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A new characterization of periodic oscillations in periodic difference equations

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Abstract

In this paper, we characterize periodic solutions of p-periodic difference equations. We classify the periods into multiples of p and nonmultiples of p. We show that the elements of the set of multiples of p follow the well-known Sharkovsky's ordering multiplied by p. On the other hand, we show that the elements of the set Γ_p of nonmultiples of p are independent in their existence. Moreover, we show the existence of a p-periodic difference equation with infinite Γ_p-set in which the maps are defined on a compact domain and agree exactly on a countable set. Based on the proposed classification, we give a refinement of Sharkovsky's theorem for periodic difference equations.

Keywords: Periodic difference equations, Sharkovsky's theorem, periodic orbits.
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1 Introduction

Consider the p-periodic difference equation

$$x_{n+1} = f(n, x_n) = f_n(x_n),\ n = 0, 1, \ldots, \tag{1.1}$$

where p is the minimal (unless mentioned otherwise) positive integer for which $f_{n+p} = f_n$, for all $n \in \mathbb{N} := \{0, 1, \ldots\}$. We assume $f_j \in \mathcal{C}(I, I)$ for all $j = 0, 1, \ldots, p-1$, where $\mathcal{C}(I, I)$ denotes the space of continuous functions on $I := [0, 1]$ endowed with the sup-norm. An orbit of Eq. (1.1) through a point $x_0 \in I$,

$$\mathcal{O}^+(x_0) := \{x_0, \overbrace{f_0(x_0)}^{x_1}, \ldots, \overbrace{f_{p-1}\cdots f_0(x_0)}^{x_p}, \overbrace{f_0 f_{p-1}\cdots f_0(x_0)}^{x_{p+1}}, \ldots\} \tag{1.2}$$

is called r-periodic (or forms a r-cycle) if r is the smallest positive integer for which $x_{n+r} = x_n, \forall n \in \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integers. It is worth stressing

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here that cycles of Eq. (1.1) are ordered sets, and we treat them as such throughout this paper. Although one can use the starting time to be $n = n_0$, we always use $n_0 = 0$. Define the set of periodic points of Eq. (1.1) as $Per(f_n, p) :=$

$$\{x_0 \in [0,1] : \mathcal{O}^+(x_0) = \{x_0, x_1, \ldots\} \text{ is a periodic orbit of Eq. (1.1)}\}.$$

If $p = 1$, then we have the autonomous case $x_{n+1} = f(x_n)$ and we use $Per(f, 1)$. Throughout this paper, a r-cycle of a map $f(x)$ is meant to be a r-cycle of the autonomous equation $x_{n+1} = f(x_n)$. Let $\mathcal{P}(f_n, p)$ be the set of minimal periods of Eq. (1.1). We let M_p and Γ_p denote the minimal periods of multiples and nonmultiples of p respectively.

One of the most interesting problems concerning Eq. (1.1) is characterizing its periodic orbits [2, 4, 5, 6, 8, 9, 10, 12, 14, 15]. For more information on the significance of periodic orbits of periodic difference equations in population biology, we refer the reader to [10, 11, 12, 13, 15, 17, 18]. On the other hand, the autonomous form of Eq. (1.1) ($p = 1$) is becoming a classical topic; however, for readers from other disciplines, the papers of Sharkovsky [20] and Li & Yorke [19] "Period three implies chaos" deserve to be acknowledged and recommended for a historical background reading. Let $f \in \mathcal{C}(I, I)$, the fascinating result of Sharkovsky's [20] states that if f has a periodic point of period k, then it has a periodic point of period r for all $k \preceq r$ in the following order:

$$\begin{aligned}
&3 \prec 5 \prec 7 \prec \cdots \\
&2 \cdot 3 \prec 2 \cdot 5 \prec 2 \cdot 7 \prec \ldots \\
&\qquad \vdots \\
&2^n \cdot 3 \prec 2^n \cdot 5 \prec 2^n \cdot 7 \prec \ldots \\
&\qquad \vdots \\
&\ldots \prec 2^n \prec \ldots \prec 2^2 \prec 2 \prec 1.
\end{aligned} \tag{1.3}$$

In [4], AlSharawi et al. extended Sharkovsky's theorem to the p-periodic difference equation in (1.1). However, the given extension does not give the specific periods assured by a periodic orbit. Alves [5, 6] approached the problem using the Zeta function and gave certain characteristics of periodic solutions when the set of intersections between the maps is finite. Cánovas and Linero [8] focused on the case $p = 2$ and described the forcing between periodic solutions. However, as can be observed in [4, 2], the case $p = 2$ is a special case since positive integers are either multiples of 2 or relatively primes with 2. In [2], AlSharawi classified the elements of $\mathcal{P}(f_n, p)$ as multiples and non-multiples of p. He showed that it is possible to determine the set Γ_p using combinatorial arguments on the common points between the maps f_j. For instance, if the maps f_j are rational, then the set Γ_p is finite. On the other hand, examples where the set Γ_p is infinite are available. In particular, such examples can be easily constructed if the domain is noncompact or the set of overlaps of the functions f_j has a positive Lebesgue measure. The question whether there is an example with an infinite Γ_p set under the conditions that the domain is compact and the maps f_j intersect on a set of zero Lebesgue measure was left open in [2]. In this paper, we give an affirmative answer to this question. Moreover, we give a refinement of Sharkovsky's theorem for periodic difference equations [4].

