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The Beverton–Holt model with periodic and conditional harvesting

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In this theoretical study, we investigate the effect of different harvesting strategies on the discrete Beverton–Holt model in a deterministic environment. In particular, we make a comparison between the constant, periodic and conditional harvesting strategies. We find that for large initial populations, constant harvest is more beneficial to both the population and the maximum sustainable yield. However, periodic harvest has a short-term advantage when the initial population is low, and conditional harvest has the advantage of lowering the risk of depletion or extinction. Also, we investigate the periodic character under each strategy and show that periodic harvesting drives population cycles to be multiples (period-wise) of the harvesting period.

Keywords: Beverton–Holt model; periodic harvesting; conditional harvesting; optimal harvesting; periodic discrete systems; periodic solutions

AMS Subject Classification: 39A11; 92D25; 92B99

1. Introduction

In a world where the human population increased by approximately 77 million in 2007 according to the US Census Bureau [31], food demands are rapidly increasing more than ever. This high demand coupled with aggressive fishing techniques is having a disastrous impact on fish populations. Many marine fish show a concerning decline in their abundance to the extent of difficult, or even impossible recovery. The collapse of the Atlantic cod (*Gadus morhua*) in Eastern Canada in 1992 is a well-known example [26], whereas a more recent example is the collapse of salmon stocks in California's major watershed as reported in The New York Times on 13 March 2008 [2]. Such a collapse often leads to one of the two non-favourable alternatives, either a complete moratorium on harvesting or population extinction. We refer the interested reader to the work of Hutchings and Reynolds [20] for exact figures and some details about collapse, recovery and extinction of fish populations. There is no doubt that harvesting plays a major role in fish-population decline, though one can attribute some other significant factors. It is our belief that thoughtful and well-articulate harvesting strategies can help us achieve our current needs without compromising our future needs.

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In the past three decades, scientists and researchers focused on different harvesting strategies in models described by differential equations [3,9,18,30,32,33]. Brauer and Sanchez investigated the effect of constant rate harvesting [6], and periodic harvesting in periodic environments [7] on the asymptotic behaviour of several continuous population models. In both cases, a great deal of attention was given to the continuous logistic model. The maximum sustainable yield (MSY) for the continuous logistic model under different harvesting strategies was widely investigated in the literature [3,9,18,33]. Let us note that Zhang *et al.* [33] argued that continuous harvesting under the logistic model $x'(t) = rx(t)(1 - x(t)/K)$ is superior when compared with an impulsive harvesting, and Dong *et al.* [15] found that continuous harvesting under the periodic Gompertz model $x'(t) = r(t)x(t) \ln K(t)/x(t)$ can be more advantageous. Recently, there has been a growing interest on the dynamics of discrete models and harvesting strategies thereon. Sinha and Parthasarathy [28] considered the discrete Ricker model with constant depletion rate $x_{n+1} = x_n \exp(r(1 - x_n)) - h$ and used numerical results to show that populations exhibiting chaotic oscillations are not necessarily vulnerable to extinction. Chau [8] considered the effect of periodic harvesting or feeding on the discrete Ricker model and the host–parasite model

$$x_{n+1} = x_n e^{r(1-x_n/K)-ay_n}, \quad y_{n+1} = x_n(1 - e^{-ay_n}),$$

and provided numerical results to conclude that harvesting tends to destabilize the dynamics, whereas feeding has a stabilizing effect. In a more general setting, Schreiber [27] investigated the dynamics of the discrete logistic, Ricker and Beverton–Holt models with constant rate depletion. Berezansky and Braverman [4] investigated the asymptotic behaviour of the stochastic Beverton–Holt equation $x_{n+1} = a_n x_n / (1 + b_n x_n)$ under constant and proportional impulsive harvesting. Tang *et al.* [29] considered the periodic Beverton–Holt model under periodic impulsive harvesting, which is a special case of periodic harvesting, and investigated the effect of seasonal environment on the MSY.

In this article, we focus on the theoretical aspects of constant, periodic and conditional harvesting in the discrete Beverton–Holt model in a deterministic environment. In particular, we investigate the effect of these different strategies on the population oscillations and on the MSY. This article is organized as follows: In the next section, we discuss the MSY in the case of constant harvesting. In Section 3, we discuss the MSY in the case of periodic harvesting and investigate the periodic and stability character of solutions. As an illustrative case, we discuss the Beverton–Holt equation with 2-periodic harvest. In Section 4, we discuss the optimal harvest, and the existence and stability of 2-periodic solutions in the case of conditional harvesting, which is better known as the threshold catch policy [10,11,25]. Finally, we summarize the conclusion of this article in Section 5.

2. Constant harvesting

Consider the classical Beverton–Holt model [5], and assume a constant rate harvesting \tilde{h} to obtain

$$y_{n+1} = f(y_n) = \frac{aKy_n}{K + (a-1)y_n} - \tilde{h}, \quad (1)$$

where K is the population carrying capacity, $a > 1$ is the inherent growth rate, and $\tilde{h} > 0$ is a constant representing the intensity of harvesting due to fishing or hunting. The dynamics of this equation are simple to analyse, and can be found in the literature (see, for instance, [22–24,27]). However, for the sake of comparison in the next sections, we discuss its dynamics here. We prefer to use a ‘nonclassical’ approach, which was used for the Riccati equation in [12]. This approach

helps us to deal with the periodic case in the next section. To simplify our writing, we reduce the number of parameters by rescaling. Let $y_n = \alpha x_n$, $\alpha = K/(a-1)$ and $h = \tilde{h}/\alpha$, so Equation (1) takes the form

$$x_{n+1} = f(x_n) = \frac{ax_n}{1+x_n} - h. \quad (2)$$

Here we emphasize that h does not depend on the density of the population. Observe that we cannot harvest more than a regardless of the initial population, and by taking the initial population into account, we cannot harvest more than $h_{\max} := ax_0/(1+x_0)$. Thus our mathematical analysis is concerned with $0 < h < h_{\max} < a$. Now write Equation (2) as

$$x_{n+1}z_{n+1} = (a-h)x_n - h, \quad \text{where } z_{n+1} = 1 + x_n,$$

which is equivalent to the matrix equation

$$z_{n+1} \begin{bmatrix} 1 \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -h & a-h \end{bmatrix} \begin{bmatrix} 1 \\ x_n \end{bmatrix} := A \begin{bmatrix} 1 \\ x_n \end{bmatrix}. \quad (3)$$

By mathematical induction, the explicit solution of Equation (2) is given by x_n , where

$$\left(\prod_{j=1}^n z_j \right) \begin{bmatrix} 1 \\ x_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ x_0 \end{bmatrix}. \quad (4)$$

