractional differential equations have been the focus of attention of many researchers in the past two decades. It is believed that the application of fractional calculus goes back to 1823 when Abel provided the first application for fractional calculus with his tautochrone problem. The problem deals with the determination of an object such that the time it takes to fall under the influence of gravity is independent of the starting position [23]. The applications of fractional calculus can be found in almost every branch of natural and social sciences, in finance and economics, in health sciences and engineering [23–27].

Most of the numerical methods applied to ordinary differential equations have been modified to provide approximate solutions to fractional differential equations. For example, Abdulaziz et al. [28] used the Homotopy Perturbation Method (HPM) to solve fractional Initial Value Problems (IVPs). The variational iteration method was modified from its original form to solve various linear and nonlinear fractional differential equations [29, 30]. Sakar et al. [31] obtained highly accurate approximation by proposing a Legendre reproducing kernel method (L-RKM) to solve fractional Bratu-type equations. Other methods that have been used to solve fractional differential equations include Sinc-Galerkin Method [32], Quasi-Newton's Method (QNM) [33], Bezier Curve Method (BCM) [34]. Other methods for solving fractional differential equations can be viewed in [35] and the references therein.

In this thesis, a recently introduced iterative method based on the generalization of the variational iteration method to BVPs [36] is presented for the solution of multiple fractional differential equations. The convergence of the method will be established and its efficiency and the accuracy will be tested by the means of comparison with other methods. The thesis is outlined as follows: In section 2, we present and describe the Generalized Variational Iteration Method (GVIM). Section 3 includes the numerical experiments, and finally, in section 4 we will summarize our findings.

Chapter 2: Literature Review

2.1 Gamma Function

The gamma function is a generalization of the factorial to non-integer values. It was introduced by Leonhard Euler in 1729. The gamma function is defined by

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx. \tag{2.1}$$

The integral converges for $n > 0$ and diverges for $n \leq 0$. The graph of gamma function is shown in Figure 2.1

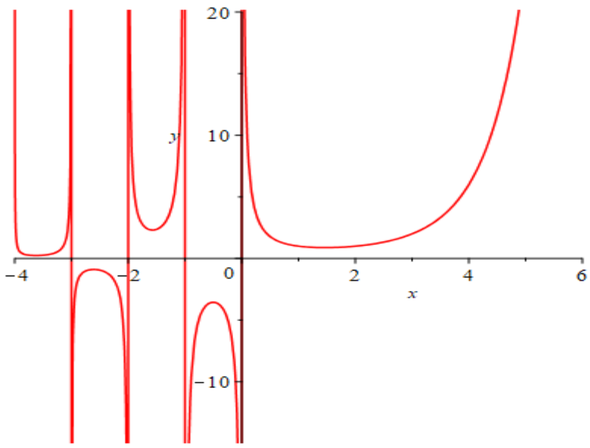

Figure 2.1: Gamma Function.

2.1.1 Properties of Gamma Function:

1. $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx = (n-1)!$

Substituting n with $p+1$, yields

$$\Gamma(p+1) = \int_0^\infty x^{p} e^{-x} dx = p! \tag{2.2}$$

Moreover, if we let

$u = x^p$, $dv = e^{-x}dx$, $du = px^{p-1}dx$ and $v = -e^{-x}$

we obtain

$$\begin{aligned}\Gamma(p+1) &= -x^p e^{-x}\Big|_0^\infty - \int_0^\infty x^{p-1} p(-e^{-x})dx \\ &= p\int_0^\infty x^{p-1}e^{-x}dx = p\,\Gamma(p).\end{aligned}$$

Hence

$$\Gamma(p+1) = p\,\Gamma(p) \tag{2.3}$$

Equation (2.3) called the recursion relation for the Γ function.

2.2 Mittag-Leffler Function

Mittag-Leffler invented and studied Mittag-Leffler's function in a series of papers from 1899–1904. It is a generalization of the exponential function and appears in the solution of fractional differential equations and fractional integral equations. Its importance lies behind its direct involvement in problems of biology, physics, applied sciences, and engineering. Mittag-Leffler defined the one parameter Mittag-Leffler function in 1903 by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \quad \alpha > 0. \tag{2.4}$$

where $z \epsilon C$. The two parameter Mittag-Leffler function for $\alpha > 0, \beta > 0$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)} \tag{2.5}$$

is a generalization of (2.4).This generalization was later studied by Wiman in 1905, and Humbert and Agarwal in 1953.

Example:

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{1}{k}(e^z - 1). \tag{2.6}$$

2.3 Fractional Calculus and Fractional Differential Equations

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. It was introduced in 1695 when L'Hôpital asked in a letter to Leibniz for the n^{th}-derivative, $\dfrac{D^n x}{Dx^n}$, of the linear function $f(x) = x$. L'Hôpital wanted to know the result for the derivative of order $n = \dfrac{1}{2}$. Leibniz replied that "It will lead to a paradox, from which one day useful consequences will be drawn." However, in 1819 and for the first time the study of non-integer order derivatives appeared in the literature by Lacroix who found a formula for the derivative of arbitrary order for monomials. He extended the n^{th} integer-order derivative defined by

$$\frac{d^n}{dx^n}x^p = \frac{p!}{(p-n)!}x^{p-n}, \tag{2.7}$$

to the derivative of an arbitrary order α by replacing the factorial with the Gamma function.

Example:

$$\frac{d^2}{dx^2}x^6 = \frac{6!}{(6-2)!}x^{6-2} = 30x^4. \tag{2.8}$$

The extended version is,

$$D^\alpha x^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}x^{p-\alpha}. \tag{2.9}$$

Example:

$$D^{\frac{1}{2}}x^2 = \frac{\Gamma(2+1)}{\Gamma\left(2-\frac{1}{2}+1\right)}x^{2-\frac{1}{2}}. \tag{2.10}$$

In addition, Fourier found the arbitrary derivatives for the sine and cosine functions in 1822. For n^{th} derivative we have,

$$\frac{d^n}{dx^n}\sin(bx) = b^n \sin\left(bx + \frac{n\pi}{2}\right) \tag{2.11}$$

and

$$\frac{d^n}{dx^n}\cos(bx) = b^n \cos\left(bx + \frac{n\pi}{2}\right), \tag{2.12}$$

where b is a constant.

Example:

$$\begin{aligned}\frac{d^2}{dx^2}\sin(3x) &= 3^2\ \sin(3x+\pi)\\ \frac{d^2}{dx^2}\cos(3x) &= 3^2\ \cos(3x+\pi)\end{aligned} \tag{2.13}$$

Extending (2.11) and (2.12) to the derivative of arbitrary order α yields,

$$D^\alpha \sin(bx) = b^\alpha \sin\left(bx + \frac{\alpha\pi}{2}\right) \tag{2.14}$$

and

$$D^\alpha \cos(bx) = b^\alpha \cos\left(bx + \frac{\alpha\pi}{2}\right), \tag{2.15}$$

where b is a constant.

Example:

$$D^{\frac{1}{2}} \sin(3x) = 3^{\frac{1}{2}} \sin\left(3x + \frac{\pi}{4}\right)$$
$$D^{\frac{1}{2}} \cos(3x) = 3^{\frac{1}{2}} \cos\left(3x + \frac{\pi}{4}\right) \tag{2.16}$$

In 1823, Abel provided the first application for fractional calculus with his tautochrone problem. Between 1832 and 1855, Liouville found the arbitrary derivative for the exponential function. He extended the nth derivative defined by

$$\frac{d^n}{dx^n} e^{bx} = b^n e^{bx}, \tag{2.17}$$

where b is a constant to the derivative of an arbitrary order α. The extended version is

$$D^{\alpha} e^{bx} = b^{\alpha} e^{bx}, \tag{2.18}$$

where b is a constant.

Example:

$$D^{\frac{1}{2}} e^{4x} = 4^{\frac{1}{2}} e^{4x} = 2e^{4x} \tag{2.19}$$

Liouville also found the fractional derivative of x^{-p} and was the first mathematician to study fractional calculus in depth. Riemann-Liouville fractional derivative is defined by

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-s)^{n-\alpha-1} f(s)\, ds, \tag{2.20}$$

where n is a positive integer with the property that $n - 1 < \alpha \leq n$.

Because the previous definition was difficult to apply. Caputo developed the Caputo fractional derivative in 1967. He reformulated Riemann-Liouville fractional derivative so he could use standard (integer order) initial conditions when working with fractional differential equations. Caputo fractional derivative is defined by

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \tag{2.21}$$

where n is a positive integer with the property that $n - 1 < \alpha \leq n$.

Example:

We will evaluate $D^{\frac{1}{2}}[c]$, where c is a constant, using the previous two definitions of the fractional derivative.

- Using Riemann-Liouville fractional derivative, we obtain

$$\begin{aligned} D^{\frac{1}{2}}[c] &= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{d}{dx} \int_0^x (x-s)^{-\frac{1}{2}} (c)\, ds \\ &= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{d}{dx} \left(2cx^{\frac{1}{2}}\right) = \frac{c}{\sqrt{\pi}} x^{-\frac{1}{2}}. \end{aligned}$$

- Using Caputo fractional derivative, we obtain

$$D^{\frac{1}{2}}[c] = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^x (x-s)^{-\frac{1}{2}} f'(c)\; ds = 0.$$

Chapter 3: Methodology

3.1 The Variational Iteration Method

3.1.1 Derivations: VIM is an iterative method that was first introduced by He [1]. This scheme reduces the size of computational work and in many instances gives rapidly convergent successive approximations of the exact solution if such a solution exists. The VIM constructs a correction functional using a general Lagrange multiplier that is designed for IVPs. The errors obtained using VIM deteriorate as we increase the value of x over the entire domain. In this section, we will derive the Lagrange multiplier for certain classes of differential equations.

(I) $u' + f\left(u, u'\right) = 0$

Applying the variational iteration method yields

$$u_{n+1}\left(t\right) = u_n\left(t\right) + \int_0^t \lambda\left(s\right)\left[\left(u_n\right)_s + \tilde{f}_n\left(u_n, u_n'\right)\right] ds. \tag{3.1}$$

where $\left(u_n\right)_s$ represents the derivative of u with respect to s and $\tilde{f}$ represents the fixed variation.