2 A refinement of Sharkovsky's theorem for periodic difference equations

For two positive integers p and q, let $\mathrm{lcm}(p,q)$ and $\gcd(p,q)$ denote the least common multiple and greatest common divisor respectively. Let $\mathcal{A}_{p,q}$ be the set defined by

$$\mathcal{A}_{p,q} = \{r \in \mathbb{Z}^+ : \ \mathrm{lcm}(r,p) = pq\}.$$

The p-Sharkovsky's ordering, as defined in [4], is given by

$$\begin{aligned}
&\mathcal{A}_{p,3} \prec \mathcal{A}_{p,5} \prec \mathcal{A}_{p,7} \prec ... \\
&\mathcal{A}_{p,2\cdot 3} \prec \mathcal{A}_{p,2\cdot 5} \prec \mathcal{A}_{p,2\cdot 7} \prec \dots \\
&\vdots \\
&\mathcal{A}_{p,2^n\cdot 3} \prec \mathcal{A}_{p,2^n\cdot 5} \prec \mathcal{A}_{p,2^n\cdot 7} \prec \dots \\
&\vdots \\
&\cdots \prec \mathcal{A}_{p,2^n} \prec ... \prec \mathcal{A}_{p,2^2} \prec \mathcal{A}_{p,2} \prec \mathcal{A}_{p,1}.
\end{aligned} \tag{2.1}$$

It is obvious that this ordering reduces to the original Sharkovsky's ordering given in (1.3) when $p=1$, which we refer to by the 1-Sharkovsky's ordering. We use $r_1 \preceq r_2$ to mean $r_1 = r_2$ or r_2 follows r_1 in the 1-Sharkovsky's ordering, while $r_1 \leq r_2$ carries the well-known (less than or equal) meaning. The next result is needed in the sequel.

Theorem 2.1 (AlSharawi et al. [4]). *Consider Eq. (1.1) with $f_i \in \mathcal{C}(I,I)$, $i = 0,1,...,p-1$. If $\mathcal{P}(f_n,p) \cap \mathcal{A}_{p,\ell} \neq \phi$ for some $\ell \in \mathbb{Z}^+$, then $\mathcal{P}(f_n,p) \cap \mathcal{A}_{p,q} \neq \phi$ for all $\ell \preceq q$ in the 1-Sharkovsky's ordering.*

Let us agree to say an interval is nontrivial if it has positive length. It is possible for Eq. (1.1) to be of minimal period p on the interval I, but reduces to a periodic equation of shorter period on a nontrivial subinterval of I. In such case, one can treat Eq. (1.1) depending on the new shorter period and the partitioned domain. However, we consider this scenario to be a degenerate one and avoid it throughout this paper. To clarify the notion, we give a formal definition followed by an example.

Definition 2.1. *A p-periodic difference equation of the form (1.1) is called degenerate if it reduces to a periodic equation of shorter period on a nontrivial subinterval of I, and it is called non-degenerate when such a scenario does not happen.*

Example 2.1. *Consider*

$$f_0(x) = \begin{cases} 4x(1-x), & 0 \le x \le 1 \\ x-1, & 1 < x \le 2 \end{cases} \qquad \text{and} \qquad f_1(x) = \begin{cases} f_0(x), & 0 \le x \le 1 \\ \frac{1}{2} f_0(x), & 1 < x \le 2, \end{cases}$$

then $x_{n+1} = f_{n \bmod 2}(x_n)$ *is* 2*-periodic on the interval* $[0,2]$*, but in fact, it reduces to a* 1*-periodic equation on* $[0,1]$.

To better understand the forcing between the periods of periodic solutions of Eq. (1.1), we give the following result:

Theorem 2.2. *Let* $p > 1$ *be a positive integer. For each* $r \in \mathbb{Z}^+ \setminus \{mp : m \in \mathbb{Z}^+\}$*, there exists a* p*-periodic difference equation in the form (1.1) with* $\Gamma_p = \{r\}$.

Proof. Given $r \in \mathbb{Z}^+ \setminus \{mp : m \in \mathbb{Z}^+\}$ and let $a_j := 1 - \frac{1}{j}$, $j = 1, \ldots, r$. Define the map $f : I \to I$ by

$$f(x) = \begin{cases} a_{j+1} + \frac{a_{j+2}-a_{j+1}}{a_{j+1}-a_j}(x - a_j), & a_j \le x \le a_{j+1}, j = 1, \ldots, r-2 \\ \frac{a_r}{a_{r-1}-a_r}(x - a_{r-1}) + a_r, & a_{r-1} \le x \le a_r \\ 0, & a_r \le x \le 1. \end{cases}$$

For each $j = 0, \ldots, p-1$, define the map $f_j : I \to I$ by

$$f_j(x) = f(x) + \frac{j}{p}(1 - f(x)) \left| \sin\left(\frac{\pi}{1 - f(x)}\right) \right|.$$

Now, it is straightforward to observe (i) $C_r := \{a_1, a_2, \ldots, a_r\}$ is an r-cycle of each map $f_j(x)$; (ii) $f_j \in \mathcal{C}(I, I)$; (iii) $f_j(a_i) = f(a_i)$ for all $j = 0, \ldots, p-1$ and $i = 1, \ldots, r$. Next, define $d := \gcd(r, p)$ and recall how phase shifts of cycles of Eq. (1.1) are defined (cf. [2]), then orbits of Eq. (1.1) that start with $a_j, j = 1, 2, \ldots, d$ give us d r-cycles. Finally, take $j \ne i$ then $f_j(x) = f_i(x)$ only when $x \in \{a_1, a_2, \ldots, a_r\}$. Therefore, $\Gamma_p = \{r\}$. $\square$

Theorem 2.2 has the significance of showing that periodic solutions with periods in Γ_p are generic characteristics of the intersections between the maps and not a result of the iterations. Thus, we have no forcing relation within the elements of Γ_p. When assuming $\Gamma_p = \phi$, this observation together with Theorem 2.1 shows that a forcing relation within the elements of M_p is as follows: The existence of kp-cycle implies the existence of a rp-cycle for all $k \prec r$ in the 1-Sharkovsky's ordering. An elaborative example here is the p-periodic logistic equation $x_{n+1} = \mu_n x_n(1 - x_n)$ [3]. Now, it remains to understand the forcing relation between the elements of Γ_p and the elements of M_p. For achieving this objective, we appeal to the notion of a digraph of a cycle used by Straffin [21] and developed by several others [16, 7, 1]. Let $C_r := \{a, f(a), f^2(a), \ldots, f^{r-1}(a)\}$ be a r-cycle of the autonomous equation $x_{n+1} = f(x_n)$, $f \in \mathcal{C}(I, I)$. Rearrange the elements of C_r in an increasing order, say