Now, simple computations reveal that $\det(A) = a > 0$, $\text{tr}(A) = 1 + a - h$, the eigenvalues of A are

$$\lambda_j := \frac{\text{tr}(A)}{2} + \frac{(-1)^j}{2} \sqrt{(\text{tr}(A))^2 - 4\det(A)}, \quad j = 1, 2, \quad (5)$$

and the equilibrium solutions of Equation (2) are

$$\beta_1 := \lambda_1 - 1 \quad \text{and} \quad \beta_2 := \lambda_2 - 1. \quad (6)$$

Obviously, since $\lambda_1 \lambda_2 = \det(A) > 0$ and $\lambda_1 + \lambda_2 = \text{tr}(A) > 0$ for $h < h_{\max}$, then both eigenvalues are positive whenever they are real. If $h \neq (\sqrt{a}-1)^2$, then $\lambda_1 \neq \lambda_2$, and a diagonalization of A implies that $A^n = S\Lambda^n S^{-1}$, where

$$\Lambda := \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{and} \quad S := \begin{bmatrix} 1 & 1 \\ \lambda_1 - 1 & \lambda_2 - 1 \end{bmatrix}.$$

On the other hand, if $h = (\sqrt{a}-1)^2$, then $\lambda_1 = \lambda_2 = \sqrt{a}$. Write $A = SJS^{-1}$, where

$$J := \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \text{and} \quad S := \begin{bmatrix} -(\sqrt{a}-1) & 1 \\ -(\sqrt{a}-1)^2 & 0 \end{bmatrix},$$

then $A^n = SJ^n S^{-1}$. Now, the explicit solution of Equation (2) is given by

$$x_n = \frac{\sqrt{a}x_0 + (\sqrt{a}-1)(x_0 - (\sqrt{a}-1))n}{\sqrt{a} + (x_0 - (\sqrt{a}-1))n} \quad (7)$$

if $h = (\sqrt{a}-1)^2$ and by

$$x_n = \frac{(\lambda_1 - 1)(\lambda_2 - 1)(\mathcal{R}^n - 1) + (\lambda_2 - 1 - (\lambda_1 - 1)\mathcal{R}^n)x_0}{\mathcal{R}^n(\lambda_2 - 1) - \lambda_1 + 1 + (1 - \mathcal{R}^n)x_0} \quad (8)$$

if $h \in (0, (\sqrt{a}-1)^2) \cup ((\sqrt{a}-1)^2, h_{\max})$, where $\mathcal{R} := \lambda_1 \lambda_2^{-1}$.

Next, we assume the MSY as a management objective and proceed to show that it is given by

$$h = h_{\text{th}} := (\sqrt{a} - 1)^2 \quad \text{when } x_0 > \tilde{x}_{\text{th}} := (\sqrt{a} - 1). \quad (9)$$

Let us agree to use optimal harvesting for the MSY. Thus we call a constant harvesting optimal if it is maximized, and the initial population survives indefinitely. Observe that $f(x)$ in Equation (2) is monotonic with x -intercept at $z_0 = h/(a - h)$. It is not possible for a population to be less than z_0 , and that is equivalent to $h < h_{\text{max}}$. Thus, we proceed with the assumption $x_0 > z_0$. If $0 < h < h_{\text{th}}$, then $0 < \mathcal{R} < 1$ and from the explicit solution in Equation (8), we obtain

$$\lim_{n \rightarrow \infty} x_n = \frac{(\lambda_2 - 1)(1 - \lambda_1 + x_0)}{1 - \lambda_1 + x_0} = \beta_2.$$

However, when $z_0 < x_0 < \beta_1$, x_n goes negative before it converges to β_2 . In this case, we investigate the first instance where the numerator or denominator of Equation (8) becomes nonpositive. This implies

$$\begin{aligned} n \geq n_0 &:= \frac{\log((\lambda_2 - 1)(\lambda_1 - 1 - x_0)/(\lambda_1 - 1)(\lambda_2 - 1 - x_0))}{\log \lambda_1 - \log \lambda_2} \quad \text{or} \\ n \geq n_1 &:= \frac{\log((\lambda_1 - 1 - x_0)/(\lambda_2 - 1 - x_0))}{\log \lambda_1 - \log \lambda_2}, \end{aligned} \quad (10)$$

respectively. Thus, a population x_0 needs $\min\{\lceil n_0 \rceil, \lceil n_1 \rceil\}$ generations to perish, where $\lceil \cdot \rceil$ represents the ceiling integer. When $h = h_{\text{th}}$, $\mathcal{R} = 1$, and from the explicit solution in Equation (7), we obtain

$$\lim_{n \rightarrow \infty} x_n = \frac{\tilde{x}_{\text{th}}(x_0 - \tilde{x}_{\text{th}})}{x_0 - \tilde{x}_{\text{th}}} = \tilde{x}_{\text{th}}.$$

As before, when $z_0 < x_0 < \tilde{x}_{\text{th}}$, x_n goes negative before it converges to \tilde{x}_{th} . In particular, x_n in Equation (7) becomes nonpositive in the first instance when

$$n \geq m_0 := \min \left\{ 1, \frac{x_0}{\tilde{x}_{\text{th}}} \right\} \frac{\sqrt{a}}{\tilde{x}_{\text{th}} - x_0}.$$

Thus, a population x_0 needs $\lceil m_0 \rceil$ generations to perish. On the other hand, if $x_0 > \tilde{x}_{\text{th}}$, then x_n converges to \tilde{x}_{th} through a positive orbit. Finally, if $h_{\text{th}} < h < h_{\text{max}}$, then λ_1 and λ_2 are nonreal; however, the explicit solution in Equation (8) is still valid, and the population dies out in finite time. Figure 1 sums up the three possible scenarios.

In conclusion, the intensity of harvesting is bounded between 0 and h_{max} . One has to decide whether to preserve stability and keep the population surviving or to maximize harvesting with a certain control over the number of generations before forcing the population to perish. The latter scenario can be of particular interest in farms or habitats with finite resources, in which the farm

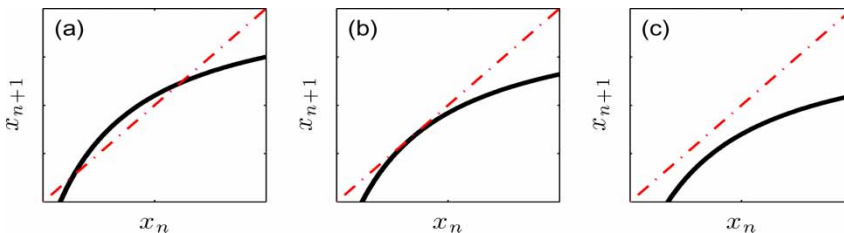


Figure 1. (a) $0 < h < h_{\text{th}}$, (b) $h = h_{\text{th}}$, (c) $h_{\text{th}} < h < a x_0 / (1 + x_0)$.

