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CAN HIGH ORDER CONVERGENCE OF EUROPEAN OPTION PRICES BE ACHIEVED WITH COMMON CRR-TYPE BINOMIAL TREES?

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Abstract. Considering European call options, we prove that CRR-type binomial trees systematically underprice the value of these options, when the spot price is not near the money. However, we show that, with a volatility premium to compensate this mispricing, any arbitrarily high order of convergence can be achieved, within the common CRR-type binomial tree framework.

1. Introduction and Setting

Let volatility σ, and risk free rate r be the standard parameters in the Black-Scholes model, and consider a European call option with maturity T and strike K, with K expressed in the form $K = S_0 \exp(\alpha\, r\, T)$ for $\alpha = \ln(K/S_0)\, r^{-1}T^{-1}$, where S_0 is the spot price of the underlying asset. Also let $\left\{S^{(n)}\right\}_{n\in\mathbb{N}}$ denote risk neutral binomial schemes such that, at every positive time t in $(T/n)\mathbb{N}$, the random walk $S^{(n)}$ has a probability $p(n)$ of jumping from its current state $S_t^{(n)}$ to the state $S_t^{(n)}u(n)$, and a probability $1-p(n)$ of jumping to the state $S_t^{(n)}d(n)$. We will say that risk neutral binomial schemes are of the CRR-type if $u(n) = \exp(\sigma\sqrt{T/n}+\lambda(n)\,T/n)$ and $d(n) = \exp(-\sigma\sqrt{T/n}+\lambda(n)\,T/n)$, for some bounded real valued function $\lambda(n)$.

Let $C(n) := C(\varphi, n)$ be the price of a European option with payoff φ under the CRR-type scheme and let $C_0 := C_0(\varphi)$ be the price of the same option in the Black-Scholes model. Considering call options in [4] and digital options in [5], Diener and Diener showed how *coefficients* $C_\ell(n)$ can be explicitly calculated such that

$$C(n) = C_0 + \sum_{\ell=2}^{i_0} C_\ell(n)\, n^{-\frac{\ell}{2}} + \mathcal{O}(n^{-\frac{i_0+1}{2}}). \tag{1.1}$$

Analyzing the convergence behavior of binomial schemes to calculate option prices has been a popular topic, in particular for the *European*, *American*, *Continuously Paying*, *Lookback*, *Digital*, *Game*, and *Barrier* option types. Let us mention [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27]. There is a vast literature about binomial trees: we refer to Joshi [12] for an exhaustive and detailed description of binomial trees and how they relate to one another.

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The natural questions of *smoothing and accelerating* the convergence has been the subject of several papers published in top journals over recent years. The question of *how fast* a convergence can be achieved with binomial trees was answered for vanilla European options by Joshi [11] for n odd, and his work was extended by Xiao [28] to n even. Eclipsing any other result in the literature, they showed that, *using special trees*, any arbitrarily high order of convergence can be attained. Unlike the *CRR-type trees*, these trees have exactly half of their nodes above the strike. In contrast, the best acceleration and smoothing results obtained so far for *CRR-type trees*, are $bn^{-1} + o\left(n^{-1}\right)$ in [2], and, *in the best case*, $o\left(n^{-1}\right)$ in [14].

This raises the natural question of whether or not high order convergence can be achieved with common *CRR-type trees*? Remarkably, as we show in this paper, the answer to this question is 'no' because *CRR-type trees* systematically underprice European options when the underlying is away from the strike! Indeed, we prove that, for *CRR-type trees with constant volatility*, systematic high order convergence is impossible to achieved, because these trees uniformly *underprice* European call options by an error of magnitude $m_2/n + O\left(n^{-3/2}\right)$ when the spot price is not near enough the money! To the best of our knowledge, this remarkable fact was unknown so far.

However, we describe in this paper a method to give the binomial scheme an appropriate "volatility premium" to offset that bias, allowing to achieve any arbitrarily high order convergence in the binomial scheme, after the drift parameter $\lambda(n)$ as been chosen carefully. While the method to reach high order of convergence in [11] and [28] depends on the value of the strike K for constructing the binomial tree, our approach applies to any situation —regardless of the existence of a strike— where an expansion of the error exists, such as for the general payoffs in [21].

For simplicity we restrict our attention here to call options in the setting of [4], described now. Let $i_0 \geq 2$ be an integer, and let $\overrightarrow{\lambda} = (\lambda_2, \ldots, \lambda_{i_0})$. Consider binomial schemes of the form

$$
\begin{aligned}
\mathrm{u}(n,\ \overrightarrow{\lambda}) &= \exp\left(\sigma\sqrt{\frac{T}{n}} + \lambda_2\,\sigma^2\frac{T}{n} + \sum_{\ell=3}^{i_0}\lambda_\ell\frac{2\,\sigma}{T}\sqrt{\frac{T}{n}}^{\ell}\right),\\
\mathrm{d}(n,\ \overrightarrow{\lambda}) &= \exp\left(-\sigma\sqrt{\frac{T}{n}} + \lambda_2\,\sigma^2\frac{T}{n} + \sum_{\ell=3}^{i_0}\lambda_\ell\frac{2\,\sigma}{T}\sqrt{\frac{T}{n}}^{\ell}\right),\\
\mathrm{p}(n,\ \overrightarrow{\lambda}) &= \frac{\exp(\frac{r\,T}{n}) - \mathrm{d}(n,\ \overrightarrow{\lambda})}{\mathrm{u}(n,\ \overrightarrow{\lambda}) - \mathrm{d}(n,\ \overrightarrow{\lambda})},
\end{aligned}
$$