$\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{r-1}$, and define the closed intervals

$$I_j = [\hat{a}_j, \hat{a}_{j+1}],\ j = 0, \ldots, r-2. \tag{2.2}$$

Now, define a directed graph whose vertices are $\{I_j,\ j = 0, \ldots, r-2\}$ such that $I_i \to I_j$ (i.e., there is an edge from I_i to I_j) if $I_j \subseteq f(I_i)$. Now, let $L(x)$ be the piecewise linear function that connects the points $(\hat{a}_j, f(\hat{a}_j)),\ j = 0, 1, \ldots, r-1$. What is the relationship between the digraph of $L(x)$ and the digraph of $f(x)$? Obviously, the digraph of $L(x)$ is a sub-digraph of the one that belongs to $f(x)$ (cf. Page 852 in [8]). However, since this is a fundamental fact in our latter results and the proof is not explicitly written in [8], we write it in the following proposition:

Proposition 2.1. *Let $f \in \mathcal{C}(I, I)$. Suppose $C_r := \{a, f(a), f^2(a), \ldots, f^{r-1}(a)\}$ is a r-cycle of f with associated digraph G. The digraph of this r-cycle under the piecewise linear map $L(x)$ that connects the points $(a_j, f(a_j)),\ j = 0, 1, \ldots, r-1$ is a sub-digraph of G.*

Proof. Order the elements of the r-cycle from smallest to largest as $\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{r-1}$ and define I_j as in Eq. (2.2). Then

$$L(I_j) = [\min\{f(\hat{a}_j), f(\hat{a}_{j+1})\}, \max\{f(\hat{a}_j), f(\hat{a}_{j+1})\}]$$

and $L(I_j) \subseteq f(I_j)$. Therefore, if $I_k \subseteq L(I_j)$ for some k and j (i.e., $I_j \to I_k$ in the digraph of L), then $I_k \subseteq L(I_j) \subseteq f(I_j)$, and consequently $I_j \to I_k$ in G. □

Before we proceed, we give the definition of what we call Straffin's loop.

Definition 2.2. *Set $\hat{I}_0 = I_0$, where I_0 as defined in Eq. (2.2), and define recursively $\hat{I}_n$ to be the interval which has $f^n(\hat{a}_0)$ as one endpoint and is contained in $f(\hat{I}_{n-1}), n \in \mathbb{Z}^+$. The obtained sequence of intervals defines a loop in the associated digraph, which we call a Straffin's loop.*

We are interested in the relationship between the cycles of f and the cycles of the digraph. Cycles in the digraph are named in terms of the vertices. To avoid confusion between cycles of difference equations and cycles of digraphs, we use loops to call cycles of a digraph, and we always use minimal periods. Thus, an r-loop means a non-repetitive cycle of period r in the digraph. It is obvious that a r-loop gives rise to a r-cycle, but the converse is not obvious. In fact, the converse is not necessarily true. We extract the following result from [21]:

Lemma 2.1. *Suppose that $x_{n+1} = f(x_n)$ has a k-cycle for some positive integer $k > 1$. Each of the following holds true for the associated digraph of this k-cycle.*

(i) At least one of the vertices has a 1-loop.
(ii) If $k \geq 3$ is odd, then the diagraph contains a k-loop.
(iii) The existence of a m-loop in the diagraph implies the existence of a m-cycle for $f(x)$.

Before we give the next result for periodic difference equations, we need to define cycle sharing between maps and decomposable loops in diagraphs.

Definition 2.3. *Two maps $f, g \in \mathcal{C}(I, I)$ are said to share the same r-cycle $C_r = \{c_0, c_1, \dots, c_{r-1}\}$ if $c_{j+1 \bmod r} = f^j(c_0) = g^j(c_0)$ for all $j \in \mathbb{N}$.*

Definition 2.4. *Let G be a digraph associated with a r-cycle of a map F. We call a m-loop of G decomposable into two sub-loops, say m_1-loop and m_2-loop if $m_1 + m_2 = m$, the vertices of the two sub-loops are exactly the vertices of the m-loop, and the union of the intervals forming the m_1-loop does not equal the union of the intervals forming the m_2-loop.*

Lemma 2.2. *Suppose that $F_0, \dots, F_{q-1} \in \mathcal{C}(I, I)$ share the same r-cycle,*

$$C_r := c_0 \ \rightarrow \ c_1 \ \rightarrow \ c_2 \ \rightarrow \ c_3 \ \rightarrow \cdots \rightarrow \ c_{r-1}$$

such that $r > 2$. If the digraphs of C_r under the effect of $F_j,\ j = 0, 1, \dots, q-1$ have a common sub-digraph that has a r-loop, which is decomposable into two sub-loops, then the periodic equation $x_{n+1} = F_{n \bmod q}(x_n)$ (q is not necessarily the minimal period) has qrm-cycles for all positive integers m.

Proof. Rearrange the elements of the r-cycle in an increasing order, say $\hat{c}_0, \hat{c}_1, \dots, \hat{c}_{r-1}$, and as in the paragraph preceding Lemma 2.1, define the closed intervals $I_j = [\hat{c}_j, \hat{c}_{j+1}],\ j = 0, \dots, r-2$. For each map F_k, define a directed graph whose vertices are $\{I_j,\ j = 0, \dots, r-2\}$. Since the C_r cycle is shared by all maps, then the order of the elements of C_r is preserved by all maps. Now, since the r-loop is common in all digraphs and decomposable into two sub-loops, say r_1-loop and r_2-loop, then there must be at least one common vertex between the two sub-loops (a bridge vertex), say I_{i^*}. Start from I_{i^*}, and trace the r_1-loop qm times using the maps $F_0, F_1, F_2, \dots, F_{(qr_1-1) \bmod q}$, i.e., guard the edge that connects I_{i^*} to I_{i^*+1} in the r_1-loop by the map F_0, then guard the edge that connects I_{i^*+1} to I_{i^*+2} by the map F_1, ... etc. Observe that $F_{(qr_1-1) \bmod q}$ will get you back to the bridge vertex I_{i^*}, and we can cross to the r_2-loop using the map $F_{qmr_1} = F_0$, then as in the r_1-loop, trace the r_2-loop qm times. This procedure, defines a chain of length $qmr_1 + qmr_2 = qrm$.

$$I_{i^*} \xrightarrow{F_0} I_{i^*+1} \xrightarrow{F_1} I_{i^*+2} \xrightarrow{F_2} \cdots \xrightarrow{F_{qmr_1-1}} I_{i^*} \xrightarrow{F_0} \cdots$$

where the indexes of F in this chain are (mod q). From the structure of the chain, and our definition of a decomposable loop, the chain gives a cycle of length qrm for the periodic equation $x_{n+1} = F_{n \bmod q}(x_n)$. Now, it remains to clarify that it is a qrm-cycle, i.e., qrm is the minimal period. We have two segments in the chain that we formed, one is coming from the r_1-loop and the other is coming from the r_2-loop; however, Definition 2.4 assures us that we have a vertex in one of the segments that does not show up on the second one. Thus, we have a qrm-loop, and consequently a qrm-cycle regardless whether or not q is the minimal period for the periodic equation. $\square$

A reader might think that the condition on all maps to agree on the same cycle as a stringent condition, but in fact, it is a tangible one as can be seen in Theorems 2.3, 2.4 and 2.5. We emphasize the significance of the other conditions of Lemma 2.2 in the next remark.