or habitat will be closed after a certain number of years or generations. On the other hand, if the indefinite survival of the population is the ultimate objective, then the main deciding factors are the initial population x_0 and the intensity of harvesting h . If $x_0 \geq \tilde{x}_{\text{th}}$, then it is obvious that $h_{\text{opt}} := h_{\text{th}}$ is the optimal amount of harvesting; however, if $x_0 < \tilde{x}_{\text{th}}$, then the optimal amount of harvesting is achieved by forcing β_1 to agree with x_0 , i.e., solve $x_0 = \beta_1$ for h to obtain

$$h_{\text{opt}} := \frac{x_0(a-1-x_0)}{1+x_0}. \quad (11)$$

At this point, it is worth mentioning that h_{opt} in Equation (11) gives a possibility of harvesting from day 1. However, one would argue that it is better not to harvest till the population exceeds \tilde{x}_{th} . This argument lies under the strategy of conditional harvesting, but if we allow ourselves to lucubrate here, we find the time needed for the population to exceed \tilde{x}_{th} before any harvest,

$$x_n = \frac{(a-1)a^n x_0}{a-1+(a^n-1)x_0} \geq \tilde{x}_{\text{th}},$$

which implies that

$$n \geq n_0 := \log_a \left(\frac{a-1-x_0}{x_0\sqrt{a}} \right).$$

One can immediately ask whether one should wait for the population to grow beyond \tilde{x}_{th} before harvesting. If $x_0 < \tilde{x}_{\text{th}}$, and we chose to wait for n_0 seasons before harvesting h_{th} constantly, then the total harvesting for $m > n_0$ seasons would yield $(m-n_0)h_{\text{th}}$. Obviously, one should wait if $(m-n_0)h_{\text{th}} \geq mh_{\text{opt}}$, or equivalently

$$m > m_0 := \frac{n_0 h_{\text{th}}}{h_{\text{th}} - h_{\text{opt}}} = \frac{n_0 h_{\text{th}}(x_0 + 1)}{h_{\text{th}} + x_0(x_0 - 2\sqrt{a} + 2)}.$$

Now, let us summarize the facts of constant harvesting in the following theorem.

THEOREM 2.1 *In Equation (2), let the indefinite survival of the population be the management objective, and define $\tilde{x}_{\text{th}} = \sqrt{a} - 1$.*

- (i) *If $x_0 \geq \tilde{x}_{\text{th}}$, then $h_{\text{th}} = (\sqrt{a} - 1)^2$ is the optimal intensity of harvesting.*
- (ii) *If $x_0 < \tilde{x}_{\text{th}}$, then $h_{\text{th}} := (x_0(a-1-x_0))/(1+x_0)$ is the optimal intensity of harvesting.*

3. Periodic harvesting

In this section, we allow different quotas of harvesting at different breeding periods, but we force periodicity on harvesting. Thus, we deal with the p -periodic equation

$$x_{n+1} = \frac{ax_n}{1+x_n} - h_n =: f_n(x_n), \quad n \in \mathbb{N} := \{0, 1, 2, \dots\}, \quad (12)$$

where h_n is the intensity of harvesting at generation n and $h_{n+p} = h_n, \forall n \in \mathbb{N}$. It is worth mentioning that Tang *et al.* [29] have considered periodic impulsive harvesting on the Beverton–Holt model, which is basically a special form of periodic harvesting. Also, we note that Equation (12) is a special case of the p -periodic Riccati equation. The dynamics of the 2-periodic Riccati equation was investigated by Grove *et al.* in [19]; however, the results in [19] are specific for the case $p = 2$ and do not address the issue of optimal harvesting. Although one can attempt to generalize the

approach given in [19] to tackle the problem of this article, we find it more convenient to follow the matrix approach used in Section 2.

To motivate our discussion and emphasize the significance of periodic harvesting, let us give the following example:

Example 3.1 Fix $a = 4$ and let the initial population be $x_0 = 2/3$. Under constant harvesting, $h_{th} = 1$, $\tilde{x}_{th} = \sqrt{a} - 1 = 1 > x_0$, and from Equation (11), $h_{th} = 14/15$. Now, let us have periodic harvesting that keeps $x_0 = 2/3$ surviving. Can we obtain an average harvest larger than $14/15$? Indeed, if we alternate between $h_0 = 3/5$ and $h_1 = 4/3$, then the average is $29/30$, which is larger than $14/15$. At the same time, our initial population x_0 survives indefinitely.

Example 3.1 shows that periodic harvesting does have some advantages. Our purpose in this section is to compare the dynamics of periodic harvesting with that of constant harvesting. The next two results show how population cycles evolve under periodic harvesting. But first, let us list some simple characteristics of the maps f_j in Equation (12).

- (i) $0 \leq h_j \leq h_{j,\max} := ax_j/(1+x_j)$, where $h_{j,\max}$, $0 \leq j \leq p-1$, are determined depending on the initial population.
- (ii) $0 \leq x < y$ implies $f_j(x) < f_j(y)$ and $f_i \circ f_j(x) < f_i \circ f_j(y)$ for all $i, j = 0, \dots, p-1$.
- (iii) $h_i < h_j$ implies $f_j(x) < f_i(x)$ for all $x \geq 0$.
- (iv) The map $G_j(x, h_0, h_1, \dots, h_j) = f_j \circ f_{j-1} \circ \dots \circ f_0(x)$ is increasing in x and decreasing in the h components.

LEMMA 3.2 *In a single-species population with nonoverlapping generations, periodic harvesting drives population cycles to be multiples (period-wise) of the phase period.*

Proof Consider $x_{n+1} = f(x_n)$ and force a p -periodic harvest to obtain $x_{n+1} = f_n(x_n) = f(x_n) - h_n$. Observe that the maps $f_j(y)$, $j = 0, \dots, p-1$, are vertical shifts of $f(x)$, and consequently, two maps overlap with equal quotas of harvesting and do not intersect otherwise. On the other hand, if there exists a cycle of period r , which is not a multiple of p , then the maps $f_0, f_d, f_{2d}, \dots, f_{p-d}$ must intersect at r/d points, where d is the greatest common divisor between r and p (cf. [1]). Since this condition is not achievable under periodic harvesting, the proof is complete. ■

Define the maps $g_0(x) = Id(x) = x$ and $g_n(x) = f_{(n-1) \bmod p}(g_{n-1}(x))$ for all $n \in \mathbb{Z}^+$. The orbits of Equation (12) are in the form

$$\mathcal{O}(x_0) = \{x_0, g_1(x_0), g_2(x_0), \dots, g_{p-1}(x_0), \dots\}. \quad (13)$$