and set

$$
\begin{aligned}
\mathrm{a}(n,\ \overrightarrow{\lambda}) &= \frac{\ln\left(\frac{K}{S_0}\right) - n\ln\left(\mathrm{d}(n,\ \overrightarrow{\lambda})\right)}{\ln\left(\mathrm{u}(n,\ \overrightarrow{\lambda})\right) - \ln\left(\mathrm{d}(n,\ \overrightarrow{\lambda})\right)},\\
\overline{\kappa}(n,\ \overrightarrow{\lambda}) &= frac\left(\mathrm{a}(n,\ \overrightarrow{\lambda})\right).
\end{aligned}
$$

Note that $\mathrm{a}(n,\ \overrightarrow{\lambda})$ simplifies to

$$\mathrm{a}(n,\ \overrightarrow{\lambda}) = \frac{T}{2}\left(\frac{T}{n}\right)^{-1} - \frac{\lambda_2\sigma^2 T - \alpha r T}{2\sigma}\left(\frac{T}{n}\right)^{-\frac{1}{2}} - \sum_{\ell=3}^{i_0}\lambda_\ell\sqrt{\frac{T}{n}}^{\ell-3}. \tag{1.2}$$

The result below is obtained by using the method for computing the asymptotic expansion of call options in powers of $n^{-1/2}$ described in [4]. We note that the multivariate polynomials $\mathcal{P}_\ell$ of (1.4) are of degree one in λ_ℓ, for $\ell \geq 3$. The proof is in the appendix.

Theorem 1. *For every integer $i_0 \geq 2$, the value $C(n,\ \overrightarrow{\lambda})$ of the call option in the binomial schemes described above can be written as*

$$C(n,\ \overrightarrow{\lambda}) = C(n,\ \overrightarrow{\lambda}, \overline{\kappa}(n,\ \overrightarrow{\lambda}))$$

with

$$C(n,\ \overrightarrow{\lambda}, \kappa) = C_0 + \sum_{\ell=2}^{i_0} C_\ell\left(\overrightarrow{\lambda},\ \kappa\right) n^{-\frac{\ell}{2}} + \mathcal{O}\left(n^{-\frac{i_0+1}{2}}\right), \tag{1.3}$$

where, for $\ell = 2, ..., i_0$, the functions C_ℓ have the form

$$C_\ell\left(\overrightarrow{\lambda},\ \kappa\right) = \exp(-\frac{T\,(-2\,\alpha\, r + \sigma^2 + 2\, r)^2}{8\,\sigma^2})\,\mathcal{P}_\ell(\overrightarrow{\lambda},\ \kappa), \tag{1.4}$$

and $\mathcal{P}_\ell$ is a multivariate polynomial in $\sigma^{-1}, \sigma, \lambda_2, \ldots, \lambda_\ell, \kappa$. The terms $\mathcal{O}(n^{-(i_0+1)/2})$ are uniform over $0 \leq \kappa \leq 1$, $\mathcal{L}^{-1} \leq \sigma \leq \mathcal{L}$, and $|\lambda_\ell| \leq \mathcal{L}$, $\ell = 2, ..., i_0$, for any real number $\mathcal{L} > 0$. For $\ell \geq 3$, $\mathcal{P}_\ell$ is a polynomial of degree one when seen as a function of λ_ℓ, and the coefficient of λ_ℓ in $\mathcal{P}_\ell$ is

$$-\frac{1}{3}\frac{\sqrt{2}}{\sqrt{\pi}} S_0 T^{\frac{\ell}{2}}\left(-3\lambda_2\sigma^2 + 2r + r\alpha\right). \tag{1.5}$$

It is sometimes convenient to write $\mathcal{P}_\ell\left(\lambda_2, \ldots, \lambda_\ell, \kappa\right) := \mathcal{P}_\ell(\overrightarrow{\lambda},\ \kappa)$ and we will use a similar convention for C and the C_ℓ's. The polynomials $\mathcal{P}_\ell$ are calculated using a Maple worksheet similar to the one available on Diener and Diener's webpage. For instance, specializing to $i_0 = 3$, one gets

$$\mathcal{P}_2\left(\lambda_2,\ \kappa\right) = -\frac{\sqrt{2T}\,S_0}{96\sqrt{\pi}\,\sigma}\mathrm{P}_2\left(\lambda_2,\ \kappa\right), \tag{1.6}$$

$$\mathcal{P}_3\left(\lambda_2,\ \lambda_3,\ \kappa\right) = -\,\frac{\sqrt{2}\ S_0}{3\sqrt{\pi}}\mathrm{P}_3\left(\lambda_2,\ \lambda_3,\ \kappa\right),$$