Remark 2.1. *Although $r > 1$ is necessary for the digraph approach to be well defined, we show its necessity in general. Let $r = 1$ and define $F_j(x) = x$, $j = 0, \ldots, q-2$, $F_{q-1}(x) = 1 - x$. The functions $F_j(x)$ share the 1-cycle $\{\frac{1}{2}\}$, but the q-periodic equation $x_{n+1} = F_n(x_n)$ has no q-cycles. In fact, all non-equilibrium orbits are of period $2q$. About the significance of a r-loop in the digraph: Let $r = 2$, and consider $F_0(x) = 1 - x, F_1(x) = 1 - x$ and $F_2(x) = (x-1)^2$. Then, $F_j \in \mathcal{C}(I, I)$ and $\{0, 1\}$ is a 2-cycle for each map F_j. It is a simple matter to see that the unique 2-cycles of the 3-periodic equation $x_{n+1} = F_{n \bmod 3}(x_n)$ are $\{0, 1\}$ and $\{1, 0\}$, and its unique 3-cycle in $[0, 1]$ is given by $\{\frac{1}{2}(3 - \sqrt{5}), \frac{1}{2}(\sqrt{5} - 1), \frac{1}{2}(3 - \sqrt{5})\}$. Since*

$$F_2 \circ F_1 \circ F_0 \circ F_2 \circ F_1 \circ F_0(x) - x = x(x-1)(x^2 - 3x + 1),$$

we conclude that the nonautonomous equation has no 6-cycles. Furthermore, one can show that the 3-periodic equation has no r-cycle for all $r \in \mathcal{A}_{3,q}$, $q \neq 1, 2$.

Remark 2.2. *We note that the order of the elements of C_r is crucial in Lemma 2.2. To illustrate this, consider*

$$f_0(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} < x \le 1 \end{cases} \qquad \text{and} \qquad f_1(x) = \begin{cases} 1 - 2x & 0 \le x \le \frac{1}{2} \\ x - \frac{1}{2} & \frac{1}{2} < x \le 1. \end{cases}$$

Observe that $f_0, f_1 \in \mathcal{C}(I, I)$. Furthermore, $\{0, \frac{1}{2}, 1\}$ is a 3-cycle for both maps, but the combinatorial structure is different, i.e.,

$$0 \xrightarrow{f_0} \frac{1}{2} \xrightarrow{f_0} 1 \xrightarrow{f_0} 0 \qquad \text{and} \qquad 0 \xrightarrow{f_1} 1 \xrightarrow{f_1} \frac{1}{2} \xrightarrow{f_1} 0.$$

In fact, $x_{n+1} = f_{n \pmod 2}(x_n)$ has no 3-cycle nor a 6-cycle. To prove this simple fact, it suffices to observe that $\mathcal{A}_{p,2} = \{4\} \prec \mathcal{A}_{p,1} = \{1, 2\}$, and that the 2-periodic equation has no 4-cycles because $f_1(f_0(x)) \le x$ for all $x \in I$ and consequently $Per(f_1 \circ f_0, 1) = \{1\}$. However, there are many 2-cycles. Indeed, $\{1, 0\}$ and $\{x, x + \frac{1}{2}\}$, $x \in [0, \frac{1}{2}]$ are all the possible 2-cycles.

Corollary 2.1. *Suppose that the non-degenerate p-periodic equation in (1.1) has a r-cycle for some odd number $r > 1$. If $\gcd(r, p) = 1$, then Eq. (1.1) has prm-cycles for all positive integers m.*

Proof. Since r and p are relatively prime, then all maps $f_0, \ldots, f_{p-1}$ share the same r-cycle. Also, since r is odd, Lemma 2.1 assures the existence of a r-loop, and we can take it from the common sub-digraph assured by Proposition 2.1. Furthermore, we can take it to be the Straffin's loop (See Definition 2.2). This Straffin's loop is decomposable into two sub-loops. Now, Lemma 2.2 completes the proof. □

The existence of a k-loop in the diagraph of a k-cycle plays a major rule in our proof of Lemma 2.2. As Lemma 2.1 shows, this k-loop is available at our disposal when $k > 1$ is odd; however, when k is a power of 2, further assumptions are needed. For instance, define $f(x)$ to be the piece-wise linear map that connects the points $(0, 2), (1, 3), (2, 1), (3, 0)$, then we have a 4-cycle. Define $I_0 = [0, 1], I_1 = [1, 2]$ and $I_2 = [2, 3]$. As Figure 1 shows, the associated digraph has no 4-loop.

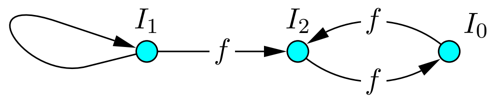

Figure 1: The digraph associated with the 4-cycle $0 \to 2 \to 1 \to 3$ of the piece-wise linear map f.

Nevertheless; we use Lemma 2.2 to obtain a result when the shared cycle is of length $r = 2^m(2k + 1)$. But first, we clarify the notion of gluing certain maps together to reduce the periodicity of Eq. (1.1). For instance, if Eq. (1.1) is 6-periodic, then we can glue each two consecutive maps as $F_0 := f_1 f_0, F_1 := f_3 f_2, F_2 := f_5 f_4$ to obtain a periodic equation of period 3 (not necessarily minimal). In general, we have the following:

Lemma 2.3. *If we have a non-degenerate p-periodic difference equation of the form given in Eq. (1.1), and we define*

$$F_j := f_{((j+1)k-1) \bmod p} \circ f_{((j+1)k-2) \bmod p} \circ \cdots \circ f_{jk \bmod p}, \; j = 0, 1, \ldots$$

for some positive integer k, then the equation $x_{n+1} = F_n(x_n)$ is q-periodic for some q that divides $\frac{p}{\gcd(p,k)}$.