Also, as in Equation (3), associate each map $f_j(x)$ with a matrix \mathbf{A}_j , i.e.,

$$\mathbf{A}_j := \begin{bmatrix} 1 & 1 \\ -h_j & a - h_j \end{bmatrix}.$$

Define $\mathcal{A}_0 = I$, $\mathcal{A}_n = \mathbf{A}_{n-1}\mathcal{A}_{n-1}$ and $X_0 = [1 \ x_0]^T$, then the equivalent form of orbit (13) in the matrix notation is given by

$$\mathcal{O}(X_0) = \{X_0, \mathcal{A}_1 X_0, \mathcal{A}_2 X_0, \dots, \mathcal{A}_{p-1} X_0, \dots\}. \quad (14)$$

THEOREM 3.3 Define \mathcal{A}_p as in Equation (14) and $F_j(x) = f_{p+j-1} \circ \cdots \circ f_{j+1} \circ f_j(x)$, $0 \leq j \leq p-1$. Each of the following holds true for Equation (12):

- (i) If $\text{tr}(\mathcal{A}_p) > 2a^{p/2}$, then there are two p -cycles, one is stable and the other is unstable. Furthermore, if $\bar{x}_0 < \bar{y}_0$ are the fixed points of $F_0(x)$, then $C_1 := \{\bar{x}_0, g_1(\bar{x}_0), \dots, g_{p-1}(\bar{x}_0)\}$ is the unstable cycle and $C_2 := \{\bar{y}_0, g_1(\bar{y}_0), \dots, g_{p-1}(\bar{y}_0)\}$ is the stable one.
- (ii) If $\text{tr}(\mathcal{A}_p) = 2a^{p/2}$, then Equation (12) has one p -cycle and it is semi-stable. Furthermore, if \bar{x}_0 is the fixed point of $F_0(x)$, then the p -cycle satisfies

$$\prod_{j=0}^{p-1} (g_{j+1} \bmod_p(\bar{x}_0) + h_j) = a^{p/2} \prod_{j=0}^{p-1} g_j(\bar{x}_0). \quad (15)$$

- (iii) If $\text{tr}(\mathcal{A}_p) < 2a^{p/2}$, then Equation (12) has no periodic orbits.

Proof To prove part (i), observe that \mathcal{A}_p is the matrix associated with $F_0(x)$. $F_0(x)$ has two fixed points if and only if the eigenvalues of \mathcal{A}_p are distinct reals. The eigenvalues of \mathcal{A}_p are given by

$$\frac{\text{tr}(\mathcal{A}_p)}{2} \pm \frac{\sqrt{\text{tr}(\mathcal{A}_p)^2 - 4 \det(\mathcal{A}_p)}}{2},$$

where $\det(\mathcal{A}_p) = a^p$ by induction. Now let $\bar{x}_0 < \bar{y}_0$ be the fixed points of $F_0(x)$ and invoke Lemma 3.2 to obtain the minimal period, consequently C_1 and C_2 are the two p -cycles. The stability of \bar{y}_0 under $F_0(x)$ follows from a simple cobweb diagram, and since $f_j(x)$, $0 \leq j \leq p-1$, are continuous, then \bar{y}_0 is stable under $F_0(x)$ if and only if C_2 is stable in Equation (12). Now, parts (ii) and (iii) are clear. Equation (15) follows from $F'_0(x) = 1$ at $x = \bar{x}_0$ and the orbit in (13). ■

The next corollary is straightforward.

COROLLARY 3.4 Denote the fixed points of $f_j(x)$ by $\beta_{j,1}, \beta_{j,2}$, and define h_{th} as before. Each of the following holds true.

- (i) If $h_j > h_{\text{th}}$ for all $j = 0, \dots, p-1$, then the population dies out in finite time.
- (ii) If $h_j \leq h_{\text{th}}$ for all $j = 0, \dots, p-1$, then the population survives if $x_0 \geq \max\{\min\{\beta_{j,1}, \beta_{j,2}\}, j = 0, 1, \dots, p-1\}$.
- (iii) $F_0(\bar{x}) = \bar{x}$ if and only if $F_j(g_j(\bar{x})) = g_j(\bar{x})$ for all $j = 0, \dots, p-1$.

We extend the definition of optimal harvesting to the periodic case as follows:

DEFINITION 3.5 The amount of harvesting in Equation (12) is called optimal if the average harvesting $h_{\text{av}} = 1/p \sum_{j=0}^{p-1} h_j$ is maximized and the population survives indefinitely.

LEMMA 3.6 Let h_{av} be the average harvest and the initial population be sufficiently large. $\text{tr}(\mathcal{A}_p) = 2a^{p/2}$ is a necessary condition for h_{av} to be optimal.

Proof From Theorem 3.3, $\text{tr}(\mathcal{A}_p) \geq 2a^{p/2}$; otherwise, the population eventually dies out. If $\text{tr}(\mathcal{A}_p) > 2a^{p/2}$, then $F_0(x) = x$ has two fixed points. From the fact that F_0 is decreasing in h_j , $j = 0, \dots, p-1$, we can increase the harvest for some h_j without the risk of extinction. Thus, h_{av} cannot be optimal without having $\text{tr}(\mathcal{A}_p) = 2a^{p/2}$. ■

THEOREM 3.7 Consider Equation (12), let $\text{tr}(\mathcal{A}_p) = 2a^{p/2}$ and $C_p := \{\bar{x}_0, \dots, \bar{x}_{p-1}\}$ be the p -cycle. The average $\frac{1}{p} \sum_{j=0}^{p-1} h_j/x_{j+1}$ is maximized when $h_j = h_{\text{th}}$, $j = 0, 1, \dots, p-1$.

Proof From Equation (12), in Theorem 3.3, C_p satisfies

$$\prod_{j=0}^{p-1} (x_{j+1} + h_j) = a^{p/2} \prod_{j=0}^{p-1} x_j, \quad \text{or equivalently} \quad \prod_{j=0}^{p-1} \left(1 + \frac{h_j}{x_{j+1}}\right) = a^{p/2}.$$

Now, let $y_j = h_j/(x_{j+1})$. Using Lagrange multipliers,

$$y_{\text{av}} = \frac{1}{p} \sum_{j=0}^{p-1} y_j \quad \text{subject to} \quad \prod_{j=0}^{p-1} (1 + y_j) = a^{p/2}$$

is maximized at $y_j = \sqrt{a} - 1$, $\forall j$, and consequently $h_j = (\sqrt{a} - 1)x_{j+1}$. Substitute h_j in Equation (12) to obtain $x_{j+1} = \sqrt{a}x_j/(1 + x_j)$, but this equation has no cycles other than the fixed points 0, $\sqrt{a} - 1$. Hence $x_j = \sqrt{a} - 1 = \tilde{x}_{\text{th}}$ and $h_j = (\sqrt{a} - 1)^2 = h_{\text{th}}$. ■

Next, assume the initial population is sufficiently large, and let us compare p -periodic harvest $\{h_0, h_1, \dots, h_{p-1}\}$ versus a constant harvest with the same average; i.e., a constant harvest with $h_{\text{av}} = 1/p \sum_{j=0}^{p-1} h_j$.