where

$$\begin{aligned}\mathrm{P}_2\left(\lambda_2,\ \kappa\right) &= \sigma^4\,T^2 \ -\ 32\,\lambda_2\,\sigma^2\,T^2\,r \ +\ 12\,T^2\,r^2 \ +\ 4\,\alpha^2\,r^2\,T^2 \\ &\quad + 8\,\alpha\,r^2\,T^2 \ +\ 12\ \sigma^2\,T - \ 96\,T\,\sigma^2\,\kappa \ +\ 24\,\lambda_2^2\,\sigma^4\,T^2 \\ &\quad + 96\,T\,\sigma^2\,\kappa^2 \ -\ 16\,\alpha\,r\,T^{\ 2}\,\lambda_2\,\sigma^2, \\ \mathrm{P}_3\left(\lambda_2,\ \lambda_3,\ \kappa\right) &= -\ 4\ \kappa^3\,r\,T \ +\ 4\,\alpha\,r\,T\,\kappa^3 \ +\ 6\,\kappa^2\,r\,T \ -\ \alpha\,r\,T\,\lambda_3 \\ &\quad + 2\,\alpha\,r\,T\,\kappa + \ 3\ \lambda_2\,\sigma^2\,T\,\lambda_3 - \ 2\,\kappa\,r\,T \ -\ 2\,\lambda_3\,r\,T \\ &\quad - 6\,\alpha\,r\,T\,\kappa^2.\end{aligned}$$

2. Acceleration with a constant volatility

Given i_0 and n, we describe in this section a method allowing to map the parameters $(\lambda_2^{(n)},\ldots,\lambda_{i_0}^{(n)}) =: \overrightarrow{\lambda}_n$ of the random walk $S_t^{(n)}$ into the coefficients $C_\ell(\overrightarrow{\lambda}_n, \overline{\kappa}(n, \overrightarrow{\lambda}_n))$ of $n^{-\ell/2}$ in (1.3), in such a way that $\{\overrightarrow{\lambda}_n\}$ remains bounded and that, for every n, $C_\ell(\overrightarrow{\lambda}_n, \overline{\kappa}(n, \overrightarrow{\lambda}_n)) = 0$, for $\ell = 2, ..., i_0$. As a result, (1.3) reduces to $C(n, \overrightarrow{\lambda}_n) = C_0 + \mathcal{O}(n^{-(i_0+1)/2})$, and a convergence of order $\mathcal{O}(n^{-(i_0+1)/2})$ is achieved.

First, we consider the coefficient $C_2(\lambda_2, \kappa)$. In order for it to vanish, one must have $\mathcal{P}_2(\lambda_2, \kappa)$ vanishing. This is a quadratic equation in λ_2, yielding

$$\lambda_2 = \frac{8Tr + 4\alpha rT \pm \sqrt{D(\kappa)}}{12T\sigma^2},$$

where

$$D(\kappa) \overset{def}{=} -8T^2r^2(\alpha-1)^2 - 6\sigma^4T^2 - 72T\sigma^2 + 576T\sigma^2\kappa(1-\kappa).$$

We choose (arbitrarily) the "+" solution and define the function

$$\lambda_2^f(\kappa) \overset{def}{=} \frac{8Tr + 4\alpha rT + \sqrt{D(\kappa)}}{12T\sigma^2}.$$

Now in order to have $\mathcal{P}_3(\lambda_2, \lambda_3, \kappa)$ vanishing, it suffices to have

$$\lambda_3 = \frac{-2\kappa r(2\kappa-1)(\kappa-1)(\alpha-1)}{3\lambda_2\sigma^2 - (2+\alpha)r},$$

and we define the function

$$\lambda_3^f(\kappa) \overset{def}{=} \frac{-2\kappa r(2\kappa-1)(\kappa-1)(\alpha-1)}{3\lambda_2^f(\kappa)\sigma^2 - (2+\alpha)r}.$$

Continuing this way, that is isolating λ_ℓ in the equation $\mathcal{P}_\ell(\lambda_2,\ldots,\lambda_\ell,\kappa) = 0$, and substituting λ_j by $\lambda_j^f(\kappa)$, for $j = 2, ..., \ell-1$, one defines functions $\lambda_\ell^f(\kappa)$, for $\ell = 2, ..., i_0$. This is easily done since, for $\ell \geq 3$, $\mathcal{P}_\ell$ is of degree one in λ_ℓ. Because for $\ell \geq 3$ the coefficient of λ_ℓ in $\mathcal{P}_\ell$ is (1.5), it follows that $\lambda_\ell^f(\kappa)$ has the form

$$\lambda_\ell^f(\kappa) = \frac{Q_\ell\left(\sigma, \lambda_2^f(\kappa), ..., \lambda_{\ell-1}^f(\kappa), \kappa\right)}{3\lambda_2^f(\kappa)\sigma^2 - 2r - r\alpha},$$

for some multivariate polynomial Q_ℓ. In order to keep $\lambda_\ell^f(\kappa)$ bounded, it suffices to keep the denominator in the right hand side of the above equation away from zero as a function of κ. Solving $3\lambda_2^f(\kappa)\sigma^2 - 2r - r\alpha = 0$ yields $D(\kappa) = 0$, for which the solution is

$$\kappa = \frac{1}{2} \pm \frac{1}{24\sigma}\sqrt{\Delta_D},$$

with

$$\Delta_D \overset{def}{=} 72\sigma^2 - 8Tr^2 + 16\alpha r^2T - 8\alpha^2r^2T - 6\sigma^4T.$$