Proof. The periodicity of the formed equation here is affected by the rotation of the indices and by the action of the individual maps. First, we handle the indices. From the fact that $\{0, k, 2k, \cdots\}$ is a subgroup of $(\mathbb{Z}_p, +)$ (the additive group of congruence classes modulo p) of order $\frac{p}{\gcd(p,k)}$, we obtain a periodic equation of (not necessarily minimal) period $\frac{p}{\gcd(p,k)}$. Now, we come to the effect of the individual maps. This may create a period shorter than $\frac{p}{\gcd(p,k)}$. However, the minimal period must be a divisor of $\frac{p}{\gcd(p,k)}$. $\square$

The following example illustrates Lemma 2.3:

Example 2.2. *(i) Define $f_j(x) = x^{j+1}, x \in [0, 1], j = 0, 1, \ldots, 3$ and consider k in Lemma 2.3 to be 2, then the obtained equation $x_{n+1} = F_n(x_n)$ is 2-periodic.*

(ii) Define the maps $f_0, f_2 \in \mathcal{C}([0, 1]$ to be monotonic such that $f_0 \neq f_2$, then define $f_1 = f_0^{-1}$ and $f_3 = f_2^{-1}$. Again, consider k in Lemma 2.3 to be 2, then the obtained equation $x_{n+1} = F_n(x_n)$ is 1-periodic.

The next theorem is tailored to address the case when the period of the shared cycle is a power of 2 times an odd number.

Theorem 2.3. *Suppose that $f_0, \ldots, f_{q-1} \in \mathcal{C}(I, I)$ share the same r-cycle,*

$$C_r := c_0 \; \rightarrow \; c_1 \; \rightarrow \; c_2 \; \rightarrow \; c_3 \; \rightarrow \cdots \rightarrow \; c_{r-1}$$

for some $r = 2^m(2k+1), m \geq 0, k \geq 1$. Let $d^* = \gcd(2^m, q)$ and $F_j = f_{(j+1)2^m-1} \circ \cdots \circ f_{j2^m}$, $j = 0, 1, \ldots$. The periodic equation $x_{n+1} = F_{n \bmod q}(x_n)$ (q is not necessarily minimal) has a $\frac{q}{d^*}(2k+1)m$-cycles for all $m \in \mathbb{Z}^+$.

Proof. If $m = 0$, then r is odd, $d^* = 1$ and $F_j = f_j$ for all $j = 0, 1, \ldots, q-1$. Thus, all maps share the $(2k+1)$-cycle. As in the proof of Corollary 2.1, we obtain the required result. Now, assume that $m \geq 1$. If $q = 1$, then we have the autonomous case and the result is obvious. Next, assume $q > 1$ and consider the maps F_j as defined in the statement of the theorem. Since $F_j^t(c_0) = c_{2^m t}$ for all $j = 0, 1, \ldots, q-1$, then the maps $F_0, F_1, \ldots, F_{q-1}$ share a cycle of minimal period $2k+1$. By Lemma 2.1, we have a decomposable $(2k+1)$-loop. Now, use Lemma 2.2 to obtain $\frac{q}{d^*}(2k+1)m$-cycles for the periodic equation $x_{n+1} = F_n(x_n)$. □

It is possible to achieve a r-loop in a r-cycle in certain classes of cycles regardless whether or not r is odd. A simple example here is the class of monotonic cycles. A r-cycle $C_r := \{a, f(a), f^2(a), \ldots, f^{r-1}(a)\}$ of $x_{n+1} = f(x_n)$ is called monotonic if

$$a < f(a) < \cdots < f^{r-1}(a) \qquad \text{or} \qquad a > f(a) > \cdots > f^{r-1}(a).$$

It is straight forward to draw the associated diagraph, and observe that a decomposable r-loop always exists. Furthermore, the digraphs of both cases are isomorphic. In Figure 2, we give the digraph of a monotonic 4-cycle.

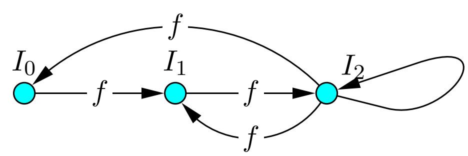

Figure 2: The digraph of a monotonic 4-cycle. Observe that $I_2 \to I_2 \to I_2 \to I_1 \to I_2$ is a decomposable loop, and $I_0 \to I_1 \to I_2 \to I_2 \to I_0$ is the decomposable Straffin's loop.

We generalize this notion to p-periodic difference equations as follows: Let $C_r := \{c_0, c_1, \ldots, c_{r-1}\}$ be a r-cycle of Eq. (1.1). Define $d := \gcd(r, p)$. We call C_r monotonic if $\{c_0, c_d, c_{2d}, \ldots, c_{r-d}\}$ is a monotonic $\frac{r}{d}$-cycle with respect to the map $F_0 = f_{d-1} \cdots f_1 f_0$. Now, we give the following result:

Theorem 2.4. *Suppose that the non-degenerate p-periodic equation in (1.1) has a monotonic r-cycle. Let $d = \gcd(r, p)$ and $F_j = f_{(j+1)d-1} \circ \ldots \circ f_{jd}$, $j = 0, 1, \ldots, \frac{p}{d} - 1$. If $\frac{r}{d} > 2$, then the periodic equation $x_{n+1} = F_n(x_n)$ has a m-cycle for all $m \in \{\frac{p}{d}\left(\frac{r}{d} + j\right) : \; j \in \mathbb{N}\}$. Furthermore, the intersections of the maps f_j determine the exact length of those cycles with respect to Eq. (1.1).*