Expanding the parenthesis inside the integral yields

$$u_{n+1}\left(t\right) = u_n\left(t\right) + \int_0^t \lambda\left(s\right)\left(u_n\right)_s ds + \int_0^t \lambda\left(s\right)\tilde{f}_n\left(u_n, u_n'\right) ds. \tag{3.2}$$

Integrating the first integral by parts implies

$$\begin{aligned}\int_0^t \lambda\left(s\right)\left(u_n\right)_s ds &= \lambda\left(s\right) u_n\Big|_0^t - \int_0^t \lambda'\left(s\right) u_n\left(s\right) ds \\ &= \lambda\left(t\right) u_n(t) - \lambda\left(0\right) u_n\left(0\right) - \int_0^t \lambda'\left(s\right) u_n\left(s\right)\, ds.\end{aligned}$$

Substituting it into (3.2) yields

$$\begin{aligned}u_{n+1}\left(t\right) &= u_n\left(t\right) + \lambda\left(t\right) u_n(t) - \lambda\left(0\right) u_n\left(0\right) - \int_0^t \lambda'\left(s\right) u_n\left(s\right)\, ds \\ &+ \int_0^t \lambda\left(s\right)\tilde{f}_n\left(u_n, u_n'\right)\, ds.\end{aligned}$$

Taking the variation δ, which means the derivative with respect to u yields

$$\delta u_{n+1}(t) = \delta u_n(t) + \lambda(t)\,\delta u_n(t) - \lambda(0)\,\delta u_n(0) - \delta\int_0^t \lambda'(s)\,u_n(s)\,ds$$
$$+ \delta\int_0^t \lambda(s)\,\tilde{f}_n(u_n, u_n')\,ds.$$

Note that $\delta\tilde{f} = 0$, because the f is assumed to be fixed iteration, and $\delta u_n(0) = 0$. Therefore, we obtain

$$\delta u_{n+1}(t) = \delta u_n(t) + \lambda(t)\,\delta u_n(t) - \delta\int_0^t \lambda'(s)\,u_n(s)\,ds. \tag{3.3}$$

Or equivalently

$$\delta u_{n+1}(t) = \delta u_n(t)\,[1 + \lambda(t)] - \delta\int_0^t \lambda'(s)\,u_n(s)\,ds. \tag{3.4}$$

We set the variation $\delta u_{n+1}(t) = 0$ to obtain the following stationary conditions

$$\lambda'(s) = 0,$$
$$1 + \lambda(s)|_{s=t} = 0.$$

Solving the above system for λ of equations implies

$\lambda'(s) = As = 0$, i.e $A = 0$ and $\lambda(s) = -1$.

Substituting the $\lambda(s)$ back in (3.1) yields

$$u_{n+1}(t) = u_n(t) - \int_0^t \left[(u_n)_s + \tilde{f}_n(u_n, u_n')\right] ds. \tag{3.5}$$

(II) $u' + \alpha u + f(u, u') = 0$

Applying the variational iteration method yields

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\left[(u_n)_s + \alpha u_n + \tilde{f}_n(u_n, u_n')\right] ds. \tag{3.6}$$

Expanding the parenthesis inside the integral yields

$$\begin{aligned} u_{n+1}(t) &= u_n(t) + \int_0^t \lambda(s)\,(u_n)_s\,ds + \int_0^t \lambda(s)\,\alpha u_n(s) \\ &+ \int_0^t \lambda(s)\,\tilde{f}_n(u_n, u_n')\,ds. \end{aligned} \tag{3.7}$$

Integrating the first integral by parts implies

$$\begin{aligned}\int_0^t \lambda(s)(u_n)_s\, ds &= \lambda(s)\, u_n|_0^t - \int_0^t \lambda'(s)\, u_n\, ds \\ &= \lambda(t)\, u_n(t) - \lambda(0)\, u_n(0) - \int_0^t \lambda'(s)\, u_n\, ds.\end{aligned} \tag{3.8}$$

Substituting it into (3.7) yields

$$\begin{aligned}u_{n+1}(t) &= u_n(t) + \lambda(t)\, u_n(t) - \lambda(0)\, u_n(0) - \int_0^t \lambda'(s)\, u_n\, ds \\ &+ \int_0^t \lambda(s)\, \alpha u_n(s) ds + \int_0^t \lambda(s)\, \tilde{f}_n(u_n, u_n')\, ds.\end{aligned}$$

Taking the variation with respect to u_n yields

$$\begin{aligned}\delta u_{n+1}(t) &= \delta u_n(t) + \lambda(t)\, \delta u_n(t) - \lambda(0)\, \delta u_n(0) - \delta \int_0^t \lambda'(s)\, u_n\, ds \\ &+ \delta \int_0^t \lambda(s) \alpha u_n(s) + \delta \int_0^t \lambda(s)\, \tilde{f}_n(u_n, u_n')\, ds.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\delta u_{n+1}(t) &= \delta u_n(t) + \lambda(t)\, \delta u_n(t) - \delta \int_0^t \lambda'(s)\, u_n\, ds \\ &+ \delta \int_0^t \alpha \lambda(s) u_n(s)\, ds.\end{aligned} \tag{3.9}$$

Or, equivalently

$$\delta u_{n+1}(t) = \delta u_n(t)\left[1 + \lambda(t)\right] - \delta \int_0^t (\lambda'(s) - \alpha \lambda_s) u_n(s)\, ds. \tag{3.10}$$

We set the variation $\delta u_{n+1}(t) = 0$ to obtain the following stationary conditions

$$\lambda'(s) - \alpha \lambda(s) = 0,$$

$$1 + \lambda(s)|_{s=t} = 0.$$

Solving the above system of equations implies

$\lambda(s) = Ae^{\alpha s}$, $1 + Ae^{\alpha t} = 0$, and hence $A = -e^{-\alpha t}$.Therefore $\lambda(s) = -e^{\alpha}(s - t)$.

Substituting the $\lambda(s)$ back in (3.6) yields

$$u_{n+1}(t) = u_n(t) - \int_0^t e^{\alpha(s-t)} \left[(u_n)_{ss} + \tilde{f}_n(u_n, u_n') \right] ds. \tag{3.11}$$

(III) $u'' + f(u, u', u'') = 0$

Similar to part(I) we obtain the following formula

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s) \left[(u_n)_{ss} + \tilde{f}_n(u_n, u_n', u_n'') \right] ds. \tag{3.12}$$

Applying the variation with respect to u_n yields

$$\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda(s) \left[(u_n)_{ss} + \tilde{f}_n(u_n, u_n', u_n'') \right] ds \\
&= \delta u_n(t) + \delta \int_0^t \left[\lambda(s)(u_n)_{ss} + \lambda(s) \tilde{f}_n(u_n, u_n', u_n'') \right] ds \\
&= \delta u_n(t) + \delta \int_0^t \lambda(s)(u_n)_{ss} + \delta \int_0^t \lambda(s) \tilde{f}_n(u_n, u_n', u_n'')\, ds
\end{aligned}$$

where $\delta \tilde{f}_n(u, u', u'') = 0$.

Integrating by parts twice the above equation implies

$$\begin{aligned}
\int_0^t \lambda(s)(u_n)_{ss}\, ds &= \lambda(s)(u_n)_s - \int_0^t \lambda'(s)(u_n)_s\, ds \\
&= \lambda(s)(u_n)_s - \lambda'(s) u_n(s)\big|_0^t + \int_0^t \lambda''(s) u_n(s)\, ds \\
&= \lambda(t)(u_n)_s - \lambda'(t) u_n(t) - \lambda(0)(u_n)_s + \lambda'(0) u_n(0) \\
&+ \int_0^t \lambda''(s) u_n(s)\, ds \\
&= \lambda(t)(u_n)_s - \lambda'(t) u_n(t) + \int_0^t \lambda''(s) u_n(s)\, ds.
\end{aligned}$$

Substituting it back into the original equation yields

$$\begin{aligned}\delta u_{n+1}(t) &= \delta u_n(t) + \lambda(t)\,\delta(u_n)_s + \lambda'(t)\,\delta u_n(t) \\ &+ \delta\int_0^t \lambda''(s)\,u_n(s)\,ds.\end{aligned} \tag{3.13}$$

Equivalently we can write

$$\begin{aligned}\delta u_{n+1}(t) &= [1-\lambda'(t)]\,\delta u_n(t) + \lambda(t)\,\delta(u_n)_s \\ &+ \delta\int_0^t \lambda''(s)\,u_n(s)\,ds.\end{aligned} \tag{3.14}$$

We set the variation $\delta u_{n+1}(t) = 0$ to obtain the following stationary conditions

$$\lambda''(s) = 0 \quad a \le s \le t,$$

$$1 - \lambda'(s)\big|_{s=t} = 0$$

$$\lambda(s)\big|_{s=t} = 0.$$

Solving the above system of equations gives us

$\lambda''(s) = As + B = 0$, $1 - A = 0$ and $At + B = 0$. Hence $A = 1$, $B = -t$ and $\lambda(s) = s - t$.

Substituting the $\lambda(s)$ back in (3.12) yields,

$$u_{n+1}(t) = u_n(t) + \int_0^t (s-t)\left[(u_n)_{ss} + \tilde{f}_n(u_n, u_n', u_n'')\right]ds. \tag{3.15}$$

(IV) $u'' + \alpha^2 u + f(u, u', u'') = 0$

Repeating part (3) steps yields,

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\left[(u_n)_{ss} + \alpha^2 u_n + \tilde{f}_n(u_n, u_n', u_n'')\right]ds. \tag{3.16}$$

Taking the variation with respect to u_n implies

$$
\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda(s) \left[(u_n)_{ss} + \alpha^2 u_n + \tilde{f}_n(u_n, u_n', u_n'') \right] ds \\
&= \delta u_n(t) + \delta \int_0^t \left[\lambda(s)(u_n)_{ss} + \lambda(s)\alpha^2 u_n + \lambda(s)\tilde{f}_n(u_n, u_n', u_n'') \right] ds \\
&= \delta u_n(t) + \delta \int_0^t \lambda(s)(u_n)_{ss}\, ds + \delta \int_0^t \alpha^2 \lambda(s) u_n\, ds \\
&+ \delta \int_0^t \lambda(s) \tilde{f}_n(u_n, u_n', u_n'')\, ds
\end{aligned}
$$

where $\delta \tilde{f}_n(u, u', u'') = 0$.