Hence, for any $0 < \mathfrak{f} < 1/2$, restricting the values of κ to the interval

$$I_0 \overset{def}{=} \left[\frac{1}{2} - \mathfrak{f}(1 \wedge \frac{1}{24\sigma}\sqrt{\Delta_D}), \frac{1}{2} + \mathfrak{f}(1 \wedge \frac{1}{24\sigma}\sqrt{\Delta_D})\right],$$

guarantees that, when $\Delta_D > 0$, the functions $\lambda_\ell^f(\kappa)$ are all real-valued and bounded on I_0, for $\ell = 2, ..., i_0$.

Recall that

$$\overline{\kappa}(n,\ \lambda_2, ..., \lambda_{i_0}) = frac\left(\frac{n}{2} - \frac{\lambda_2\sigma^2 T - \alpha r T}{2\sigma\sqrt{T}}\sqrt{n} - \sum_{\ell=3}^{i_0}\lambda_\ell\sqrt{\frac{T}{n}}^{\ell-3}\right) \tag{2.1}$$

and define the function

$$\overline{\kappa}^f\left(n,\kappa\right) \stackrel{def}{=} \overline{\kappa}\left(n, \lambda_2^f\left(\kappa\right), ..., \lambda_{i_0}^f\left(\kappa\right)\right).$$

If, for all n sufficiently large, we can solve the equation

$$\kappa_n = \overline{\kappa}^f\left(n,\kappa_n\right),$$

with $\kappa_n \in I_0$, then, setting

$$\lambda_\ell^{(n)} \stackrel{def}{=} \lambda_\ell^f\left(\kappa_n\right),$$

for $\ell = 2, ..., n$, and defining

$$\overrightarrow{\lambda}_n \stackrel{def}{=} \left(\lambda_2^{(n)}, ..., \lambda_{i_0}^{(n)}\right)$$

one gets $\kappa_n = \overline{\kappa}(n,\ \overrightarrow{\lambda}_n)$ and $C_\ell(\overrightarrow{\lambda}_n,\ \overline{\kappa}(n,\ \overrightarrow{\lambda}_n)) = 0$, for $\ell = 2, ..., i_0$, so that

$$C(n,\ \overrightarrow{\lambda}_n) = C_0 + \mathcal{O}(n^{-\frac{i_0+1}{2}}),$$

as wanted.

A glance at (2.1) reveals that solving $\kappa = \overline{\kappa}^f\left(n,\kappa\right)$, is the same as solving $\mathring{\kappa}^f\left(n,\kappa\right) \in \mathbb{N}$, where

$$\mathring{\kappa}^f\left(n,\kappa\right) \stackrel{def}{=} \frac{n}{2} - \frac{\lambda_2^f\left(\kappa\right)\sigma^2 T - \alpha r T}{2\sigma\sqrt{T}}\sqrt{n} - \sum_{\ell=3}^{i_0}\lambda_\ell^f\left(\kappa\right)\sqrt{\frac{T}{n}}^{\ell-3} - \kappa.$$

Note that for sufficiently large values of n, $\mathring{\kappa}^f\left(n,\kappa\right)$ behaves (as a function of $\kappa \in I_0$) as $n/2 - (\lambda_2^f\left(\kappa\right)\sigma^2 T - \alpha r T)\sqrt{n}/(2\sigma\sqrt{T})$, and it is obvious that, as n tends to infinity, the number of solutions $\kappa_n \in I_0$ to $\mathring{\kappa}^f\left(n,\kappa_n\right) \in \mathbb{N}$ tends to infinity. It is trivial to find such solutions numerically in a logarithmic time by exploiting the intermediate value theorem.

Note also that $\overrightarrow{\lambda}_n$ exists if and only if there is a subinterval I of $(0,1)$ on which the concave parabola $D\left(\kappa\right) > 0$. Because $D\left(\kappa\right)$ has the form $a\kappa\left(1-\kappa\right) - b$, where $a, b > 0$, its real roots, if any, have to be in the interval $(0,1)$, as $D\left(\kappa\right)$ is negative for κ outside $(0,1)$. Clearly $D\left(\kappa\right)$ has real roots when its discriminant is positive, which occurs when

$$\Delta_D = 72\sigma^2 - 8Tr^2 + 16\alpha r^2 T - 8\alpha^2 r^2 T - 6\sigma^4 T > 0. \tag{2.2}$$

This condition will be satisfied in most practical circumstances. Indeed, finding the interval in α for which condition (2.2) holds, and substituting this into $S_0 = K\exp(-\alpha\, r\, T)$, one gets that an arbitrarily fast convergence can be achieved when the spot price is in the interval

$$K\exp\left(-Tr \pm 3\sigma\sqrt{T}\sqrt{1 - \frac{\sigma^2 T}{12}}\right).$$

In practical applications, it will almost always be the case that $\sigma^2 T/12$ is negligible and that $3\sqrt{T}\sigma$ is several times larger than Tr. Suppose that, for some $a > 0$,

$$3\sigma\sqrt{T}\sqrt{1 - \frac{\sigma^2 T}{12}} = \left(1 + a\right)Tr.$$

Then arbitrarily fast convergence for CRR-type random walks can be achieved when the spot price is in the interval

$$\left[Ke^{-(a+2)Tr}, Ke^{aTr}\right] \approx \left[K\left(1-(a+2)\,Tr\right), K\left(1+aTr\right)\right].$$

Typically a will be *big enough* so that this interval is *large enough* for *most* practical applications.