LEMMA 3.8 *Let $h_{\text{av}} = 1/p \sum_{j=0}^{p-1} h_j$. If $\text{tr}(\mathcal{A}_p) \geq 2a^{p/2}$, then $h_{\text{av}} \leq h_{\text{th}}$. Furthermore, $h_{\text{av}} = h_{\text{th}}$ if and only if $h_j = h_{\text{th}}$ for each j .*

Proof By Theorem 3.3, $\text{tr}(\mathcal{A}_p) \geq 2a^{p/2}$ assures the existence of a stable p -cycle, say $\{x_0, x_1, \dots, x_{p-1}\}$. Now,

$$\frac{1}{p} \sum_{j=0}^{p-1} x_{j+1} \mod p = \frac{1}{p} \sum_{j=0}^{p-1} \left(\frac{ax_j}{1+x_j} - h_j \right) = -h_{\text{av}} + \frac{1}{p} \sum_{j=0}^{p-1} \frac{ax_j}{1+x_j},$$

which implies

$$h_{\text{av}} = \frac{1}{p} \sum_{j=0}^{p-1} \left(\frac{ax_j}{1+x_j} - x_j \right).$$

Consider the function $g(t) = (t(a - 1 - t))/(1 + t)$. Since $\max_{t>0} g(t) = (\sqrt{a} - 1)^2 = h_{\text{th}}$, then $h_{\text{av}} < h_{\text{th}}$. Finally, $h_{\text{av}} = h_{\text{th}}$ holds only when the p -periodic equation reduces to an autonomous one. ■

THEOREM 3.9 *Assume the initial population is sufficiently large. Populations governed by the Beverton–Holt model attenuate under periodic harvesting, i.e., constant harvesting $h_{\text{av}} \leq h_{\text{th}}$ is more beneficial to the population compared with periodic harvesting with average $1/p \sum_{j=0}^{p-1} h_j = h_{\text{av}}$.*

Proof If $\text{tr}(\mathcal{A}_p) < 2a^{p-1}$, then the population attenuates anyway. So we consider $\text{tr}(\mathcal{A}_p) \geq 2a^{p-1}$ and let the stable cycle (or semi-stable in case $\text{tr}(\mathcal{A}_p) = 2a^{p-1}$) be $\{x_0, x_1, \dots, x_{p-1}\}$.

Define $x_{av} = 1/p \sum_{j=0}^{p-1} x_{j+1}$ and use Equation (12) to write

$$x_{av} = -h_{av} + \frac{1}{p} \sum_{j=0}^{p-1} \frac{ax_j}{1+x_j}.$$

Define $g_1(t) = t/(1+t)$ and use Jensen's inequality on g_1 (cf. [21]) to obtain

$$x_{av} + h_{av} < a \frac{x_{av}}{1+x_{av}}. \quad (16)$$

Now consider a constant harvest of h_{av} in Equation (2) and let \bar{y} be the associated stable equilibrium, we obtain

$$h_{th} = \frac{a\bar{y}}{1+\bar{y}} - \bar{y}. \quad (17)$$

Keep in mind that $h_{av} \leq h_{th}$ if and only if $\bar{y} \geq (\sqrt{a} - 1)$. From Equations (16) and (17), we obtain

$$\frac{a\bar{y}}{1+\bar{y}} - \bar{y} < \frac{ax_{av}}{1+x_{av}} - x_{av}.$$

From the facts that $\bar{y} \geq (\sqrt{a} - 1)$ and $g_2(t) := at/(1+t) - t$ is decreasing on the interval $[\sqrt{a} - 1, a - 1]$, we obtain $\bar{y} > x_{av}$, and this completes the proof. ■

The next corollary became obvious.

COROLLARY 3.10 *If the initial population is larger than the threshold \tilde{x}_{th} , then the optimal harvest is the constant harvest; i.e., $h_i = h_j = h_{th}$.*

Theorem 3.9 and Corollary 3.10 show that constant harvest optimizes the MSY (compared with periodic harvest) and benefits the surviving population. So one might wonder about the advantages of periodic harvesting. As shown in Example 3.1, the advantage of periodic harvesting lies in short-term considerations in the sense that populations are given time to recover before increasing the harvesting quota. Indeed, an articulate periodic harvesting makes 'a larger' basin of attraction (Figure 2). The basin of attraction of a p -cycle

$$C_p := \{\bar{x}_0, g_1(\bar{x}_0), \dots, g_{p-1}(\bar{x}_0)\}$$

of Equation (12) is defined as

$$W^s(C_p) = \{x_0 \in \mathbb{R}^+ : \lim_{n \rightarrow \infty} x_{np} = \bar{x}_0\}.$$

If $\bar{x}_0 < \bar{y}_0$ are the fixed points of $F_0(x)$, then the basin of attraction of the stable cycle C_p of Equation (12) is given by $W^s(C_p) = \{x \in \mathbb{R}^+ : x > \bar{x}_0\}$. A small harvest h_0 gives the population a chance to grow, then the intensity of harvest can be increased as the population grows to obtain a total harvest (for the short term) more than the constant harvest. Once the population exceeds the threshold \tilde{x}_{th} , then one might switch to the constant harvest policy.

We illustrate the results of this section by considering the 2-periodic case in the next example.

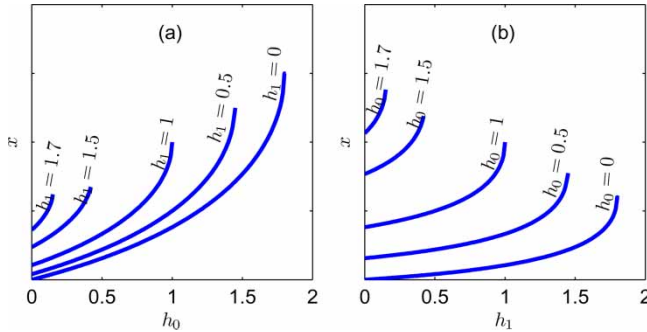


Figure 2. (a and b) Curves of the fixed point \bar{x}_0 for different values of h_0 and h_1 when $a = 4$. The interval (\bar{x}_0, ∞) on the vertical axis gives the basin of attraction for different choices of 2-periodic harvesting. A small initial harvesting quota (small h_0) gives a larger basin of attraction, and thus benefits low level populations.