Carrying out integration by parts twice for the above equation yields

$$
\begin{aligned}
\int_0^t \lambda(s)(u_n)_{ss}\, ds &= \lambda(s)(u_n)_s - \int_0^t \lambda'(s)(u_n)_s\, ds \\
&= \lambda(s)(u_n)_s - \lambda'(s)u_n(s)\Big|_0^t + \int_0^t \lambda''(s) u_n(s)\, ds \\
&= \lambda(t)(u_n)_s - \lambda'(t)u_n(t) - \lambda(0)(u_n)_s + \lambda'(0)u_n(0) \\
&+ \int_0^t \lambda''(s) u_n(s)\, ds \\
&= \lambda(t)(u_n)_s - \lambda'(t)u_n(t) + \int_0^t \lambda''(s) u_n(s)\, ds.
\end{aligned}
$$

Substituting it back we obtain the following

$$
\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \delta\left[\lambda(t)(u_n)_s\right] - \delta\left[\lambda'(t)u_n(t)\right] \\
&+ \delta \int_0^t \lambda''(s) u_n(s)\, ds + \delta \int_0^t \alpha^2 \lambda(s) u_n(s)\, ds.
\end{aligned} \tag{3.17}
$$

or equivalently

$$
\begin{aligned}
\delta u_{n+1}(t) &= \left[1 - \lambda'(t)\right] \delta u_n(t) + \delta\left[\lambda(t)(u_n)_s\right] \\
&+ \delta \int_0^t \lambda''(s) u_n(s)\, ds + \delta \int_0^t \alpha^2 \lambda(s) u_n(s)\, ds.
\end{aligned} \tag{3.18}
$$

By setting the variation $\delta u_{n+1}(t) = 0$ we obtain the following stationary conditions,

$$\lambda''(s) + \alpha^2 \lambda(s) = 0,$$

$$1 - \left.\lambda'(s)\right|_{s=t} = 0$$

$$\left.\lambda(s)\right|_{s=t} = 0.$$

Solving the above system of equations yields

$1 + \alpha\left(A\sin(\alpha t) - B\cos(\alpha t)\right) = 0$, and $A\cos(\alpha t) + B\sin(\alpha t) = 0$

Hence $A = \frac{-\sin(\alpha t)}{\alpha}$, $B = \frac{\cos(\alpha t)}{\alpha}$ and $\lambda(s) = \frac{1}{\alpha}\sin(\alpha(s-t))$.

Substituting the $\lambda(s)$ back in (3.16) yields

$$u_{n+1}(t) = u_n(t) + \frac{1}{\alpha}\int_0^t sin(\alpha(s-t))\left[(u_n)_{ss} + \alpha^2 u_n + \tilde{f}_n(u_n, u_n', u_n'')\right] ds. \tag{3.19}$$

(V) $u'' - \alpha^2 u + f(u, u', u'') = 0$

Applying the variational iteration method,we obtain

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\left[(u_n)_{ss} - \alpha^2 u_n + \tilde{f}_n(u_n, u_n', u_n'')\right] ds. \tag{3.20}$$

Taking the variation with respect to u_n

$$\begin{aligned} \delta u_{n+1}(t) &= \delta u_n(t) + \delta\int_0^t \lambda(s)\left[(u_n)_{ss} - \alpha^2 u_n + \tilde{f}_n(u_n, u_n', u_n'')\right] ds \\ &= \delta u_n(t) + \delta\int_0^t \left[\lambda(s)(u_n)_{ss} - \lambda(s)\alpha^2 u_n + \lambda(s)\tilde{f}_n(u_n, u_n', u_n'')\right] ds \\ &= \delta u_n(t) + \delta\int_0^t \lambda(s)(u_n)_{ss}\, ds \\ &- \lambda(s)\left[\delta\int_0^t \alpha^2 u_n\, ds + \delta\int_0^t \tilde{f}_n(u_n, u_n', u_n'')\, ds\right] \end{aligned}$$

where $\delta\tilde{f}_n(u, u', u'') = 0$.

Integrating by parts twice the above equation gives us

$$
\begin{aligned}
\int_0^t \lambda(s)(u_n)_{ss}\, ds &= \lambda(s)(u_n)_s - \int_0^t \lambda'(s)(u_n)_s\, ds \\
&= \lambda(s)(u_n)_s - \lambda'(s)\, u_n(s)\big|_0^t + \int_0^t \lambda''(s)\, u_n(s)\, ds \\
&= \lambda(t)(u_n)_s - \lambda'(t)\, u_n(t) - \lambda(0)(u_n)_s + \lambda'(0)\, u_n(0) \\
&+ \int_0^t \lambda''(s)\, u_n(s)\, ds \\
&= \lambda(t)(u_n)_s - \lambda'(t)\, u_n(t) + \int_0^t \lambda''(s)\, u_n(s)\, ds.
\end{aligned}
$$

Substituting it back yields

$$
\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \delta\left[\lambda(t)(u_n)_s\right] - \delta\left[\lambda'(t)\, u_n(t)\right] \\
&+ \delta \int_0^t \lambda''(s)\, u_n(s)\, ds - \delta \int_0^t \alpha^2 \lambda(s)\, u_n(s)\, ds.
\end{aligned}
\tag{3.21}
$$

or equivalently

$$
\begin{aligned}
&\delta u_{n+1}(t) = \left[1 - \lambda'(t)\right] \delta u_n(t) + \delta\left[\lambda(t)(u_n)_s\right] \\
&+\delta \int_0^t \lambda''(s)\, u_n(s)\, ds - \delta \int_0^t \alpha^2 \lambda(s)\, u_n(s)\, ds.
\end{aligned}
\tag{3.22}
$$

By setting the variation $\delta u_{n+1}(t) = 0$ we obtain the following stationary conditions

$$
\begin{aligned}
&\lambda''(s) - \alpha^2 \lambda(s) = 0, \\
&1 - \lambda'(s)\big|_{s=t} = 0 \\
&\lambda(s)\big|_{s=t} = 0.
\end{aligned}
$$

Solving the above system of equations yields the following,

$1 + \alpha\left(Be^{-\alpha t} - Ae^{\alpha t}\right) = 0$, and $Ae^{\alpha t} + Be^{-\alpha t} = 0$. Hence, $A = \frac{e^{-\alpha t}}{2\alpha}$, $B = \frac{-e^{\alpha t}}{2\alpha}$ and $\lambda(s) = \frac{1}{2\alpha}\left(e^{\alpha(s-t)} - e^{\alpha(t-s)}\right)$.

Substituting the $\lambda(s)$ back in (3.20) yields

$$
u_{n+1}(t) = u_n(t) + \frac{1}{2\alpha} \int_0^t \left(e^{\alpha(s-t)} - e^{\alpha(t-s)}\right) \left[(u_n)_{ss} - \alpha^2 u_n + \tilde{f}_n\left(u_n, u_n', u_n''\right)\right] ds.
\tag{3.23}
$$

(VI) $u''' + f\left(u, u', u'', u'''\right) = 0$

Applying the variational iteration method,we obtain

$$u_{n+1}\left(t\right) = u_n\left(t\right) + \int\limits_0^t \lambda\left(s\right)\left[\left(u_n\right)_{sss} + \tilde{f}_n\left(u_n, u_n', u_n'', u_n'''\right)\right] ds. \tag{3.24}$$

Taking the variation with respect to u_n

$$\begin{aligned}
\delta u_{n+1}\left(t\right) &= \delta u_n\left(t\right) + \delta\int\limits_0^t \lambda\left(s\right)\left[\left(u_n\right)_{sss} + \tilde{f}_n\left(u_n, u_n', u_n''\right)\right] ds \\
&= \delta u_n\left(t\right) + \delta\int\limits_0^t \left[\lambda\left(s\right)\left(u_n\right)_{sss} + \lambda\left(s\right)\tilde{f}_n\left(u_n, u_n', u_n'', u_n'''\right)\right] ds \\
&= \delta u_n\left(t\right) + \delta\int\limits_0^t \lambda\left(s\right)\left(u_n\right)_{sss} ds + \delta\int\limits_0^t \lambda\left(s\right)\tilde{f}_n\left(u_n, u_n', u_n'', u_n'''\right) ds
\end{aligned}$$

where $\delta\tilde{f}_n(u, u', u'', u''') = 0$.

Integrating by parts thrice the above equation yields,

$$\begin{aligned}
\int\limits_0^t \lambda\left(s\right)\left(u_n\right)_{sss} ds &= \lambda\left(s\right)\left(u_n\right)_{ss} - \int\limits_0^t \lambda'\left(s\right)\left(u_n\right)_s ds \\
&= \lambda\left(s\right)\left(u_n\right)_{ss} - \lambda'\left(s\right)_s + \int\limits_0^t \lambda''\left(s\right)u_n\left(s\right) ds \\
&= \lambda\left(s\right)\left(u_n\right)_{ss} - \lambda'\left(s\right)\left(u_n\right)_s + \lambda''\left(s\right)\left(u_n\right)\left(s\right)\Big|_0^t \\
&- \int\limits_0^t \lambda'''\left(s\right)\left(u_n\right)\left(s\right) ds \\
&= \lambda\left(t\right)\left(u_n\right)_{ss}\left(t\right) - \lambda'\left(t\right)\left(u_n\right)_s\left(t\right) + \lambda''\left(t\right)\left(u_n\right)\left(t\right) \\
&- \lambda\left(0\right)\left(u_n\right)_{ss}\left(0\right) + \lambda'\left(0\right)\left(u_n\right)_s\left(0\right) - \lambda''\left(0\right)\left(u_n\right)\left(0\right) \\
&- \int\limits_0^t \lambda'''\left(s\right)\left(u_n\right)\left(s\right) ds \\
&= \lambda\left(t\right)\left(u_n\right)_{ss}\left(t\right) - \lambda'\left(t\right)\left(u_n\right)_s\left(t\right) + \lambda''\left(t\right)\left(u_n\right)\left(t\right) \\
&- \int\limits_0^t \lambda'''\left(s\right)u_n\left(s\right) ds.
\end{aligned}$$

Substituting it back,we obtain the following

$$\begin{aligned}\delta u_{n+1}(t) &= \delta u_n(t) + \delta\left[\lambda(t)(u_n)_{ss}\right] + \delta\left[\lambda'(t)(u_n)_s\right] \\ &+ \delta\left[\lambda''(t)(u_n)(t)\right] - \delta\int\limits_0^t \lambda'''(s)(u_n)(s)\,ds.\end{aligned} \tag{3.25}$$

or equivalently,

$$\begin{aligned}\delta u_{n+1}(t) &= \delta u_n\left[1+\lambda''(t)\right] + \delta\left[\lambda(t)(u_n)_{ss}\right] - \delta\left[\lambda'(t)(u_n)_s\right] \\ &- \delta\int\limits_0^t \lambda'''(s)\,u_n(s)\,ds.\end{aligned} \tag{3.26}$$

By setting the variation $\delta u_{n+1}(t) = 0$ we obtain the following stationary conditions,

$$\lambda'''(s) = 0$$

$$1 + \lambda''(s)\big|_{s=t} = 0$$

$$\lambda'(s)\big|_{s=t} = 0$$

$$\lambda(s)\big|_{s=t} = 0.$$

Solving the above system of equations yields the following

$As^2 + Bs + C = 0$, $1 - 2A = 0$, $2At + B = 0$ and $At^2 + Bt + C = 0$. Hence, $A = \frac{1}{2}$, $B = -t$, $C = \frac{1}{2}t^2$ and $\lambda(s) = \frac{1}{2}s^2 - ts + \frac{1}{2}t^2 = \frac{1}{2}(s-t)^2$.