Proof. Since we have a monotonic r-cycle of Eq. (1.1), say $C_r = \{c_0, c_1, \ldots, c_{r-1}\}$, then $F_0 = f_{d-1} \cdots f_1 f_0$ has a monotonic $\frac{r}{d}$-cycle. Furthermore, the maps $F_0, F_1, \ldots, F_{\frac{p}{d}-1}$ share the same monotonic $\frac{r}{d}$-cycle; namely $\{c_0, c_d, c_{2d}, \ldots, c_{r-d}\}$. For the monotonic $\frac{r}{d}$-cycle, we have a common $\frac{r}{d}$-loop which is decomposable into 1-loop and $(\frac{r}{d} - 1)$-loop. By a construction analogous to that given in the proof of Lemma 2.2, we obtain $\frac{p}{d}(M_1 + (\frac{r}{d} - 1)M_2)$-cycles for the periodic equation $x_{n+1} = F_{n \bmod \frac{p}{d}}(x_n)$ ($\frac{p}{d}$ is not necessarily the minimal period), where $M_1, M_2 \in \mathbb{Z}^+$ representing the number of times you loop in the 1-loop and $(\frac{r}{d}-1)$-loop respectively. This implies $\frac{p}{d}(\frac{r}{d} + j)$-cycles for the periodic equation $x_{n+1} = F_n(x_n)$. Finally, after the unfolding of the maps F_j, we obtain cycles for Eq. (1.1), and if the minimal period of any one of those cycles is not in the form $p(\frac{r}{d} + j)$ then the intersections between the maps f_j determine the exact length. In particular, if the formed cycle is $C_{r^*} = \{c_0^*, c_1^*, \ldots, c_{r^*-1}^*\}$ and r^* is not a multiple of p, then $f_0, f_{\hat{d}}, f_{3\hat{d}}, \cdots f_{p-\hat{d}}$ must intersect at the points $c_0^*, c_{\hat{d}}^*, c_{2\hat{d}}^*, \ldots c_{r^*-\hat{d}}^*$, where $\hat{d} := \gcd(r^*, p)$. □

The next result gives a refinement of Theorem 2.1 in certain cases. Notice that $qp \in \mathcal{A}_{p,q}$, so besides to know that the cluster $\mathcal{A}_{p,\ell}$ forces the existence of a period in the cluster $\mathcal{A}_{p,q}$ if $\ell \preceq q$, we are able to determine a such period inside this latter cluster under certain conditions as we illustrate in Example 2.3.

Theorem 2.5. *Consider the non-degenerate p-periodic equation in (1.1) with $f_i \in \mathcal{C}(I, I)$, $i = 0, 1, ..., p-1$, and suppose there exists a r-cycle for some r belongs to the cluster $\mathcal{A}_{p,\ell}$, where $\ell \geq 3$ is an odd number. Let $d := \gcd(r, p)$. Each of the following holds true:*

(i) If $d = 1$ or p, then Eq. (1.1) has a ℓp-cycle.

(ii) If $1 < d < p$ and $d^2 < p$, then Eq. (1.1) has a r^*-cycle for some r^* divides $p\ell$ and $r^* \neq r$ with $\gcd(r^*, p) > 1$.

Proof. Let $C_r = \{c_0, c_1, \ldots, c_{r-1}\}$ be a r-cycle of Eq. (1.1) and suppose that $r \in \mathcal{A}_{p,\ell}$ for some odd number $\ell \geq 3$. Notice that $r = d\ell$ as a consequence of the property $\mathrm{lcm}(a, b) \cdot \gcd(a, b) = ab,\ a, b \in \mathbb{Z}^+$. (i) The case $d = p$ is obvious since r must be ℓp. So, we proceed with $d = 1$. In this case, $r = \ell$ and the maps $f_j,\ j = 0, 1, \ldots, p-1$ share the cycle C_r. Since r is odd, then we have a common r-loop, and consequently Eq. (1.1) has a ℓp-cycle. Next, we proceed to prove (ii). Define the maps $F_j = f_{(j+1)d-1} \circ \ldots \circ f_{dj},\ j = 0, 1, \ldots, \frac{p}{d} - 1$. The maps $F_j,\ j = 0, 1, \ldots, \frac{p}{d} - 1$ share a ℓ-cycle. Since ℓ is odd, then we have a common decomposable ℓ-loop and Lemma 2.2 implies the existence of a $\frac{p}{d}\ell$-cycle for the periodic equation $x_{n+1} = F_{n \bmod \frac{p}{d}}(x_n)$, say $B_{\frac{\ell p}{d}} := \{b_0, F_0(b_0), F_1 F_0(b_0), \ldots\}$. Now, unfold the maps F_j into their initial components (i.e. $f_j,\ j = 0, 1, \ldots, p-1$) by expanding each element into d-elements where the order is preserved as follows:

$$\begin{array}{rcl} b_0 & \Rightarrow & b_0, f_0(b_0), f_1 f_0(b_0), \ldots, f_{d-2} \ldots f_0(b_0) \\ F_0(b_0) & \Rightarrow & F_0(b_0), f_d F_0(b_0), f_{d+1} f_d F_0(b_0), \ldots, f_{2d-2} \ldots f_d(F_0(b_0)) \\ \vdots & \Rightarrow & \vdots \end{array}$$

This process gives a cycle of length ℓp (unknown whether it is minimal or not) for Eq. (1.1). However, denote the minimal period by r^*, then it is at least $\frac{p}{d}\ell$ and it must divide $p\ell$. Since $d^2 < p$ then $r = \ell d < \ell \frac{p}{d}$, and consequently $r^* > r$. Also, $\gcd(r^*, p) > 1$; otherwise, $\gcd(r^*, p) = 1$ and the fact that r^* divides $p\ell$ implies r^* divides ℓ, which is not possible since $r^* \geq \ell \frac{p}{d}$ and $\frac{p}{d} > 1$. $\square$

Remark 2.3. *The conditions in part (ii) of Theorem 2.5 are sufficient conditions. The question of finding a necessary and sufficient condition for this particular case remains open for further investigations. However, we can conjecture, that* $r \in \mathcal{A}_{p,\ell}$ *forces the existence of a* ℓp*-cycle whenever* ℓ *takes the form* $2^k(2t+1)$, *where* $k \geq 0$ *and* $t \geq 1$.