Example 3.11 Consider the 2-periodic case. Obviously, here either $h_0 < h_1$ or $h_1 < h_0$. Lemma 3.6 leads us to focus our attention on the condition $\text{tr}(\mathcal{A}_2) = 2a$ and from which we obtain h_1 as a function of h_0 ,

$$h_1 = \frac{(a-1)^2 - h_0(a+1)}{a+1-h_0}, \quad 0 \leq h_0 \leq \frac{(a-1)^2}{a+1}. \quad (18)$$

From this fact, we obtain

$$h_{\text{av}} = \frac{1}{2} \frac{(a-1-h_0)(a-1+h_0)}{1+a-h_0}.$$

The maximum average is achieved at

$$h_0 = (\sqrt{a} - 1)^2 = h_{\text{th}},$$

which implies that $h_0 = h_1 = h_{\text{th}}$. On the other hand, if

$$0 \leq h_0 \leq \frac{(a-1)^2}{a+1}, \quad 0 \leq h_1 \leq \frac{(a-1)^2 - h_0(a+1)}{a+1-h_0}, \quad (19)$$

then we identify the explicit form of the 2-cycles assured by Theorem 3.3. Indeed, the solution of $f_1 f_0(x) = x$ is given by

$$\bar{x}_0 = \frac{h_0 - 1 + \lambda_1}{1 + a - h_0} \quad \text{and} \quad \bar{y}_0 = \frac{h_0 - 1 + \lambda_2}{1 + a - h_0}, \quad (20)$$

where λ_1 and λ_2 are the eigenvalues of $\mathcal{A}_2 = A_1 A_0$. The fixed point \bar{x}_0 gives the unstable 2-cycle $\{\bar{x}_0, \bar{x}_1\}$, where

$$\bar{x}_1 = f_0(\bar{x}_0) = \frac{h_1 - 1 + \lambda_1}{1 + a - h_1}, \quad (21)$$

and \bar{y}_0 gives the stable 2-cycle (semi-stable when $\text{tr}(\mathcal{A}_2) = 2a$) $\{\bar{y}_0, \bar{y}_1\}$, where

$$\bar{y}_1 = f_0(\bar{y}_0) = \frac{h_1 - 1 + \lambda_2}{1 + a - h_1}. \quad (22)$$

Now assume the initial population is sufficiently large, and let us compare 2-periodic harvesting $\{h_0, h_1\}$ versus a constant harvest with the same average; i.e. a constant harvest

$h = h_{av} = (h_0 + h_1)/2$. After simplifications, we write the attractor in Equation (22) as

$$\bar{y}_j = \frac{1}{2}(a - h_{j+1(\text{mod } 2)} - 1) + \frac{1}{2(1 + a - h_j)}\sqrt{\text{tr}(\mathcal{A}_2)^2 - 4a^2}, \quad j = 0, 1, \quad (23)$$

and $\text{tr}(\mathcal{A}_2) = 1 + a^2 - (1 + a)(h_0 + h_1) + h_0h_1$. On the other hand, the attractor under constant harvesting h_{av} is given by the fixed point

$$x_{av} := \frac{1}{2}(a - h_{av} - 1) + \frac{1}{2}\sqrt{(1 + a - h_{av})^2 - 4a}.$$

Now

$$2x_{av} - (\bar{y}_0 + \bar{y}_1) = \sqrt{(1 + a - h_{av})^2 - 4a} - \left(\frac{1 + a - h_{av}}{(1 + a - h_0)(1 + a - h_1)} \right) \sqrt{\text{tr}(\mathcal{A}_2)^2 - 4a^2}.$$

Since multiplying by the conjugate of this quantity will not affect its sign, we multiply by the conjugate to obtain

$$\frac{a(h_0 - h_1)^2}{(1 + a - h_0)(1 + a - h_1)} \geq 0.$$

Thus $x_{av} \geq 1/2(\bar{y}_0 + \bar{y}_1)$ as assured by Theorem 3.9. In the above discussion, we neglected the role of the initial population; however, as we mentioned before, periodic harvesting can be beneficial to the basin of attraction (Figure 2). It is possible for a population x_0 to survive under periodic harvesting $\{h_0, h_1\}$ while dying out under constant harvesting of the same average, see region R_p in Figure 3. Also, it is possible for a large initial population x_0 to die out under periodic harvesting while surviving under constant harvesting of the same average, see regions R_c in Figure 3.

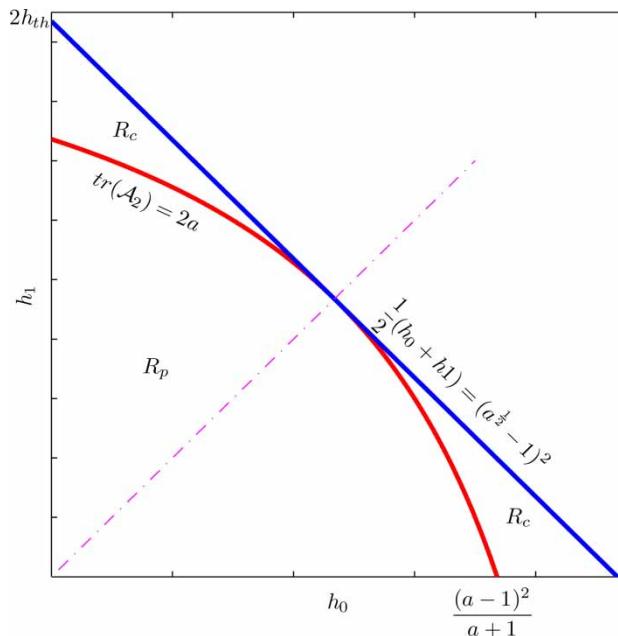


Figure 3. Define R_p to be the region bounded by $h_0 = 0$, $h_0 = h_1$ and $\text{tr}(\mathcal{A}_2) = 2a$ and R_c to be the region in the first quadrant bounded between $\text{tr}(\mathcal{A}_2) = 2a$ and $h_0 + h_1 = 2h_{th}$. In comparing a 2-periodic harvest $\{h_0, h_1\}$ with a constant harvest of the same average, periodic harvesting $(h_0, h_1) \in R_p$ benefits small initial populations. Otherwise, constant harvesting is more beneficial. In region R_c , periodic harvesting forces extinction while constant harvesting keeps survival.

4. Conditional harvesting

It is perfectly logical to allow harvesting only when a population exceeds a certain level x_{th} (a threshold level). Accordingly, it is natural to consider what we call the Beverton–Holt model with conditional harvesting.

$$x_{n+1} = f(x_n) = \begin{cases} \frac{ax_n}{1+x_n}, & x_n < x_{th}, \\ \frac{ax_n}{1+x_n} - h, & x_n \geq x_{th}. \end{cases} \quad (24)$$

For the motivation behind considering conditional harvesting, we refer the reader to Clark and Hare [10] and Lande *et al.* [25]. The main deciding factors in this model are the initial population x_0 , the threshold level x_{th} and the intensity of harvesting h . It is tempting to suggest that conditional harvesting is the safest strategy to follow, but in fact, it has its drawbacks. In practice, the decision maker must have an accurate estimate of the population level at each step to decide whether or not to allow harvesting. Putting a halt on harvesting or limiting the harvesting level after an over harvest are hard decisions to make given their socio-economical and political impacts. While these issues are of paramount importance, we limit our attention to the periodic character and the total harvest. Let us start by ruling out some obvious scenarios. If we put the threshold x_{th} larger than $a - 1$ (which means larger than the carrying capacity K before our variable rescaling), then $x_0 \leq a - 1$ means that no harvest will take place at any time. If $x_0 > a - 1$, then harvest will only take place finitely many times before it ceases forever. Thus, such a scenario is far from being optimal. Also, if we lower the threshold x_{th} below the carrying capacity, then $h < ax_{th}/(1+x_{th}) - x_{th}$ is far from optimal too. So we focus our attention on the case $0 < x_{th} < a - 1$ and $ax_{th}/(1+x_{th}) - x_{th} \leq h < ax_{th}/(1+x_{th})$. To simplify our writing in this section, let us agree to define $f_0(x) = ax/(1+x)$ and $f_1(x) = f_0(x) - h$. The next proposition is straightforward.