Substituting the $\lambda(s)$ back in (3.24) yields,

$$u_{n+1}(t) = u_n(t) + \frac{1}{2}\int\limits_0^t (s-t)^2\left[(u_n)_{sss} + \tilde{f}_n(u, u', u'', u''')\right] ds. \tag{3.27}$$

(VII) $u^{(4)} + f\left(u, u', u'', u''', u^{(4)}\right) = 0$

Repeating the previous steps yields,

$$u_{n+1}(t) = u_n(t) + \int\limits_0^t \lambda(s)\left[(u_n)_{ssss} + \tilde{f}_n\left(u_n, u_n', u_n'', u_n''', u^{(4)}\right)\right] ds. \tag{3.28}$$

Taking the variation with respect to u_n,

$$
\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda(s)\left[(u_n)_{ssss} + \tilde{f}_n(u_n, u_n', u_n'')\right] ds \\
&= \delta u_n(t) + \delta \int_0^t \left[\lambda(s)(u_n)_{ssss} + \lambda(s)\tilde{f}_n\left(u_n, u_n', u_n'', u_n''', u^{(4)}\right)\right] ds \\
&= \delta u_n(t) + \delta \int_0^t \lambda(s)(u_n)_{ssss}\, ds + \delta \int_0^t \lambda(s)\tilde{f}_n\left(u_n, u_n', u_n'', u_n''', u^{(4)}\right) ds
\end{aligned}
$$

where $\delta \tilde{f}_n\left(u, u', u'', u''', u^{(4)}\right) = 0$.

Integrating by parts four times the above equation yields

$$
\begin{aligned}
\int_0^t \lambda(s)(u_n)_{ssss}\, ds &= \lambda(s)(u_n)_{sss} - \int_0^t \lambda'(s)(u_n)_{sss}\, ds \\
&= \lambda(s)(u_n)_{sss} - \lambda'(s)_{ss} + \int_0^t \lambda''(s)(u_n)_{ss}(s)\, ds \\
&= \lambda(s)(u_n)_{sss} - \lambda'(s)(u_n)_{ss} + \lambda''(s)(u_n)_s \\
&- \int_0^t \lambda'''(s)(u_n)_s\, ds \\
&= \lambda(t)(u_n)_{sss}(t) - \lambda'(t)(u_n)_{ss}(t) + \lambda''(s)(u_n)(s) \\
&- \lambda'''(s)(u_n)\big|_0^t + \int_0^t \lambda^{(4)}(s)(u_n)(s)\, ds \\
&= \lambda(t)(u_n)_{sss} - \lambda'(t)(u_n)_{ss} + \lambda''(t)(u_n)_s - \lambda'''(t)(u_n) \\
&- \lambda(0)(u_n)_{sss} - \lambda'(0)(u_n)_{ss} + \lambda''(0)(u_n)_s - \lambda'''(0)(u_n) \\
&+ \int_0^t \lambda^{(4)}(s)\, u_n(s)\, ds.
\end{aligned}
$$

Substituting it back,we obtain the following

$$
\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \delta\left[\lambda(t)(u_n)_{sss}\right] - \delta\left[\lambda'(t)(u_n)_{ss}\right] \\
&+ \delta\left[\lambda''(t)(u_n)_s(t)\right] - \delta\left[\lambda'''(s)(u_n)\right] + \delta \int_0^t \lambda^{(4)}(s)(u_n)(s)\, ds.
\end{aligned} \tag{3.29}
$$

or equivalently,

$$\begin{aligned}\delta u_{n+1}(t) &= \delta u_n\left[1-\lambda'''(t)\right]+\delta\left[\lambda(t)(u_n)_{sss}\right]-\delta\left[\lambda'(t)(u_n)_{ss}\right] \\ &+ \delta\left[\lambda''(t)(u_n)_s(t)\right]+\delta\int_0^t \lambda^{(4)}(s)\,u_n(s)\,ds.\end{aligned} \tag{3.30}$$

By setting the variation $\delta u_{n+1}(t)=0$ we obtain the following stationary conditions

$$\lambda^{(4)}(s)\big|_{s=t}=0$$

$$1-\lambda'''(s)\big|_{s=t}=0$$

$$\lambda''(s)\big|_{s=t}=0$$

$$\lambda'(s)\big|_{s=t}=0$$

$$\lambda(s)\big|_{s=t}=0.$$

Solving the above system of equations yields the following

$As^3+Bs^2+Cs+D=0$, $1-6A=0$, $6At+2B=0$,

$3At^2+2Bt+C=0$, and $At^3+Bt^2+Ct+D=0$. Hence,

$$A=\frac{1}{6},B=\frac{-t}{2},C=\frac{1}{2}t^2,D=\frac{-1}{6}t^3,$$

and

$$\lambda(s)=\frac{1}{6}s^3-\frac{t}{2}s^2+\frac{s}{2}t^2-\frac{1}{6}t^3=\frac{1}{6}(s-t)^3.$$

Substituting the $\lambda(s)$ back in (3.28) gives us,

$$u_{n+1}(t)=u_n(t)+\frac{1}{6}\int_0^t (s-t)^3\left[(u_n)_{ssss}+\tilde{f}_n\left(u,u',u'',u^{(4)}\right)\right]ds. \tag{3.31}$$

3.2 The Generalized Variational Iteration Method (GVIM)

It is important to note that the VIM includes the left endpoint $x=a$ but not the right endpoint $x=b$. Thus, the VIM is powerful for the initial value problems (IVPs). However, the VIM is considered as a setback when dealing with BVPs and affects the accuracy of the solution. Based on this drawback we will modify the correctional function for BVPs in order to include both endpoints, $x = a$ and $x = b$. This modification is referred to as the Generalized Variational Method (GVIM). We consider the following modified correction functional for BVPs

$$\begin{aligned} u_{n+1}(t) &= u_n(t) + \int_a^t \lambda_1(s;t)\left[L(u_n)_s + N\tilde{u}_n(s) - f(s)\right] ds \\ &+ \int_t^b \lambda_2(s;t)\left[L(u_n)_s + N\tilde{u}_n(s) - f(s)\right] ds, \end{aligned} \tag{3.32}$$

where $\lambda_1(s;t)$ and $\lambda_2(s;t)$ are two general Lagrange multipliers defined on the intervals $[a,t]$ and $[t,b]$ respectively and satisfy the homogeneous BCs at $t = b$ and $t = a$ respectively. The $\tilde{u}_n$ is a restricted variation i.e $\delta\tilde{u}_n = 0$.

Next, we will derive the proper correctional functionals for the three classes of BVPs.

(I) $u'' + f(t, u, u', u'') = 0$ where $a \le t \le b$ and subject to the boundary conditions(BCs):

$$u(a) = \alpha, u(b) = \beta. \tag{3.33}$$

The correction function in (3.32) becomes

$$\begin{aligned} u_{n+1}(t) &= u_n(t) + \int_a^t \lambda_1(s;t)\left[u_n''(s) + \tilde{f}(s, u_n(s), u_n'(s), u_n''(s))\right] ds \\ &+ \int_t^b \lambda_2(s;t)\left[u_n''(s) + \tilde{f}(s, u_n(s), u_n'(s), u_n''(s))\right] ds. \end{aligned} \tag{3.34}$$

Note that substituting $t = b$ on both sides of the equation makes the second integral zero. Thus, we require $\lambda_1(s;b) = 0$ to render the first integral to zero. Similar argument holds for λ_2. To find the optimal values of λ_1 and λ_2 we integrate by parts twice the first term within each of the integrals in (3.34) and we obtain,

$$\begin{aligned} \int_a^t \lambda_1(s;t)\, u_n''(s)\, ds &= \lambda_1(s;t)\, u_n'(s)\big|_a^t - \lambda_1'(s;t)\, u_n(s)\big|_a^t \\ &+ \int_a^t \lambda_1''(s;t)\, u_n(s)\, ds + \lambda_1(t;t)\, u_n'(t) - \lambda_1'(t;t)\, u_n(t) \\ &- \lambda_1(a;t)\, u_n'(a) + \lambda_1'(a;t)\, u_n(a) + \int_a^t \lambda_1''(s;t)\, u_n(s)\, ds. \end{aligned}$$

Similarly,

$$\begin{aligned}
\int_t^b \lambda_2(s;t)\,u_n''(s)\,ds &= \lambda_2(s;t)\,u_n'(s)\big|_t^b - \lambda_2'(s;t)\,u_n(s)\big|_t^b + \int_t^b \lambda_2''(s;t)\,u_n(s)\;ds \\
&= \lambda_2(b;t)\,u_n'(b) - \lambda_2'(b;t)\,u_n(b) - \lambda_2(t;t)\,u_n'(t) \\
&+ \lambda_2'(t;t)\,u_n(t) + \int_t^b \lambda_2''(s;t)\,u_n(s)\;ds.
\end{aligned}$$

Replacing them back in (3.34) yields

$$\begin{aligned}
u_{n+1}(t) &= [1-\lambda_1'(t;t)+\lambda_2'(t;t)]\,u_n(t) + [\lambda_1(t;t)-\lambda_2(t;t)]\,u_n'(t) \\
&- \lambda_1(a;t)\,u_n'(a) + \lambda_1'(a;t)\,u_n(a) + \lambda_2(b;t)\,u_n'(b) - \lambda_2'(b;t)\,u_n(b) \\
&+ \int_a^t \lambda_1''(s;t)\,u_n(s)\;ds + \int_t^b \lambda_2''(s;t)\,u_n(s)\;ds \\
&+ \int_a^t \lambda_1(s;t)\,\tilde{f}(s,u_n(s),u_n'(s),u_n''(s))\;ds \\
&+ \int_t^b \lambda_2(s;t)\,\tilde{f}(s,u_n(s),u_n'(s),u_n''(s))\;ds.
\end{aligned}$$

Next, we take the variation with respect to $u_n(t)$ on both sides of above equation. Taking into account that $\tilde{f}$ is a restricted variation i.e $\delta\tilde{f}(s,u_n(s),u_n'(s),u_n''(s)) = 0$, we obtain

$$\begin{aligned}
\delta u_{n+1}(t) &= [1-\lambda_1'(t;t)+\lambda_2'(t;t)]\,\delta u_n(t) + [\lambda_1(t;t)-\lambda_2(t;t)]\,\delta u_n'(t) \\
&+ \delta\left(\int_a^t \lambda_1''(s;t)\,u_n(s)\;ds\right) + \delta\left(\int_t^b \lambda_2''(s;t)\,u_n(s)\;ds\right).
\end{aligned} \tag{3.35}$$