We close this section by the following example illustrating Theorem 2.5:

Example 2.3. *Consider* $p = 6$, *and define*

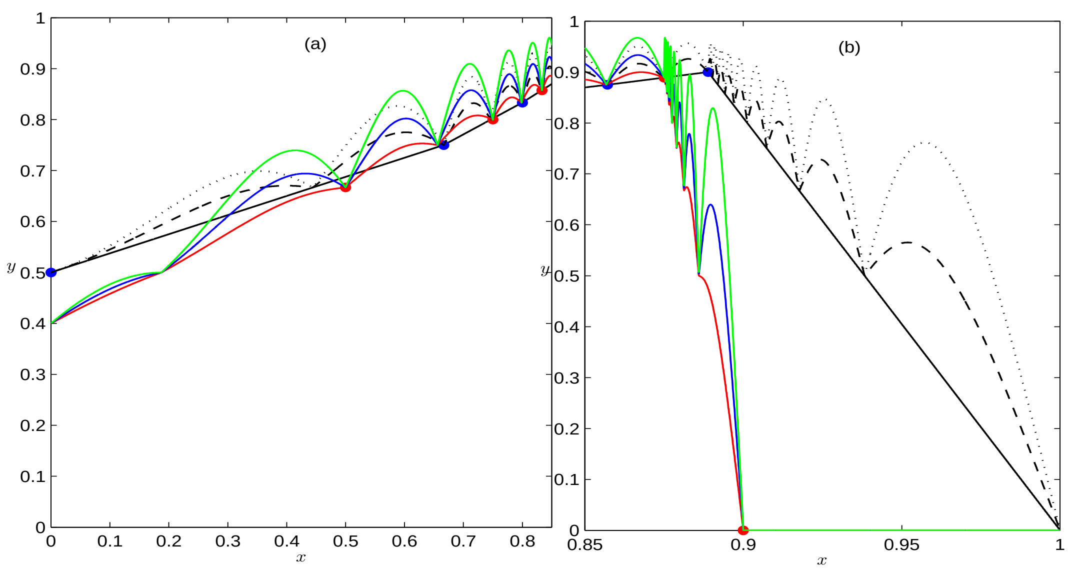

Figure 3: This figure shows the graphs of the maps in Example 2.3. f_0, f_2 and f_4 are represented by solid, dashed and dotted black respectively. f_1, f_3 and f_5 are represented by blue, red and green respectively. The graph is separated into two figures to obtain better visibility by changing the scale of the x-axis when $x \in [0.85, 1]$. The bullets represent the points (c_j, c_{j+1}) where $c_j, c_{j+1} \in C_{10}$.

$$f(x) = \begin{cases} \frac{1}{2} + \frac{3}{8}x & 0 \leq x < \frac{2}{3} \\ \frac{1}{3} + \frac{5}{8}x & \frac{2}{3} \leq x < \frac{4}{5} \\ \frac{1}{4} + \frac{35}{48}x & \frac{4}{5} \leq x < \frac{6}{7} \\ \frac{1}{5} + \frac{63}{80}x & \frac{6}{7} \leq x < \frac{8}{9} \\ \frac{81}{10} - \frac{81}{10}x & \frac{8}{9} \leq x \leq 1 \end{cases} \quad \textit{and} \quad g(x) = \begin{cases} \frac{2}{5} + \frac{8}{15}x & 0 \leq x < \frac{3}{4} \\ \frac{2}{7} + \frac{24}{35}x & \frac{3}{4} \leq x < \frac{5}{6} \\ \frac{2}{9} + \frac{16}{21}x & \frac{5}{6} \leq x < \frac{7}{8} \\ 32 - \frac{320}{9}x & \frac{7}{8} \leq x < \frac{9}{10} \\ 0 & \frac{9}{10} \leq x \leq 1. \end{cases}$$

For $j=0,2,4,$ *let*

$$f_j(x)=f(x)+\frac{j}{p}x(1-f(x))\left|\sin\left(\frac{\pi}{1-f(x)}\right)\right|$$

and for $j=1,3,5,$ *let*

$$f_j(x)=g(x)+\frac{j}{p}x(1-g(x))\left|\sin\left(\frac{\pi}{1-g(x)}\right)\right|.$$

The graphs of $f_j, j=0,1,\ldots 6$ are illustrated in Figure 3. It is straightforward to check that $C_{10}:=\{0,\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots,\frac{9}{10}\}$ is a 10-cycle of the 6-periodic equation $x_{n+1}=f_{n \bmod 6}(x_n)$. Observe that $10\in\mathcal{A}_{6,5}=\{5,10,15,30\}$. By Theorem 2.1, each cluster $\mathcal{A}_{6,\ell},\ \ell\geq 5$ contains the period of a cycle. Inside the cluster $\mathcal{A}_{6,5}$, Part (ii) of Theorem 2.5 implies the existence of a 15-cycle or a 30-cycle. Since $\gcd(6,15)=3$, then f_0 and f_3 must intersect at five elements (say c_0,c_3,c_6,c_9,c_{12}) of any 15-cycle, and if this intersection does not take place, then the forced cycle must be a 30-cycle. For clusters of the form $\mathcal{A}_{6,2^k}$; if there exists a forced cycle of length $r\in\Gamma_p\cap\mathcal{A}_{6,2^k}$, then $\gcd(r,6)=2$ and the maps f_0,f_2,f_4 must agree on 2^k points of this r-cycle. However, by solving the equations $f_0(x)=f_2(x)=f_4(x)$, one can find that $x\in f^{-1}(C_{10})$ (the pre-image of the elements of C_{10}). Furthermore, $x_0=0,\frac{2}{3},\frac{4}{5},\frac{6}{7},\frac{8}{9}$ gives us C_{10}, and f_1,f_3,f_5 do not intersect at $0,\frac{2}{3},\frac{4}{5},\frac{6}{7},\frac{8}{9}$. Therefore, the forced cycle in $\mathcal{A}_{6,2^k}$ must be of length $6\cdot 2^k$. Also, clusters of the form $\mathcal{A}_{6,2^k(2t+1)}, t\geq 1$ are handled similarly, i.e., cycles of length $2^{k+1}(2t+1)$ are excluded using the intersection argument, and the forced cycles must be of length $6\cdot 2^k(2t+1)$.