PROPOSITION 4.1 *Let $0 < x_{th} < a - 1$, $f_0(x_{th}) - x_{th} \leq h < f_0(x_{th})$ and $\{x_0, x_1, \dots, x_n, \dots\}$ be an orbit of Equation (24). Each of the following holds true.*

- (i) *The interval $I = [f_0(x_{th}) - h, f_0(x_{th})]$ is invariant, and each orbit enters I in finite time.*
- (ii) *$f_0(x_{th}) - h \leq 1/n \sum_{j=0}^{n-1} x_j \leq f_0(x_{th})$ for sufficiently large n .*

After an orbit enters the invariant set I , it oscillates around x_{th} , and the nature of its oscillation depends on the intensity of harvesting h . The more we harvest, the more time needed for the population to recover.

THEOREM 4.2 *The optimal harvest for the conditional strategy is achieved at $x_{th} = \tilde{x}_{th} = \sqrt{a} - 1$ and $h = h_{th}$.*

Proof Let $\{x_0, x_1, \dots, x_n > 0\}$, be an orbit of Equation (24). Write

$$\frac{1}{n} \sum_{j=0}^{n-1} x_j = \frac{x_0}{n} + \frac{1}{n} \sum_{j=0}^{n-2} x_{j+1} = \frac{x_0}{n} + \frac{1}{n} \left[\sum_{j=0}^{m_1-1} x_{j+1} + \sum_{j=m_1}^{n-2} x_{j+1} \right],$$

where m_1 is the number of times we harvest, and we rearrange terms if necessary. Thus

$$\frac{1}{n} \sum_{j=0}^{n-1} x_j = \frac{x_0}{n} + \frac{1}{n} \left[\sum_{j=0}^{m_1-1} \frac{ax_j}{1+x_j} - m_1 h + \sum_{j=m_1}^{n-2} \frac{ax_j}{1+x_j} \right],$$

which implies

$$h_{av} := \frac{m_1}{n}h = \frac{x_0 - x_{n-1}}{n} + \frac{1}{n} \sum_{j=0}^{n-2} \left(\frac{ax_j}{1+x_j} - x_j \right). \quad (25)$$

Since $0 < x_j < a - 1$ and $g(t) = at/(1+t) - t$ has a maximum at $t = \sqrt{a} - 1$, then we obtain

$$h_{av} < \frac{x_0 - x_{n-1}}{n} + \frac{n-1}{n} g(\sqrt{a} - 1) = \frac{x_0 - x_{n-1} - (\sqrt{a} - 1)^2}{n} + (\sqrt{a} - 1)^2. \quad (26)$$

Obviously, for sufficiently large n , $h_{av} \leq h_{th}$. Inequality (26) shows that $h_{av} < h_{th}$ in finite time whenever $x_{n-1} \leq x_0 - h_{th}$. Finally, by choosing $h = h_{th}$ and $x_{th} = \tilde{x}_{th} = \sqrt{a} - 1$, every small initial population grows beyond x_{th} in finite time and then converges to the fixed point x_{th} . Thus harvesting always takes place except possibly a finite number of times for small initial populations $x_0 < x_{th}$, and consequently $h_{av} = h_{th}$. ■

The next corollary gives the analogue of Theorem 3.9.

COROLLARY 4.3 *Populations governed by system (24) attenuate compared with constant harvesting of the same average.*

Proof Use Equation (25) and the fact that $h_{av} \leq h_{th}$, then the proof follows along the same lines as the proof of Theorem 3.9. ■

Next, we investigate the existence of 2-cycles. Obviously, any existing cycle has to be in the invariant set. Furthermore, any 2-cycle of system (24) must be a 2-cycle of the 2-periodic system $x_{n+1} = f_{n \bmod 2}(x_n)$. Thus, we set $h_0 = 0$ and $h_1 = h$ in Equations (21) and (22) to obtain

$$\bar{x}_0 = \frac{\lambda_1 - 1}{a + 1}, \quad \bar{x}_1 = \frac{h - 1 + \lambda_1}{a + 1 - h}, \quad \bar{y}_0 = \frac{\lambda_2 - 1}{a + 1} \quad \text{and} \quad \bar{y}_1 = \frac{h - 1 + \lambda_2}{a + 1 - h}, \quad (27)$$

where

$$\lambda_j = \frac{1}{2} \left(\text{tr}(\mathcal{A}_2) + (-1)^j \sqrt{\text{tr}(\mathcal{A}_2)^2 - 4a^2} \right), \quad \text{tr}(\mathcal{A}_2) = 1 + a^2 - (1 + a)h, \quad j = 0, 1.$$

Recall that inequality (19) implies $0 \leq h \leq (a - 1)^2/(a + 1)$, and from Equation (24), $0 < h < ax_{th}/(1 + x_{th}) - x_{th}$. Taking those constraints into consideration, we need to find the circumstances where $\{\bar{x}_0, \bar{x}_1\}$ or $\{\bar{y}_0, \bar{y}_1\} \subset I$. Here, we avoid writing the simple but rather tedious computations, and summarize the conclusion in the next theorem. Figures 4 and 5 illustrate the outlines for the computational proof.

THEOREM 4.4 *Consider Equation (24), and define the curves*

$$h_a(x_{th}) := \frac{(1 + a)x_{th}(a - 1 - x_{th})}{(1 + x_{th})(a - x_{th})}, \quad \tilde{h}_a(x_{th}) := \frac{(1 + a)x_{th}(a - 1 - x_{th})}{1 + (1 + a)x_{th}}.$$

Let $\bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1$ be as in (27). Each of the following holds true.