We set the variation $\delta u_{n+1}(t) = 0$ to obtain the following stationary conditions

$$1-\lambda_1'(s;t)+\lambda_2'(s;t)|_{s=t}=0,$$

$$\lambda_1(s;t)-\lambda_2(s;t)|_{s=t}=0,$$

$$\lambda_1''(s;t)=0, \quad a\le s\le t,$$

$$\lambda_2''(s;t)=0, \quad t\le s\le b.$$

Solving the above system of equations with the fact that $\lambda_1(s;a)=0$ and $\lambda_2(s;b)=0$ implies,

$$\lambda_1(s;t)=\frac{t-b}{a-b}s+\frac{a(b-t)}{a-b}, \quad a\le s\le t,$$

and

$$\lambda_2(s;t)=\frac{t-a}{a-b}s+\frac{b(a-t)}{a-b}, \quad t\le s\le b.$$

Thus, the GVIM for this case is

$$\begin{aligned} u_{n+1}(t) &= u_n(t)+\int_a^t\left(\frac{t-b}{a-b}s+\frac{a(b-t)}{a-b}\right)\left[u_n''(s)+\tilde{f}(s,u_n(s),u_n'(s),u_n''(s))\right]ds \\ &+ \int_t^b\left(\frac{t-a}{a-b}s+\frac{b(a-t)}{a-b}\right)\left[u_n''(s)+\tilde{f}(s,u_n(s),u_n'(s),u_n''(s))\right]ds. \end{aligned} \tag{3.36}$$

(II) $u''-k^2u+f(t,u,u',u'')=0$

where $a\le t\le b$ and subject to the BCs:

$$u(a)=\alpha, u(b)=\beta. \tag{3.37}$$

The correction function in (3.32) becomes

$$\begin{aligned} u_{n+1}(t) &= u_n(t)+\int_a^t\lambda_1(s;t)\left[u_n''(s)-k^2u_n(s)+\tilde{f}(s,u_n(s),u_n'(s),u_n''(s))\right]ds \\ &+ \int_t^b\lambda_2(s;t)\left[u_n''(s)-k^2u_n(s)+\tilde{f}(s,u_n(s),u_n'(s),u_n''(s))\right]ds. \end{aligned} \tag{3.38}$$

Similar to case (I) substituting $t=b$ on both sides of the equation makes the second integral zero. Thus, we require $\lambda_1(s;b)=0$ to render the first integral to zero. Similar argument holds for λ_2.

To find the optimal values of λ_1 and λ_2 we integrate by parts twice the first term within each of the integrals in (3.38) obtaining,

$$
\begin{aligned}
\int_a^t \lambda_1(s;t)\,u_n''(s)\,ds &= \lambda_1(s;t)\,u_n'(s)\big|_a^t - \lambda_1'(s;t)\,u_n(s)\big|_a^t + \int_a^t \lambda_1''(s;t)\,u_n(s)\;ds \\
&= \lambda_1(t;t)\,u_n'(t) - \lambda_1'(t;t)\,u_n(t) - \lambda_1(a;t)\,u_n'(a) \\
&+ \lambda_1'(a;t)\,u_n(a) + \int_a^t \lambda_1''(s;t)\,u_n(s)\;ds.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\int_t^b \lambda_2(s;t)\,u_n''(s)\,ds &= \lambda_2(s;t)\,u_n'(s)\big|_t^b - \lambda_2'(s;t)\,u_n(s)\big|_t^b + \int_t^b \lambda_2''(s;t)\,u_n(s)\;ds \\
&= \lambda_2(b;t)\,u_n'(b) - \lambda_2'(b;t)\,u_n(b) - \lambda_2(t;t)\,u_n'(t) \\
&+ \lambda_2'(t;t)\,u_n(t) + \int_t^b \lambda_2''(s;t)\,u_n(s)\;ds.
\end{aligned}
$$

Replacing them back in (3.38) yields,

$$
\begin{aligned}
u_{n+1}(t) &= \left[1 - \lambda_1'(t;t) + \lambda_2'(t;t)\right]u_n(t) + \left[\lambda_1(t;t) - \lambda_2(t;t)\right]u_n'(t) \\
&- \lambda_1(a;t)\,u_n'(a) + \lambda_1'(a;t)\,u_n(a) + \lambda_2(b;t)\,u_n'(b) - \lambda_2'(b;t)\,u_n(b) \\
&+ \int_a^t \left[\lambda_1''(s;t) - k^2\lambda_1(s;t)\right]u_n(s)\;ds + \int_t^b \left[\lambda_2''(s;t) - k^2\lambda_2(s;t)\right]u_n(s)\;ds \\
&+ \int_a^t \lambda_1(s;t)\,\tilde{f}\left(s, u_n(s), u_n'(s), u_n''(s)\right)\;ds \\
&+ \int_t^b \lambda_2(s;t)\,\tilde{f}\left(s, u_n(s), u_n'(s), u_n''(s)\right)\;ds.
\end{aligned}
$$

Next, we take the variation with respect to $u_n(t)$ on both sides of equation (3.38). Taking into account that $\tilde{f}$ is a restricted variation i.e $\delta\tilde{f}\left(s, u_n(s), u_n'(s), u_n''(s)\right) =$

0, we obtain

$$\delta u_{n+1}(t) = \left[1-\lambda_1'(t;t)+\lambda_2'(t;t)\right]\delta u_n(t)+\left[\lambda_1(t;t)-\lambda_2(t;t)\right]\delta u_n'(t) + \delta\left(\int_a^t \left[\lambda_1''(s;t)-k^2\lambda_1(s;t)\right]u_n(s)\,ds\right) + \delta\left(\int_t^b \left[\lambda_2''(s;t)-k^2\lambda_2(s;t)\right]u_n(s)\,ds\right). \tag{3.39}$$

We set the variation $\delta u_{n+1}(t)=0$ to obtain the following stationary conditions

$$1-\lambda_1'(s;t)+\lambda_2'(s;t)\big|_{s=t}=0,$$

$$\lambda_1(s;t)-\lambda_2(s;t)\big|_{s=t}=0,$$

$$\lambda_1''(s;t)-k^2\lambda_1(s;t)=0, \quad a\le s\le t,$$

$$\lambda_2''(s;t)-k^2\lambda_2(s;t)=0, \quad t\le s\le b.$$

Solving the above system of equations with the fact that $\lambda_1(s;a)=0$ and $\lambda_2(s;b)=0$ gives us

$$\lambda_1(s;t)=\frac{\sinh(k(b-t))\sinh(k(a-s))}{k\sinh(k(a-b))}, \quad a\le s\le t,$$

and

$$\lambda_2(s;t)=\frac{\sinh(k(a-t))\sinh(k(b-s))}{k\sinh(k(a-b))}, \quad t\le s\le b.$$

Thus, the GVIM for this case is:

$$u_{n+1}(t) = u_n(t)+\int_a^t\left(\frac{\sinh(k(b-t))\sinh(k(a-s))}{k\sinh(k(a-b))}\right)\left[u_n''(s)-k^2u_n+\tilde f(s,u_n(s),u_n'(s),u_n''(s))\right]ds + \int_t^b\left(\frac{\sinh(k(a-t))\sinh(k(b-s))}{k\sinh(k(a-b))}\right).\left[u_n''(s)-k^2u_n+\tilde f(s,u_n(s),u_n'(s),u_n''(s))\right]ds \tag{3.40}$$

(III) $u'''+f(t,u,u',u'',u''')=0$

where $a \leq t \leq b$ and subject to the BCs:

$$u(a) = \alpha, u'(a) = \beta, u(b) = \eta. \tag{3.41}$$

The correction function in (3.32) becomes

$$\begin{aligned} u_{n+1}(t) &= u_n(t) + \int_a^t \lambda_1(s;t) \left[u_n'''(s) + \tilde{f}(s, u_n(s), u_n'(s), u_n''(s), u'''(s)) \right] ds \\ &+ \int_t^b \lambda_2(s;t) \left[u_n''(s) + \tilde{f}(s, u_n(s), u_n'(s), u_n''(s), u'''(s)) \right] ds, \end{aligned} \tag{3.42}$$

Similar to the previous cases substituting $t = b$ on both sides of the equation makes the second integral zero. Thus, we require $\lambda_1(s;b) = 0$ to render the first integral to zero. Similar argument holds for λ_2. To find the optimal values of λ_1 and λ_2 we integrate by parts three times the first term within each of the integrals in (3.42) obtaining,

$$\begin{aligned} \int_a^t \lambda_1(s;t) u_n'''(s) ds &= \lambda_1(s;t) u_n''(s)\big|_a^t - \lambda_1'(s;t) u_n'(s)\big|_a^t + \lambda_1''(s;t) u_n\big|_a^t \\ &- \int_a^t \lambda_1'''(s;t) u_n(s) \, ds \\ &= \lambda_1(t;t) u_n''(t) - \lambda_1'(t;t) u_n'(t) + \lambda_1''(t;t) u_n(t) \\ &- \lambda_1(a;t) u_n''(a) + \lambda_1'(a;t) u_n'(a) - \lambda_1''(a;t) u_n(a) \\ &- \int_a^t \lambda_1'''(s;t) u_n(s) \, ds. \end{aligned}$$

Similarly,

$$
\begin{aligned}
\int_t^b \lambda_2(s;t)\, u_n'''(s)\, ds &= \lambda_2(s;t)\, u_n''(s)\big|_t^b - \lambda_2'(s;t)\, u_n'(s)\big|_t^b + \lambda_2''(s;t)\, u_n\big|_t^b \\
&- \int_t^b \lambda_2'''(s;t)\, u_n(s)\ ds \\
&= \lambda_2(b;t)\, u_n''(b) - \lambda_2'(b;t)\, u_n'(b) + \lambda_2''(b;t)\, u_n(b) \\
&- \lambda_2(t;t)\, u_n''(t) + \lambda_2'(t;t)\, u_n'(t) - \lambda_2''(t;t)\, u_n(t) \\
&- \int_t^b \lambda_2'''(s;t)\, u_n(s)\ ds.
\end{aligned}
$$