Obviously, if one wants to use our technique in a systematic manner, or to evaluate American options for instance, this limitation would be a more serious issue, and we show in section 4 how a *volatility premium* resolves the problem. But first we derive in the next section an important consequences of this limitation: *CRR-type* binomial scheme with constant volatility, such as the original CRR scheme, systematically underprice European options when the spot price is away from the strike.

3. Systemic underpricing by CRR-type trees with constant volatility

Assume that condition (2.2) is not met. More precisely, assume that there is no real solution to $\mathrm{P}_2(\lambda_2,\,\kappa) = 0$ as a function of λ_2, for every κ in $(0,1)$. Because the coefficient of λ_2^2 in $\mathrm{P}_2(\lambda_2,\,\kappa)$ is positive, that means that only a strictly positive minimum can be reached. This minimum of $\mathrm{P}_2(\lambda_2,\,\kappa)$ is in fact

$$m_2^* \overset{def}{=} \frac{1}{3}T\left(4Tr^2 - 8T\alpha r^2 + 4T\alpha^2 r^2 + 3T\sigma^4 + 36\sigma^2 - 288\sigma^2\kappa + 288\sigma^2\kappa^2\right).$$

Obviously because P_2 has no roots, that means that m_2^* remains positive if we replace $-288\sigma^2\kappa + 288\sigma^2\kappa^2$ by its minimum $-72\sigma^2$. Hence,

$$m_2^* > m_2^{**} \overset{def}{=} \frac{1}{3}T\left(4Tr^2 - 8T\alpha r^2 + 4T\alpha^2 r^2 + 3T\sigma^4 - 36\sigma^2\right) > 0.$$

If follows that no matter what is the choice of $\lambda\left(n\right)$,

$$C(n,\,\overrightarrow{\lambda}_n) < C_0 - \frac{\sqrt{2T}\,S_0}{96\sqrt{\pi}\,\sigma}\exp\left(-\frac{T\left(-2\,\alpha\,r + \sigma^2 + 2\,r\right)^2}{8\,\sigma^2}\right)\frac{m_2^*}{n} + \mathcal{O}(n^{-\frac{3}{2}}),$$

showing that, for these values of the spot price for which condition (2.2) is not met, up to a negligible term $\mathcal{O}(n^{-3/2})$, every CRR-type binomial tree with constant volatility underestimate the true price C_0 by a quantity which is larger than m_2/n, for some $m_2 > 0$.

4. Full acceleration with a volatility premium

In this section we show how, with a volatility premium, arbitrary fast acceleration can be achieved even when condition (2.2) is not met. Making the volatility σ explicit in the Black-Scholes price C_0 of the call option, we write $C_0\left(\sigma\right)$ and we use a similar convention whenever needed.

Define $c_2^*\left(\sigma\right)$ by

$$c_2^*\left(\sigma\right) \overset{def}{=} \frac{1}{3}T\left(4Tr^2 - 8T\alpha r^2 + 4T\alpha^2 r^2 + 3T\sigma^4 + 36\sigma^2\right).$$

Note that $c_2^*\left(\sigma\right)$ is obtained by replacing in the minimum m_2^* of $\mathrm{P}_2(\sigma, \lambda_2,\,\kappa)$ the quantity $-288\sigma^2\kappa + 288\sigma^2\kappa^2$ by 0. Note that m_2^* has the form $a\kappa^2 - a\kappa - b$, $a > 0$, so its minimum occurs at $\kappa = 1/2$ and its maximum value in $[0,1]$ occurs at the end points. Hence for any value of κ in the interval $(0,1)$, $c_2^*(\sigma)$ is bigger than m_2^* and thus the quadratic equation

$$\mathrm{P}_2(\sigma, \lambda_2,\, \kappa) - c_2^*(\sigma) = 0 \tag{4.1}$$

has a real-valued solution in λ_2. Since $c_2^*(\sigma)$ has no real roots in α, $c_2^*(\sigma) > 0$.

Now define $c_2(\sigma)$ by

$$c_2(\sigma) \stackrel{def}{=} \frac{\sqrt{2T}\, S_0}{96\sqrt{\pi}\,\sigma} \exp(-\frac{T\,(-2\,\alpha\, r + \sigma^2 + 2\, r)^2}{8\,\sigma^2}) c_2^*(\sigma) > 0.$$

It is easy to see that there exists $\sigma_n \to \sigma$ such that, at least for n large enough,

$$C_0(\sigma_n) = C_0(\sigma) + \frac{c_2(\sigma_n)}{n}. \tag{4.2}$$

Note that $\sigma_n > \sigma$ can be calculated by a binary search with any arbitrary precision in a logarithmic time.