- (i) *If $(x_{th}, h) \in \left\{ (x, h) : 0 < x < a - 1, \tilde{h}_a(x) < h < h_a(x) \right\}$, then there exists a unique 2-cycle, which is the stable cycle $\{\bar{y}_0, \bar{y}_1\}$.*
- (ii) *If $(x_{th}, h) \in \{(x, h) : 0 < x < a - 1, h_a(x) < h < \min\{\tilde{h}_a(x), ax/(1 + x)\}\}$, then there exists a unique 2-cycle, which is the unstable cycle $\{\bar{x}_0, \bar{x}_1\}$.*

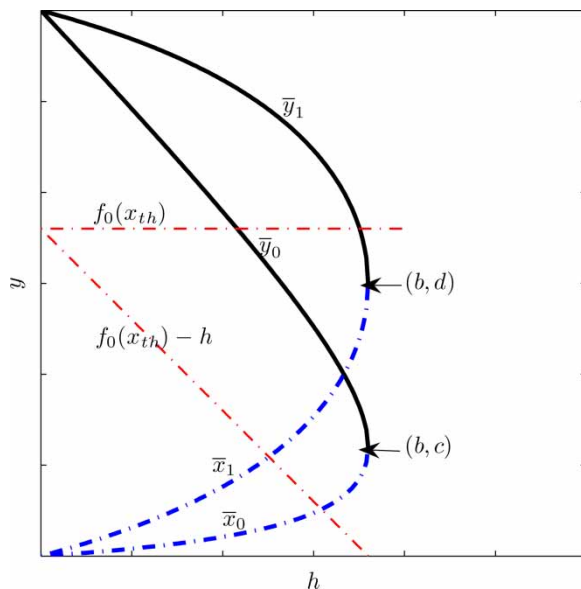


Figure 4. This figure shows the branches of the stable 2-cycle $\{\bar{y}_0, \bar{y}_1\}$ and the unstable 2-cycle $\{\bar{x}_0, \bar{x}_1\}$ of the 2-periodic system with $h_0 = 0$ and $h_1 = h$. In the graph, $b = (a-1)^2/(a+1)$, $c = (a-1)/(a+1)$ and $d = (1/2)(a-1)$. By allowing the constant x_{th} to slide along the interval $[0, a-1]$, the dashed lines $f_0(x_{th})$ and $f_0(x_{th}) - h$ show the circumstances where the cycles of the periodic harvest lie inside the invariant interval $[f_0(x_{th}) - h, f_0(x_{th})]$.

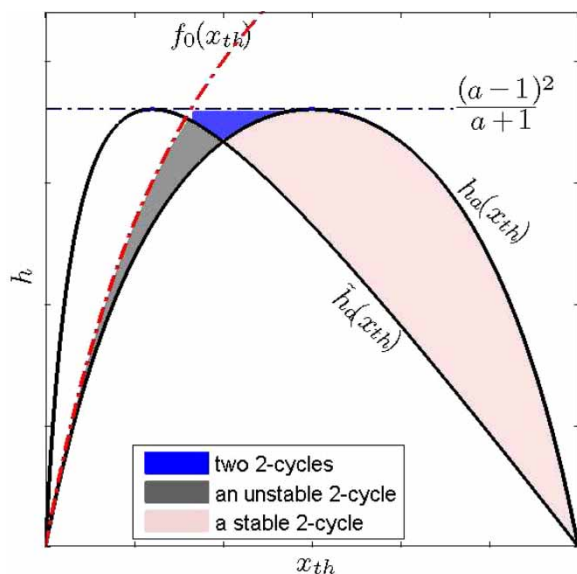


Figure 5. This figure shows the regions in the x_{th}, h plane where 2-cycles of Equation (24) exist. $h_a(x_{th})$ and $\tilde{h}_a(x_{th})$ are defined in Theorem 4.4. $h_a(x_{th})$ is found from $f_0(x_{th}) - h = \bar{y}_0$ and $f_0(x_{th}) - h = \bar{x}_0$, and $\tilde{h}_a(x_{th})$ is found from $f_0(x_{th}) = \bar{y}_1$ and $f_0(x_{th}) = \bar{x}_1$.

(iii) If

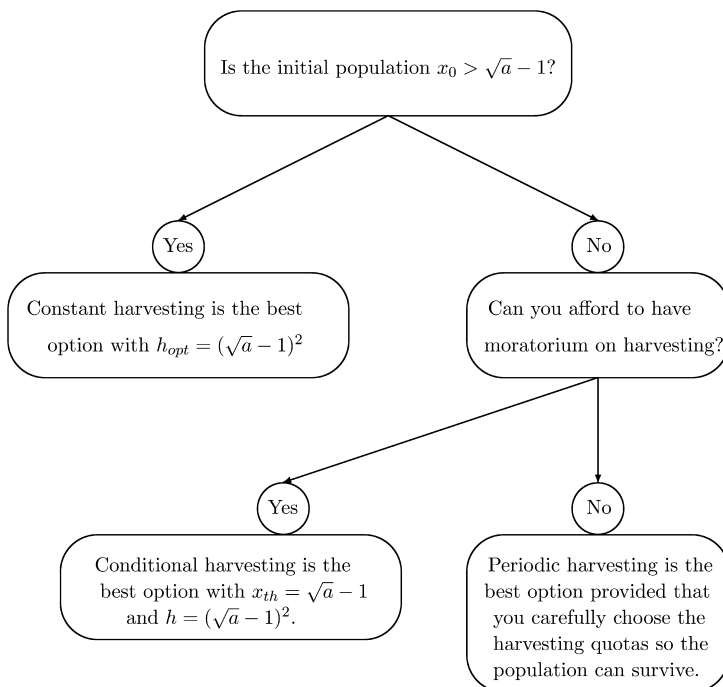
$$(x_{\text{th}}, h) \in \left\{ (x, h) : \frac{a-1}{a+1} < x < \frac{a-1}{2}, \max\{h_a(x), \tilde{h}_a(x)\} < h < \min\left\{\frac{ax}{1+x}, \frac{(a-1)^2}{a+1}\right\} \right\},$$

then two 2-cycles exist, namely, the stable one $\{\bar{y}_0, \bar{y}_1\}$ and the unstable one $\{\bar{x}_0, \bar{x}_1\}$.

Finally, we remark that the Beverton–Holt model with conditional harvesting portrays cycles of periods other than 2. An abstract analysis of its dynamics would be the topic of further investigation.

5. Conclusion

This article is concerned with constant, periodic and conditional harvesting strategies on the discrete Beverton–Holt model with constant inheritance growth rate and carrying capacity. A comparison between the three strategies leads to the conclusion summarized in the next chart. Here, it is possible to argue that theoretical aspects of these strategies are far from being applicable in reality. For instance, Clark [9] shows that MSY does not necessarily lead to an optimal policy in an economic sense. However, this work is not concerned with this issue. Finally, it is worth mentioning that Cushing and Henson [13,14] found that periodic environment does not benefit populations governed by the discrete Beverton–Holt model (see also [16,17]), whereas this study shows that populations attenuate under periodic harvesting. This motivates us in the near future to investigate the constant, periodic and conditional harvesting strategies for the periodic discrete Beverton–Holt model.



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