Replacing them back in (3.42) yields,

$$
\begin{aligned}
u_{n+1}(t) &= [1 + \lambda_1''(t;t) - \lambda_2''(t;t)]\, u_n(t) + [-\lambda_1'(t;t) + \lambda_2'(t;t)]\, u_n'(t) \\
&+ [\lambda_1(t;t) - \lambda_2(t;t)]\, u_n''(t) - \lambda_1(a;t)\, u_n''(a) + \lambda_1'(a;t)\, u_n'(a) \\
&- \lambda_1''(a;t)\, u_n(a) + \lambda_2(b;t)\, u_n''(b) - \lambda_2'(b;t)\, u_n'(b) + \lambda_2''(b;t)\, u_n(b) \\
&- \int_a^t \lambda_1'''(s;t)\, u_n(s)\ ds - \int_t^b \lambda_2'''(s;t)\, u_n(s)\ ds \\
&+ \int_a^t \lambda_1(s;t)\, \tilde{f}(s, u_n(s), u_n'(s), u_n''(s), u_n'''(s))\ ds \\
&+ \int_t^b \lambda_2(s;t)\, \tilde{f}(s, u_n(s), u_n'(s), u_n''(s), u_n'''(s))\ ds.
\end{aligned}
$$

Next, we take the variation with respect to $u_n(t)$ of both sides of the above equation. Taking into account that $\tilde{f}$ is a restricted variation i.e

$\delta f(s, u_n(s), u_n'(s), u_n''(s), u'''(s)) = 0$, we obtain

$$
\begin{aligned}
u_{n+1}(t) &= [1 + \lambda_1''(t;t) - \lambda_2''(t;t)]\, \delta u_n(t) + [-\lambda_1'(t;t) + \lambda_2'(t;t)]\, \delta u_n'(t) \\
&+ [\lambda_1(t;t) - \lambda_2(t;t)]\, \delta u_n''(t) - \delta\left(\int_a^t \lambda_1'''(s;t)\, u_n(s)\ ds\right) \\
&- \delta\left(\int_t^b \lambda_2'''(s;t)\, u_n(s)\ ds\right).
\end{aligned}
\tag{3.43}
$$

We set the variation $\delta u_{n+1}(t) = 0$ to obtain the following stationary conditions

$$1+\lambda_1''(s;t)-\lambda_2''(s;t)\big|_{s=t}=0,$$

$$-\lambda_1'(s;t)+\lambda_2'(s;t)\big|_{s=t}=0,$$

$$\lambda_1(s;t)-\lambda_2(s;t)\big|_{s=t}=0,$$

$$\lambda_1'''(s;t)=0,\quad a\le s\le t,$$

$$\lambda_2'''(s;t)=0,\quad t\le s\le b.$$

Solving the above system of equations with the fact that $\lambda_1(s;a)=0$ and $\lambda_2(s;b)=0$ gives us

$$\lambda_1(s;t)=\frac{1}{2}\frac{(a-s)(b-t)}{(a-b)^2}\left(ab-2as+at+bs-2bt+st\right),\quad a\le s\le t,$$

and

$$\lambda_2(s;t)=\frac{1}{2}\frac{(a-t)^2(b-s)^2}{(a-b)^2},\quad t\le s\le b.$$

Thus, the GVIM for this case is:

$$\begin{aligned} u_{n+1}(t) &= u_n(t)+\int_a^t\left(\frac{1}{2}\frac{(a-s)(b-t)}{(a-b)^2}\left(ab-2as+at+bs-2bt+st\right)\right) \\ &\quad \left[u_n'''(s)+\tilde{f}\left(s,u_n(s),u_n'(s),u_n''(s),u_n'''(s)\right)\right]ds \\ &+ \int_t^b\left(\frac{1}{2}\frac{(a-t)^2(b-s)^2}{(a-b)^2}\right)\left[u_n'''(s)+\tilde{f}\left(s,u_n(s),u_n'(s),u_n''(s),u_n'''(s)\right)\right]ds. \end{aligned} \tag{3.44}$$

Chapter 4: Convergence Analysis

This section includes the proof of the convergence of the GVIM. Without loss of generality, we will prove the convergence of GVIM for the following class of boundary value problem.

$$u'' + \tilde{f}\left(s, u, u', u''\right) = 0. \tag{4.1}$$

Then

$$u_{n+1}\left(t\right) = u_n\left(t\right) + \int_a^b \lambda(s,t)\left[u'' + \tilde{f}\left(s, u, u', u''\right)\right] ds. \tag{4.2}$$

Define the operator T_λ as follows

$$T_\lambda\left(u_n\right) = u_n\left(t\right) + \int_a^b \lambda(s,t)\left[u'' + \tilde{f}\left(s, u, u', u''\right)\right] ds. \tag{4.3}$$

To prove convergence it suffices to show that T_λ is a contractive mapping on $C^2([a,b])$. In other words, we will show that

$$|T_\lambda(u) - T_\lambda(v)| \leq K||u - v||. \tag{4.4}$$

where $K < 1$.

After two integration by parts of the integral in (4.3), we obtain:

$$T_\lambda\left(u\right) = \int_a^b \lambda(s,t)\left[u + \tilde{f}\left(s, u, u', u''\right)\right] ds. \tag{4.5}$$

Plugging (4.5) in (4.4) yields,

$$
\begin{aligned}
|T_\lambda(u) - T_\lambda(v)| &\leq \int_a^b |\lambda(s,t)| \left| \tilde{f}(s,u,u',u'') - \tilde{f}(s,v,v',v'') \right| ds \\
&\leq \max_{(t,s)\epsilon[a,b]} |\lambda(s,t)| \int_a^b \left| \tilde{f}(s,u,u',u'') - \tilde{f}(s,v,v',v'') \right| ds \\
&= \left| \lambda\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \right| \int_a^b \left| \tilde{f}(s,u,u',u'') - \tilde{f}(s,v,v',v'') \right| ds \\
&= \frac{b-a}{4} \int_a^b \left| \tilde{f}(s,u,u',u'') - \tilde{f}(s,v,v',v'') \right| ds.
\end{aligned}
\tag{4.6}
$$

Now, if we assume that $\tilde{f}$ satisfies a Lipshitz condition of the form

$$
\left| \tilde{f}(s,u,u',u'') - \tilde{f}(s,v,v',v'') \right| \leq N|u-v| + L|u'-v'| + M|u''-v''|, \tag{4.7}
$$

inequality (4.6) becomes

$$
\begin{aligned}
|T_\lambda(u) - T_\lambda(v)| &\leq \frac{b-a}{4} \int_a^b (N|u-v| + L|u'-v'| + M|u''-v''|)\, ds \\
&\leq \frac{b-a}{4} \max\{N,L,M\} \int_a^b (|u-v| + |u'-v'| + |u''-v''|)\, ds \\
&\leq \frac{b-a}{4} \max\{N,L,M\} \int_a^b \left(\sup_{[a,b]}|u-v| + \sup_{[a,b]}|u'-v'| + \sup_{[a,b]}|u''-v''|\right) ds \\
&= \frac{b-a}{4} \max\{N,L,M\} \int_a^b ||u-v||_{C^2([a,b])} ds \\
&= \frac{b-a}{4} \max\{N,L,M\} ||u-v||_{C^2([a,b])}\, (b-a),
\end{aligned}
\tag{4.8}
$$

where $||u||_{c^2([a,b])}$ is defined as follows:

$$
||u||_{C^2([a,b])} = \sum_{i=0}^{2} \sup \left| u^{(i)} \right|. \tag{4.9}
$$

Equivalently,

$$|T_\lambda(u) - T_\lambda(v)| < \frac{(b-a)^2}{4} max\{N, L, M\}||u - v||_{C^2([a,b])}. \tag{4.10}$$

Now if we assume that $K = \frac{(b-a)^2}{4} max\{N, L, M\} < 1$, then (68) yields

$$|T_\lambda(u) - T_\lambda(v)| < ||u - v||_{C^2([a,b])} \tag{4.11}$$

which implies that T_λ is a contraction and hence our scheme converges to the fixed point of T_λ.

Chapter 5: Numerical Results

In this section, fractional differential equations are discussed e.g., the Bratu-type equation and many other examples. The results obtained for the various examples are compared with existing numerical solutions to confirm the high accuracy and validity of the method. To start with, the proposed method will be tested for ordinary differential equation and the obtained results will be compared with results found by Variational Iteration Method (VIM).

Example 1**.** Consider the following ordinary differential equation

$$u'''(x) = u''(x) + 4u^2 e^{-2x}, \tag{5.1}$$

where $0 \leq x \leq 1$ and subject to the following boundary conditions(BCs)

$$u(0) = 1, \quad u'(0) = 2, \quad u(1) = e^2. \tag{5.2}$$

The exact solution is $u(x) = e^{2x}$. The iteration formula for the GVIM with $a = 0$ and $b = 1$ is,

$$\begin{aligned} u_{n+1} &= u_n + \int_0^x \left(\left(\frac{s^2}{2} - s \right) x^2 + xs - \frac{s^2}{2} \right) \left[u''' - u'' - 4u^2 e^{-2s} \right] ds \\ &+ \int_x^1 \left(x^2 \left(\frac{s^2}{2} - s + \frac{1}{2} \right) \right) \left[u''' - u'' - 4u^2 e^{-2s} \right] ds. \end{aligned}$$

The initial iterate is found to be $u_0 = (e^2 - 3)x^2 + 2x + 1$ which is the solution of $L[u] = 0$ subject to the BCs (5.2).

For quantitative comparison, G_n is defined for the results obtained via GVIM and V_n for the iterates of the results obtained via VIM.

Table 5.1: Residuals for Example 1 using our method and VIM.

t	V_3	G_2	G_5	G_9
0.1	2.5947	1.2738(-5)	1.9678(-8)	1.6232(-12)
0.2	9.6406	6.6717(-5)	6.0047(-8)	5.1270(-12)
0.3	20.3679	1.5125(-4)	8.1834(-8)	7.1773(-12)
0.4	33.9843	2.3230(-4)	5.7994(-8)	5.1756(-12)
0.5	49.4169	2.7381(-4)	6.5729(-9)	4.9768(-15)
0.6	65.2482	2.5373(-4)	7.7779(-8)	6.5995(-12)
0.7	79.7620	1.7578(-4)	1.1405(-7)	9.4061(-12)
0.8	91.1009	7.4143(-4)	9.5302(-8)	7.5669(-12)
0.9	97.5816	3.4495(-5)	4.1262(-8)	3.1725(-12)
1.0	98.2695	1.2000(-33)	1.2000(-33)	1.0000(-34)

Example 2. Consider the following ordinary differential equation

$$u''(x) = u^3(x) - u(x)u'(x), \tag{5.3}$$

where $1 \leq x \leq 2$ and subject to the following boundary conditions(BCs)

$$u(1) = \frac{1}{2}, \quad u(2) = \frac{1}{3}. \tag{5.4}$$

The exact solution is $u(x) = \frac{1}{x+1}$. The iteration formula for the GVIM with $a = 1$ and $b = 2$ is,

$$\begin{aligned} u_{n+1} &= u_n + \int_1^x [(s-1)(2-x)] \left[u'' - u^3 + uu'\right] ds \\ &+ \int_x^2 [(x-1)(2-s)] \left[u'' - u^3 + uu'\right] ds. \end{aligned}$$

The initial iterate is found to be $u_0 = -\frac{x}{6} + \frac{2}{3}$ which is the solution of $L[u] = 0$ subject to the BCs (5.4).