Define now

$$\mathrm{P}_2^*(\sigma, \lambda_2,\, \kappa) \stackrel{def}{=} \mathrm{P}_2(\sigma, \lambda_2,\, \kappa) - c_2^*(\sigma).$$

Multiplying each term by

$$-\frac{\sqrt{2T}\, S_0}{96\sqrt{\pi}\,\sigma} \exp(-\frac{T\,(-2\,\alpha\, r + \sigma^2 + 2\, r)^2}{8\,\sigma^2}),$$

set

$$C_2^*(\sigma, \lambda_2,\, \kappa) \stackrel{def}{=} C_2(\sigma, \lambda_2,\, \kappa) + c_2(\sigma_n).$$

Set also $\mathcal{P}_2^*$ as in (1.6) with P_2^* replacing P_2. We seek to find $\overrightarrow{\lambda}_n$ such that

$$0 = \mathcal{P}_2^*(\sigma_n,\, \overrightarrow{\lambda}_n,\, \kappa_n) = \mathcal{P}_3(\sigma_n,\, \overrightarrow{\lambda}_n,\, \kappa_n) = \ldots = \mathcal{P}_{i_0}(\sigma_n,\, \overrightarrow{\lambda}_n,\, \kappa_n),$$

which implies that

$$0 = C_2^*(\sigma_n,\, \overrightarrow{\lambda}_n,\, \kappa_n) = C_3(\sigma_n,\, \overrightarrow{\lambda}_n,\, \kappa_n) = \ldots = C_{i_0}(\sigma_n,\, \overrightarrow{\lambda}_n,\, \kappa_n),$$

from which, in particular,

$$C_2(\sigma_n,\, \overrightarrow{\lambda}_n,\, \kappa_n) = -c_2(\sigma_n). \tag{4.3}$$

Note that unlike $\mathcal{P}_2$ which we essentially replaced by $\mathcal{P}_2^*$, for $\ell \geq 3$ the coefficients $\mathcal{P}_\ell$ are kept unchanged except for the fact that σ is replaced by σ_n. To find $\overrightarrow{\lambda}_n$ we can proceed *exactly as in section 2.* Doing so we define recursively the functions $\lambda_\ell^f(\sigma_n, \kappa)$, $\ell = 2, .., i_0$, such that

$$\begin{aligned}
0 &= \mathcal{P}_2^*(\sigma_n, \lambda_2^f(\sigma_n, \kappa), \kappa),\\
0 &= \mathcal{P}_3(\sigma_n, \lambda_2^f(\sigma_n, \kappa), \lambda_3^f(\sigma_n, \kappa), \kappa),\\
&\ldots\\
0 &= \mathcal{P}_{i_0}(\sigma_n, \lambda_2^f(\sigma_n, \kappa), \ldots, \lambda_{i_0}^f(\sigma_n, \kappa), \kappa).
\end{aligned}$$

In particular, this gives

$$\lambda_2^f(\sigma, \kappa) \stackrel{def}{=} \frac{2\sqrt{T}r + \sqrt{T}r\alpha + 6\sigma\sqrt{\kappa\,(1-\kappa)}}{3\sqrt{T}\sigma^2}.$$

As in section 2, we need to keep $3\lambda_2^f(\sigma, \kappa)\,\sigma^2 - 2r - r\alpha$ away from zero. This function has roots $\kappa = 0$ and $\kappa = 1$. Restricting the values of κ to the interval

$$I_0 = [\mathfrak{f}_1, \mathfrak{f}_2],$$

for any $0 < \mathfrak{f}_1 < \mathfrak{f}_2 < 1$, guarantees that for $\ell = 2, ..., i_0$, the functions $\lambda_\ell^f(\sigma_n, \kappa)$ are all real valued and bounded.

Solving now

$$\kappa_n = \overline{\kappa}^f(n, \kappa_n),$$

with $\kappa_n \in I_0$, and defining

$$\lambda_\ell^{(n)} \stackrel{def}{=} \lambda_\ell^f(\sigma_n, \kappa_n),$$
$$\overrightarrow{\lambda}_n \stackrel{def}{=} \left(\lambda_2^{(n)}, ..., \lambda_{i_0}^{(n)}\right),$$

we get

$$0 = C_2^*(\sigma_n, \overrightarrow{\lambda}_n, \kappa_n) = C_3(\sigma_n, \overrightarrow{\lambda}_n, \kappa_n) = ... = C_{i_0}(\sigma_n, \overrightarrow{\lambda}_n, \kappa_n).$$

Therefore

$$C(n, \sigma_n, \overrightarrow{\lambda}_n) = C_0(\sigma_n) + C_2(\sigma_n, \overrightarrow{\lambda}_n, \kappa_n)n^{-1} + \mathcal{O}(n^{-\frac{i_0+1}{2}}).$$

Using (4.3) and (4.2) yields

$$\begin{aligned} C(n, \sigma_n, \overrightarrow{\lambda}_n) &= C_0(\sigma_n) - c_2(\sigma_n)\, n^{-1} + \mathcal{O}(n^{-\frac{i_0+1}{2}}) \\ &= C_0(\sigma) + \mathcal{O}(n^{-\frac{i_0+1}{2}}). \end{aligned}$$

In other words, the CRR-type binomial scheme with parameters σ_n and $\overrightarrow{\lambda}_n$ produces option prices converging at a rate $\mathcal{O}(n^{-(i_0+1)/2})$ to the true option price value $C_0(\sigma)$.