3 Infinite Γ_p sets

Unlike autonomous equations, in the above section we showed that the existence of a period $r\in\mathcal{A}_{p,3}$ does not necessarily imply the existence of all periods. However, it became obvious that most periods are located in the M_p set, which motivates us to investigate how large the set Γ_p can be. If the maps in Eq. (1.1) are rational maps, then the set Γ_p is finite [2]. In fact, one can strengthen this result and prove that if the numerators and denominators of each $f_j,\ 0\leq j\leq p-1$, are of degrees at most M and m, respectively, then either $card(\Gamma_p)\leq M+m+1$ or there exists $r_1,r_2\in\Gamma_p$ such that $\mathrm{lcm}(\gcd(r_1,p),\gcd(r_2,p))=p$. We proceed in this section to construct an example of Eq. (1.1) where the Γ_p set is infinite for any p, while the functions f_j intersect only on a countable set. This answers the question left open in [2].

If we consider the logistic function $f(x)=\mu x(1-x),\ x\in[0,1], 1+2\sqrt{2}<\mu<4$, then $Per(f,1)$ is infinite, countable and dense in I. Therefore, if a continuous function agrees with $f(x)$ on $Per(f_n,1)$, it must agree with f on I. From this simple fact, we need the periodic points with periods in Γ_p to be non-dense in I. The next lemma makes our task possible.

Lemma 3.1. *Let $\{a_n\}_{n=1}^{\infty} \subset I$ be a monotonic sequence, then there exists a continuous function $f \in \mathcal{C}(I, I)$ such that $Per(f, 1) \supset \{a_n : n \in \mathbb{Z}^+\}$.*

Proof. Without loosing the general case, we assume that $\{a_n\}$ is increasing. For each $q \in \mathbb{Z}^+$, define $\gamma(q) = \frac{1}{2}q(q-1)+1$. For $q \geq 2$, define $f_q : [a_{\gamma(q)}, a_{\gamma(q+1)-1}] \to I$ as the polygonal line joining the points $(a_{\gamma(q)}, a_{\gamma(q)+1}), \dots, (a_{\gamma(q+1)-1}, a_{\gamma(q)})$. Here, notice that $\gamma(q)+q = \gamma(q+1)$ and $\{a_{\gamma(q)}, a_{\gamma(q)+1} \dots, a_{\gamma(q+1)-1}\}$ is a q-cycle for the map f_q. Also, for $q \geq 1$, define $\ell_q : [a_{\gamma(q+1)-1}, a_{\gamma(q+1)}] \to I$ as the line segment joining the points $(a_{\gamma(q+1)-1}, a_{\gamma(q)})$ and $(a_{\gamma(q+1)}, a_{\gamma(q+1)+1})$. Since $\{a_n\} \subset I$ is increasing, we have $\lim a_n := a \leq 1$ and

$$[0, a) = [0, a_1] \cup \bigcup_{j=1}^{\infty} [a_j, a_{j+1}].$$

Define the function $\phi : [0, a) \to I$ by

$$\phi(x) = \begin{cases} x, & 0 \leq x \leq a_1 \\ f_q(x), & a_{\gamma(q)} \leq x \leq a_{\gamma(q+1)-1},\ q \geq 2 \\ \ell_q(x), & a_{\gamma(q+1)-1} \leq x \leq a_{\gamma(q+1)},\ q \geq 1. \end{cases}$$

We claim that ϕ is uniformly continuous. To show this, let $\epsilon > 0$ be given. Then there exists q^* such that

$$0 \leq a - a_{\gamma(q^*)} < \frac{\epsilon}{2}. \tag{3.1}$$

By the uniform continuity of ϕ on $[0, a_{\gamma(q^*)}]$, there exists $\delta > 0$ such that

$$|\phi(x) - \phi(y)| \leq \frac{\epsilon}{2} \text{ whenever } x, y \in [0, a_{\gamma(q^*)}] \text{ and } |x - y| < \delta. \tag{3.2}$$

Next, from the construction of $\phi(x)$, it is easy to observe that

$$|\phi(x) - \phi(y)| \leq a - a_{\gamma(q^*)} < \frac{\epsilon}{2} \text{ whenever } x, y \in [a_{\gamma(q^*)}, a). \tag{3.3}$$

Now, given $x, y \in [0, a)$ with $|x - y| < \delta$. If $x, y \in [0, a_{\gamma(q^*)}]$ or $x, y \in [a_{\gamma(q^*)}, a)$, then it is obvious from (3.2) and (3.3) that $|\phi(x) - \phi(y)| < \epsilon$. Also, if $x \in [0, a_{\gamma(q^*)}]$ and $y \in [a_{\gamma(q^*)}, a)$, then (3.2) and (3.3) imply

$$|\phi(x) - \phi(y)| \leq |\phi(x) - \phi(a_{\gamma(q^*)})| + |\phi(a_{\gamma(q^*)}) - \phi(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Finally, let $\Phi : [0, a] \to I$ be the continuous extension of ϕ at a. Hence, the required function $f : I \to I$ is defined by

$$f(x) = \begin{cases} \Phi(x), & 0 \leq x \leq a \\ \Phi(a), & a < x \leq 1. \end{cases}$$

□

Now, we give the main result of this section.

Theorem 3.1. *For each $p = 2, 3, \ldots,$ there exists a p-periodic difference equation in the form (1.1) such that the maps f_j agree on a countable set, and the set Γ_p has infinite cardinality.*

Proof. We make use of the construction in Lemma 3.1. Let $a_n = 1 - \frac{1}{n}$, and let $f(x)$ be the function assured by Lemma 3.1. For each $j = 0, \ldots, p-1$, define the maps $f_j : I \to I$ as

$$f_j(x) = f(x) + \frac{j}{p}(1 - f(x)) \left| \sin\left(\frac{\pi}{1 - f(x)}\right) \right|.$$

Then it is easy to observe that for each $j = 0, \ldots, p-1, f_j : \; I \to I, f_j(a_n) = f(a_n), \forall n \in \mathbb{Z}^+$, the maps f_j agree only on a countable set, and $Per(f_n, p) \supset \{a_n : n \in \mathbb{Z}^+\}$. In particular, $\Gamma_p = \mathbb{Z}^+ \setminus \{mp : \; m \in \mathbb{Z}^+\}$. □

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