Table 5.2: Residuals for Example 2 using our method and VIM.

t	V_5	G_2	G_5	G_{12}
1.1	0.1851	8.8038(-7)	2.0506(-10)	8.6095(-18)
1.2	0.2202	2.1969(-6)	5.4035(-10)	1.6247(-17)
1.3	0.2554	3.4111(-6)	8.9779(-10)	2.2777(-17)
1.4	0.2911	4.0648(-6)	1.1461(-9)	2.8428(-17)
1.5	0.3276	3.9440(-6)	1.1893(-9)	3.3752(-17)
1.6	0.3652	3.1004(-6)	1.0054(-9)	3.9452(-17)
1.7	0.4040	1.8096(-6)	6.5547(-10)	4.6160(-17)
1.8	0.4445	5.0613(-7)	2.6663(-10)	5.4233(-17)
1.9	0.4870	2.8357(-7)	2.1216(-12)	6.3332(-17)
2.0	0.5317	4.3700(-33)	5.4744(-26)	3.6492(-25)

Example 3. Consider the following ordinary differential equation

$$u''(x) = \frac{1}{2}\left(1 + x + u(x)\right)^3, \tag{5.5}$$

where $0 \leq x \leq 1$ and subject to the following boundary conditions(BCs)

$$u'(0) - u(0) = -\frac{1}{2}, \quad u'(1) + u(1) = 1. \tag{5.6}$$

The exact solution is $u(x) = \frac{2}{2-x} - x - 1$. The iteration formula for the GVIM with $a = 0$, $b = 1$ and $\alpha = 0.5$ is,

$$\begin{aligned} u_{n+1} &= u_n - 0.5\int_0^x \left[(1+s)\left(\frac{x}{3}-\frac{2}{3}\right)\right]\left[u'' - \frac{1}{2}(1+s+u)^3\right]ds \\ &- 0.5\int_x^1 \left[(s-2)\left(\frac{x}{3}+\frac{1}{3}\right)\right]\left[u'' - \frac{1}{2}(1+s+u)^3\right]ds. \end{aligned}$$

The initial iterate is found to be $u_0 = \frac{x}{6} + \frac{2}{3}$ which is the solution of $L[u] = 0$ subject to the BCs(5.6).

Table 5.3: Residuals for Example 3 using our method and VIM.

t	V_3	G_4	G_{15}	G_{45}
0.1	0.7432	1.1808(-3)	7.1286(-8)	8.0710(-18)
0.2	0.8430	1.2130(-3)	7.9522(-8)	3.2389(-18)
0.3	0.9735	1.3581(-3)	9.5059(-8)	7.8876(-20)
0.4	1.1462	1.5702(-3)	1.0938(-7)	1.1946(-18)
0.5	1.3788	1.8682(-3)	1.1262(-7)	2.0125(-18)
0.6	1.7010	2.2571(-3)	1.0037(-7)	3.9263(-18)
0.7	2.1656	2.7180(-3)	6.4960(-8)	7.0213(-18)
0.8	2.8740	3.1914(-3)	1.3507(-9)	7.7760(-18)
0.9	4.0462	3.5467(-3)	9.0164(-8)	1.9760(-18)
1.0	6.2369	3.5238(-3)	1.5473(-7)	1.8076(-17)

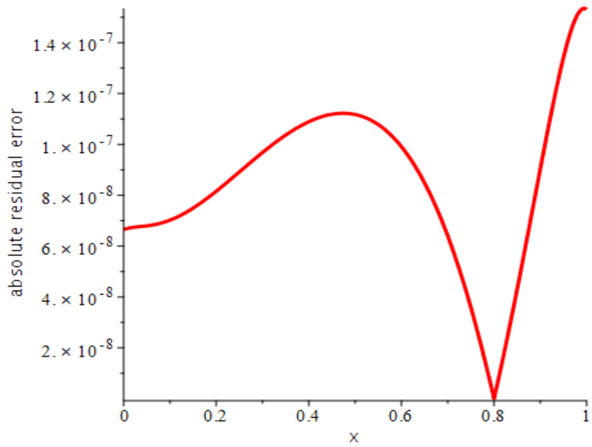

Figure 5.1: Residuals for the 15^{th} iteration in Example 3.

Tables $5.1-5.3$ confirm that the GVIM strategy is powerful, accurate and less time consuming for these ordinary differential equations when comparing the numerical results G_n obtained using our introduced procedure and the numerical results V_n obtained by VIM. Also, Figure 5.1 shows that the residuals are significantly small over the entire domain, which indicates that the method is stable.

Now, we implement the method for fractional differential equations. The numerical results of G_n obtained using our introduced procedure will be compared with existing numerical solutions to assure the validity and high accuracy of the strategy for fractional differential equations.

Example 4**.** Consider the following nonlinear fractional Bratu differential equation, which is taken from [30]

$$u^{\alpha}(x) + \lambda e^{u} = 0, \tag{5.7}$$

where $0 \leq x \leq 1$, $1 < \alpha \leq 2$ and subject to the following boundary conditions (BCs):

$$u(0) = 0, \quad u(1) = 0. \tag{5.8}$$

When $\alpha = 2$, the exact solution is $u(x) = -2\ln\left[\frac{\cosh\left(\left(x - \frac{1}{2}\right)\frac{\theta}{2}\right)}{\cosh(\frac{\theta}{4})}\right]$

where θ satisfies $\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right)$. The iteration formula for the GVIM with $a = 0$, $b = 1$ yields

$$u_{n+1} = u_n + \int_0^x (1 - x)\, s\, [u^{\alpha} + \lambda e^{u}]\, ds + \int_x^1 (1 - s)\, x\, [u^{\alpha} + \lambda e^{u}]\, ds$$

where α is the fractional derivative and $u_0 = 0$ is the initial iterate.

For quantitative comparison, G_n is defined for the iterates of the results obtained via GVIM and $L - RKM$ for the results obtained via [30]. Tables $5.4 - 5.6$ below assure that the results obtained by GVIM are accurate.

Table 5.4: Numerical solution for Example 4 using GVIM with $\lambda = 1$, compared with the method in [30] and exact solution.

t	Exact sol.$(\alpha = 2)$	$L - RKM(\alpha = 2)$	$G_4(\alpha = 2)$
0.1	0.04985	0.04985	0.04985
0.2	0.08919	0.08919	0.08919
0.3	0.11761	0.11761	0.11761
0.4	0.13479	0.13479	0.13479
0.5	0.14054	0.14054	0.14054
0.6	0.13479	0.13479	0.13479
0.7	0.11761	0.11761	0.11761
0.8	0.08919	0.08919	0.08919
0.9	0.04985	0.04985	0.04985

Table 5.5: Numerical solution for Example 4 using GVIM with $\lambda = 2$, compared with the method in [30] and exact solution.

t	Exact sol.$(\alpha = 2)$	$L - RKM(\alpha = 2)$	$G_7(\alpha = 2)$
0.1	0.11441	0.11441	0.11441
0.2	0.20642	0.20642	0.20641
0.3	0.27388	0.27388	0.27387
0.4	0.31509	0.31509	0.31508
0.5	0.32895	0.32895	0.32894
0.6	0.31509	0.31508	0.31508
0.7	0.27388	0.27388	0.27387
0.8	0.20642	0.20642	0.20641
0.9	0.11441	0.11441	0.11441

Table 5.6: Numerical solution for Example 4 using GVIM with $\lambda = 3$, compared with the method in [30] and exact solution.

t	Exact sol.$(\alpha = 2)$	$L - RKM(\alpha = 2)$	$G_{15}(\alpha = 2)$
0.1	0.21577	0.21577	0.21576
0.2	0.39432	0.39432	0.39428
0.3	0.52844	0.52844	0.52838
0.4	0.61183	0.61183	0.61176
0.5	0.64015	0.64015	00.64007
0.6	0.61183	0.61183	0.61175
0.7	0.52844	0.52844	0.52836
0.8	0.39432	0.39432	0.39425
0.9	0.21577	0.21577	0.21571

The results obtained by Tables $5.7 - 5.13$ below confirm that the GVIM strategy is reliable and effective.

Table 5.7: Residuals for Example 4 using GVIM .

t	$G_4(\alpha = 2, \lambda = 1)$	$G_7(\alpha = 2, \lambda = 2)$	$G_{15}(\alpha = 2, \lambda = 3)$
0.1	6.62220(-6)	1.44433(-5)	1.47926(-5)
0.2	1.31438(-5)	3.03426(-5)	3.40834(-5)
0.3	1.86745(-5)	4.50304(-5)	5.44463(-5)
0.4	2.23894(-5)	5.54847(-5)	7.03292(-5)
0.5	2.36985(-5)	5.92799(-5)	7.62862(-5)
0.6	2.23868(-5)	5.54847(-5)	7.02557(-5)
0.7	1.86740(-5)	4.50305(-5)	5.43476(-5)
0.8	1.31390(-5)	3.03425(-5)	3.39956(-5)
0.9	6.62000(-6)	1.44362(-5)	2.01306(-5)

Unfortunately, the residuals for L-RKM were not found in [30] and hence we can't assure that their numerical solutions are right.