5. Numerical Illustration

To demonstrate the performance of our *acceleration method* we consider the case of $i_0 = 4$. Using $\sigma = 0.5$, $T = 1$, $r = 0.05$, $S_0 = 100$ we choose K in such a way that condition (2.2) fails. This requires S_0 to be away from the strike, which is obtained for instance when $\alpha = -29$, yielding $K \approx 23.45702881$. Figure 1 compares the option convergence in our *accelerated* CRR schemes with the one in the *classical* CRR scheme where $\lambda_2 = \lambda_3 = \lambda_4 = 0$. We define the error $Err_T^n(K)$ as

$$Err_T^n(K) \stackrel{def}{=} C(n, \sigma_n, \lambda_2^{(n)}, \lambda_3^{(n)}, \lambda_4^{(n)}) - C_0.$$

As shown in Figure 2, the quantity $n^{5/2} Err_T^n(K)$ oscillates heavily but remains bounded, illustrating numerically that the convergence is of order $\mathcal{O}(n^{-5/2})$.

6. Appendix

We prove here Theorem 1. Apart from the fact the coefficient of λ_ℓ in $\mathcal{P}_\ell$ is given by (1.5), the rest is a textual application of the method described in Diener and Diener [4] with simple observations. The method is briefly summarized here.

Note first that, obviously, $\mathrm{u}(n, \overrightarrow{\lambda})$ and $\mathrm{d}(n, \overrightarrow{\lambda})$ have a convergent expansion in powers of $n^{-1/2}$, hence our binomial schemes are part of the general class described in [4]. To see that the coefficients C_ℓ have the form (1.3) with $\mathcal{P}_\ell$ a multivariate polynomial in $\sigma^{-1}, \sigma, \lambda_2, ..., \lambda_\ell, \kappa,$ we follow the method described in [4]. More specifically, the authors obtain the asymptotic expansion of $C(n)$ by replacing the

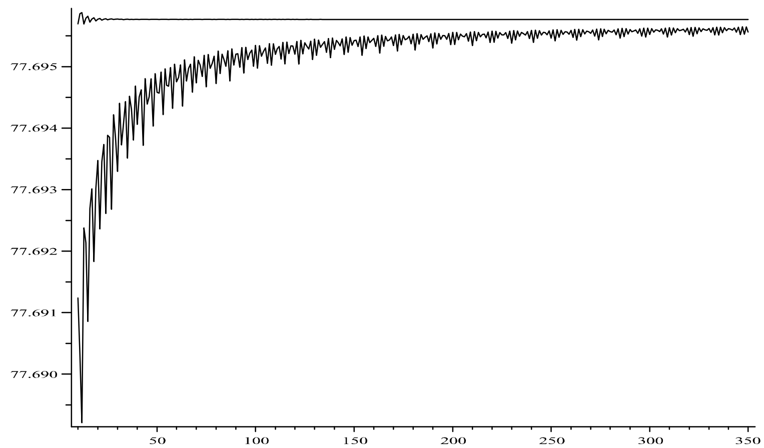

FIGURE 1. The convergence in our accelerated CRR scheme versus the classical CRR scheme. Note the underpricing of the classical CRR scheme as described in section 3.

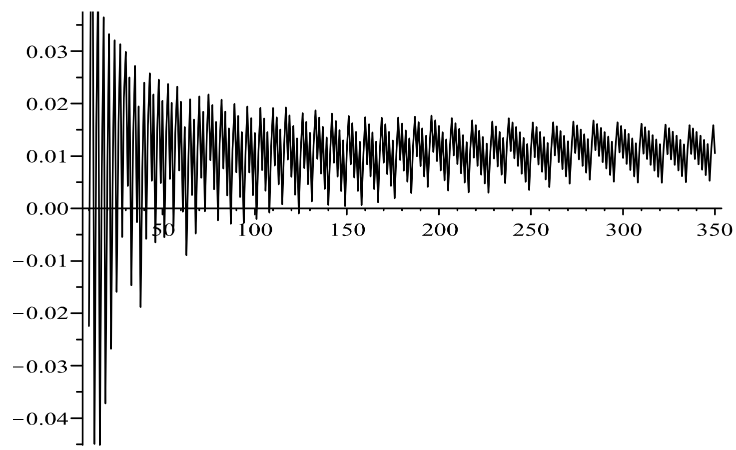

FIGURE 2. The quantity $n^{\frac{5}{2}} Err_T^n(K)$ remains bounded.