Table 5.8: Numerical solutions for Example 4 using GVIM with $\lambda = 1$, compared with the method in [30].

t	$L - RKM(\alpha = 1.9)$	$G_4(\alpha = 1.9)$	$L - RKM(\alpha = 1.8)$	$G_4(\alpha = 1.8)$
0.1	0.05334	0.05335	0.05640	0.05658
0.2	0.09450	0.09406	0.09779	0.09805
0.3	0.12256	0.12258	0.12578	0.12605
0.4	0.13902	0.139038	0.14108	0.14131
0.5	0.14358	0.14361	0.14425	0.14442
0.6	0.13649	3.0.13652	0.13586	0.13595
0.7	0.11810	0.11813	0.11655	0.11654
0.8	0.08884	0.08887	0.08700	0.08690
0.9	0.04926	0.04929	0.04796	0.04779

Table 5.9: Residuals for Example 4 using GVIM with $\lambda = 1$.

t	$G_4(\alpha = 1.9)$	$G_4(\alpha = 1.8)$
0.1	4.57883(-4)	8.02389(-3)
0.2	2.52548(-4)	6.48309(-3)
0.3	2.34297(-4)	5.10075(-3)
0.4	2.73507(-4)	4.25438(-3)
0.5	3.12099(-4)	3.56350(-3)
0.6	3.19564(-4)	2.77334(-3)
0.7	2.80064(-4)	1.76309(-3)
0.8	1.89580(-4)	5.15655(-3)
0.9	6.24001(-2)	9.09273(-3)

Table 5.10: Numerical solutions for Example 4 using GVIM with $\lambda = 2$, compared with the method in [30].

t	$L - RKM(\alpha = 1.9)$	$L - RKM(\alpha = 1.8)$	$G_2(\alpha = 1.9)$	$G_3(\alpha = 1.8)$
0.1	0.12395	0.13161	0.12122	0.13255
0.2	0.22061	0.23047	0.21570	0.23251
0.3	0.28927	0.29805	0.28295	0.30125
0.4	0.32911	0.33470	0.32227	0.33903
0.5	0.33986	0.34126	0.33338	0.34649
0.6	0.32203	0.31930	0.31661	0.32499
0.7	0.27693	0.27114	0.27299	0.27663
0.8	0.20653	0.19963	0.20417	0.20417
0.9	0.11332	0.10794	0.11232	0.11082

Table 5.11: Residuals for Example 4 using GVIM with $\lambda = 2$.

t	$G_2(\alpha = 1.9)$	$G_3(\alpha = 1.8)$
0.1	0.03499	0.01636
0.2	0.05096	0.01586
0.3	0.06138	0.01367
0.4	0.06201	0.00385
0.5	0.05220	0.01169
0.6	0.03369	0.02883
0.7	0.01020	0.04290
0.8	0.01357	0.05005
0.9	0.04436	0.04869

Table 5.12: Numerical solutions for Example 4 using GVIM with $\lambda = 3$, compared with the method in [30].

t	$L - RKM(\alpha = 1.9)$	$L - RKM(\alpha = 1.8)$	$G_2(\alpha = 1.9)$	$G_2(\alpha = 1.8)$
0.1	0.24085	0.26065	0.21203	0.20426
0.2	0.43526	0.46414	0.38106	0.36290
0.3	0.57681	0.60602	0.50356	0.47517
0.4	0.65969	0.68207	0.57615	0.53977
0.5	0.68091	0.69192	0.59692	0.55629
0.6	0.64127	0.63987	0.56605	0.52569
0.7	0.54538	0.53413	0.48589	0.45041
0.8	0.40065	0.38508	0.36083	0.33430
0.9	0.21597	0.20346	0.19666	0.18230

Table 5.13: Residuals for Example 4 using GVIM with $\lambda = 3$.

t	$G_2(\alpha = 1.9)$	$G_2(\alpha = 1.8)$
0.1	0.15336	0.54512
0.2	0.27928	0.66582
0.3	0.38391	0.77893
0.4	0.43565	0.81256
0.5	0.41562	0.73901
0.6	0.32637	0.56246
0.7	0.19229	0.31399
0.8	0.04924	0.03981
0.9	0.07016	0.21405

Example 5**.** Consider the following fractional nonlinear differential equation, which is taken from [33]

$$u^{1.5}(x) + u^3 - \frac{\Gamma(2.9)}{\Gamma(1.4)}x^{0.4} + (x^{1.9} - 1)^3 = 0, \tag{5.9}$$

where $0 \leq x \leq 1$ and subject to the following boundary conditions (BCs)

$$u(0) = -1, \quad u(1) = 0. \tag{5.10}$$

The exact solution is $u(x) = x^{1.9} - 1$. The iteration formula for the GVIM with $a = 0$, $b = 1$ yields

$$\begin{aligned} u_{n+1} &= u_n + \int_0^x (1-x)\, s \left[u^{\alpha} + u^3 - \frac{\Gamma(2.9)}{\Gamma(1.4)} s^{0.4} + (s^{1.9} - 1)^3 \right] ds \\ &+ \int_x^1 (1-s)\, x \left[u^{\alpha} + u^3 - \frac{\Gamma(2.9)}{\Gamma(1.4)} s^{0.4} + (s^{1.9} - 1)^3 \right] ds \end{aligned}$$

where $\alpha = 1.5$ and $u_0 = x^{1.5} - 1$ is the initial iterate.

Table 5.14: Absolute error for Example 5 using GVIM.

t	Absolute error
0.1	0.03037
0.2	0.04303
0.3	0.04515
0.4	0.04032
0.5	0.03157
0.6	0.02156
0.7	0.01237
0.8	0.00541
0.9	0.00131

The 1^{st} iterate results in Table 5.14 confirm that the GVIM strategy is simple, effective and more accurate. Since the above results obtained from one iteration are accurate, we can deduce that the convergence rate is fast.

Example 6**.** Consider the following fractional linear differential equation, which is taken from [32]

$$u^{1.2}(x) + \frac{3}{57} u(x) - x - \frac{3x^{2.2}}{57\Gamma(3.2)} = 0, \tag{5.11}$$

subject to the following nonhomogeneous boundary conditions (BCs)

$$u(0) = 0, \quad u(1) = \frac{1}{\Gamma(3.2)}. \tag{5.12}$$

The exact solution is $u(x) = \dfrac{x^{2.2}}{\Gamma(3.2)}$. The iteration formula for the GVIM with $a = 0$, $b = 1$, and $\alpha = 1.2$ yields,

$$\begin{aligned} u_{n+1} &= u_n + \int_0^x (1-x)\, s \left[u^{\alpha} + \frac{3}{57} u(x) - s - \frac{3s^{2.2}}{57\Gamma(3.2)} \right] ds \\ &+ \int_x^1 (1-s)\, x \left[u^{\alpha} + \frac{3}{57} u(x) - s - \frac{3s^{2.2}}{57\Gamma(3.2)} \right] ds \end{aligned}$$

Table 5.15: Absolute error for Example 6 using GVIM.

t	Absolute error
0.1	6.67230(-5)
0.2	7.48105(-5)
0.3	9.96561(-5)
0.4	5.53723(-7)
0.5	2.24204(-4)
0.6	2.87651(-4)
0.7	1.50894(-4)
0.8	3.17525(-5)
0.9	9.92607(-5)

The 50^{th} iterate results in Table 5.15 above confirm that the GVIM strategy is efficient.

Example 7**.** Consider the following fractional differential equation

$$u^{\alpha}(x) + u(x) - x^5 + x^4 - x^{3.5}\frac{128}{7\sqrt{\pi}} + x^{2.5}\frac{64}{5\sqrt{\pi}} = 0, \tag{5.13}$$

subject to the following nonhomogeneous boundary conditions (BCs)

$$u(0) = 0, \quad u(1) = 0. \tag{5.14}$$

The exact solution is $u(x) = x^5 - x^4$. The iteration formula for the GVIM with $a = 0$, $b = 1$ yields,

$$\begin{aligned} u_{n+1} &= u_n - \int_0^x (1-x)\,s\left[u^{\alpha} + u(x) - s^5 + s^4 - s^{3.5}\frac{128}{7\sqrt{\pi}} + s^{2.5}\frac{64}{5\sqrt{\pi}}\right] ds \\ &- \int_x^1 (1-s)\,x\left[u^{\alpha} + u(x) - s^5 + s^4 - s^{3.5}\frac{128}{7\sqrt{\pi}} + s^{2.5}\frac{64}{5\sqrt{\pi}}\right] ds \end{aligned}$$

where $\alpha = 1.5$ and $u_0 = x^4(x-1)$ is the initial iterate which is equal to the exact solution.

Table 5.16: Absolute error for Example 7 using GVIM.

t	Absolute error
0.1	1.79100(-41)
0.2	3.60000(-41)
0.3	5.30000(-41)
0.4	7.00000(-41)
0.5	9.00000(-41)
0.6	1.10000(-40)
0.7	7.00000(-41)
0.8	1.50000(-40)
0.9	1.00000(-40)

The 1^{st} iterate results in Table 5.16 above confirm that the GVIM strategy is simple and accurate.

Example 8**.** Consider the following fractional differential equation

$$u^{\alpha}(x) - \frac{\pi}{4}u^2(x) - \frac{\pi}{4} = 0, \tag{5.15}$$

subject to the following nonhomogeneous boundary conditions (BCs)

$$u(0) = 0, \quad u(1) = 0. \tag{5.16}$$

The iteration formula for the GVIM with $a = 0$, $b = 1$ yields

$$\begin{aligned} u_{n+1} &= u_n + \int_0^x (1-x)\, s \left[u^{\alpha} - 4u^2(x) - \frac{\pi}{4}\right] ds \\ &+ \int_x^1 (1-s)\, x \left[u^{\alpha} - 4u^2(x) - \frac{\pi}{4}\right] ds \end{aligned}$$

where $\alpha = 1.8$ and $u_0 = x$ is the initial iterate.

Table 5.17: Residuals for Example 8 using GVIM.

t	Residuals
0.1	1.20459(-3)
0.2	2.45947(-5)
0.3	3.98589(-4)
0.4	4.67306(-4)
0.5	3.81921(-4)
0.6	2.41249(-4)
0.7	9.86814(-5)
0.8	1.20077(-5)
0.9	6.53556(-5)
1.0	3.72571(-5)

Table 5.17 and Figure 5.2 show that high degree of accuracy was achieved after the 6^{th} iteration.

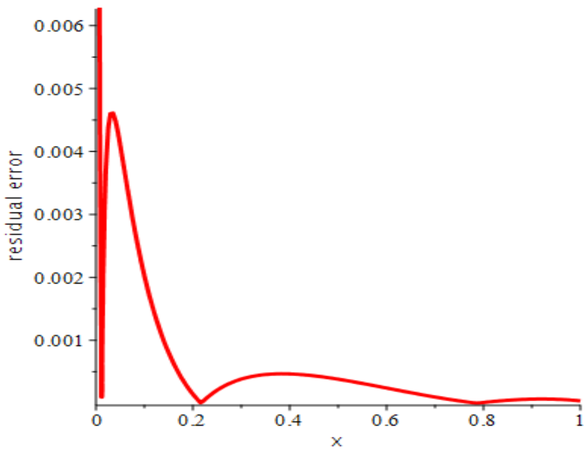

Figure 5.2: Residuals for the 6^{th} iteration in Example 8.

Chapter 6: Conclusions and Future Work

In this thesis, a recent approach based on extending the Variational Iteration Method (VIM) is implemented for the numerical solution of various Fractional Boundary Value Problems (FBVPs). The proposed Generalized Variational Iteration Method (GVIM) is particularly suitable for tackling BVPs. The achieved results on eight different examples assure that the proposed method has high accuracy compared to other methods and has a high convergence rate. It yields almost uniformly distributed errors across the interval. For future work, it will be interesting to apply the GVIM approach to other linear and nonlinear fractional differential equations.

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Vita

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