"frozen" parameter κ by $\overline{\kappa}(n, \overrightarrow{\lambda})$ in the asymptotic expansion of function $C(n,\kappa)$ which, in the notation of [4], is defined as

$$C(n,\kappa) \overset{def}{=} S_0 c(n,\kappa) I^q(n,\kappa) - K e^{-rT} c(n,\kappa) I^p(n,\kappa),$$

$$c(n,\kappa) \overset{def}{=} \frac{2^{1-n}}{\sqrt{n}} \mathrm{k}\,(n,\kappa) \binom{n}{\mathrm{k}\,(n,\kappa)},$$

$$\mathrm{k}\,(n,\kappa) \overset{def}{=} \mathrm{a}(n, \overrightarrow{\lambda}) + 1 - \kappa,$$

and where I^p is defined by [4, eq. (3.9)] and I^q is defined similarly. Diener and Diener write the expansion of $\mathrm{k}\,(n,\kappa)$ as

$$\mathrm{k}\,(n,\kappa) = \mathrm{k}_{-2} n + \mathrm{k}_{-1}\sqrt{n} + \mathrm{k}_0 + ... + \mathrm{k}_{i_0-3} n^{\frac{i_0-3}{2}}, \tag{6.1}$$

where the k_j are obtained from (1.2). Note that $\mathrm{k}_{-2}=1/2$ and $\mathrm{k}_0=1-\kappa$. Following Diener and Diener we write

$$\begin{aligned} c(n,\kappa) &= \exp((1-n)\ln 2-\frac{1}{2}\ln n+\ln(\mathrm{k}(n,\kappa)) \\ &\quad +\ln(n!)-\ln((n-\mathrm{k}(n,\kappa))!)-\ln(\mathrm{k}(n,\kappa)!)) \end{aligned}$$

and use Stirling's formula and (6.1), to see that $c(n,\kappa)$ has an asymptotic expansion in powers of $\sqrt{n}^{-1}$, where the coefficients are an exponential term, $\exp\left(-2\mathrm{k}_{-1}^2\right)$, multiplying multivariate polynomials in $\mathrm{k}_{-1},...,\mathrm{k}_{i_0-3}$. The latter translate into polynomials in $\sigma^{-1},\sigma,\lambda_2,\ldots,\lambda_{i_0},\kappa$. This asymptotic expansion is uniform not only over $0\leq\kappa\leq 1$ as pointed out in [4], but also over $\mathrm{k}_{-1},...,\mathrm{k}_{i_0-3}$ in any compact set and, therefore, over $\mathcal{L}^{-1}\leq\sigma\leq\mathcal{L}$, and $|\lambda_\ell|\leq\mathcal{L}$, $\ell=2,...,i_0$ for any $\mathcal{L}>0$. As for the asymptotic expansions of $I^p(n,\kappa)$ and $I^q(n,\kappa)$ in powers of $\sqrt{n}^{-1}$, they are treated in an analogous manner, resulting in a trivial extension of [4, theorem 3.4]. The explicit expression of each $\mathcal{P}_\ell$ needs to be calculated using a computational algebra system such as Diener and Diener's Maple worksheet, available at http://math.unice.fr/~diener, which we adapted to calculate the $\mathcal{P}_\ell$'s needed for our numerical illustration.

In order to show that $\mathcal{P}_\ell$ is of degree one when seen as functions of λ_ℓ, we introduce a new "frozen" parameter μ. More specifically, we write $C(n)=C(n,\mu(n))$, where $C(n,\mu)$ is the value of the call option in the binomial scheme

$$\mathrm{u}(n,\lambda_2,\mu)=\exp\left(\sigma\sqrt{\frac{T}{n}}+\left(\lambda_2\,\sigma^2+\mu\right)\frac{T}{n}\right),$$

$$\mathrm{d}(n,\lambda_2,\mu)=\exp\left(-\sigma\sqrt{\frac{T}{n}}+\left(\lambda_2\,\sigma^2+\mu\right)\frac{T}{n}\right),$$

and where

$$\mu(n)=\sum_{\ell=3}^{i_0}\lambda_\ell\frac{2\,\sigma}{T}\sqrt{\frac{T}{n}}^{\ell-2}.$$

Note that $C(n,\mu)$ is a particular case of $C(n,\overrightarrow{\lambda})$ with λ_2 replaced by $\lambda_2+\mu/\sigma^2$, and λ_ℓ replaced by 0 for $\ell\geq 3$. Proceeding just as in [4] we get a "new" asymptotic expansion of $C(n)$ by substituting our "frozen" parameter μ by $\mu(n)$ in the asymptotic expansion of $C(n,\mu)$. Obviously our "old" expansion of $C(n)$ given in (1.3) can be obtained by collecting together, from the "new" expansion, all the factors of $n^{-\ell/2}$ for $\ell=2,...,i_0$. Now fix $\ell\geq 3$ and concentrate on λ_ℓ. Obviously, any polynomial in $\lambda_2+\mu(n)/\sigma^2$ translates into a polynomial in $\lambda_\ell n^{-(\ell-2)/2}$. Therefore, collecting the factors of $n^{-\ell/2}$ from the "new" expansion of $C(n)$ involves only the terms for which the degree of λ_ℓ is one in the "new" $\mathcal{P}_2$, since for every $j\geq 2$, $\mathcal{P}_j$ is itself a factor of $n^{-j/2}$. Doing so, one easily gets that the coefficient of λ_ℓ in the "old" $\mathcal{P}_\ell$ is given by (1.5). Note that, clearly, for $j<\ell$, λ_ℓ does not appear in the "old" $\mathcal{P}_j$.

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