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The dynamics of some discrete models with delay under the effect of constant yield harvesting*

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Abstract

In this paper, we study the dynamics of population models of the form $x_{n+1} = x_n f(x_{n-1})$ under the effect of constant yield harvesting. Results concerning stability, boundedness, persistence and oscillations of solutions are given. Also, some regions of persistence and extinction are characterized. Pielous equation was considered as an example on these models, and a connection with a Lyness type equation has been established at certain harvesting level, which is used to give an explicit description of a persistent set.

Key words: Population models; harvesting; persistence; extinction; Pielou's equation.
Mathematics Subject Classification (2000): 92D5

1 Introduction

Difference equations of the form $x_{n+1} = x_n f(x_n), \quad n = 0, 1, 2, ...$ are used to model single species with non-overlapping generations [19], where x_n denotes the population density at discrete time n and $f(x_n)$ represents the per-capita growth rate of the population. The appropriate form of $f(x)$ is usually chosen to reflect known observation about the studied species. For instance, in fish populations with high fertility rates and low survivorship to adulthood, Beverton and Holt [5] considered $f(x) = \frac{\mu K}{K+(\mu-1)x}$, where $\mu > 1$ is interpreted as the growth rate per generation and K is the carrying capacity of the environment. Straightforward re-scaling gives the simpler one parameter equation $x_{n+1} = \frac{bx_n}{1+x_n}$. Interestingly, the classical Beverton-Holt (BH) model can be derived using several arguments and approaches [39, 8, 13, 10]. The BH model has been generalized to several more sophisticated models such as the Hassell model and the Maynard Smith-Slatkin model in order to make the model better accommodate certain observations on real populations. Another prototype model of single species with non-overlapping

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generations is the Ricker model [29, 27], which is obtained when $f(x) = exp(r(1-\frac{x}{k})),\ r, k > 0$.

In populations with substantial maturation time to sexual maturity, certain delay effect must be included in the function $f(x)$, which motivates considering difference equations with delay [22] of the form

$$x_{n+1} = x_n f(x_{n-m}), \quad n = 0, 1, 2, ..., m \geq 1. \tag{1.1}$$

For instance, Pielou [28] (page 80) suggested taking $f(x_{n-m}) = \frac{\mu K}{K+(\mu-1)x_{n-m}}$ in order to account for oscillations in certain populations. When the delay is one and after re-scaling, Pielou's equation takes the form

$$y_{n+1} = \frac{by_n}{1 + y_{n-1}}. \tag{1.2}$$

Eq. (1.2) was studied in [17, 20] where it was shown that when $b > 1$, every positive solution converges to the positive equilibrium $\bar{y} = b - 1$. The main objective of this paper is to study the effect of harvesting on the dynamics of Eq. (1.1) with delay one, i.e., we consider the difference equation

$$x_{n+1} = x_n f(x_{n-1}) - h, \tag{1.3}$$

where h is a fixed harvesting quota and $f(x)$ obeys certain assumptions. In particular, $f(x)$ is $C^2([0, \infty))$ and satisfies the following conditions:

(C1) $f(0) = b > 1$.
(C2) $f(x)$ is strictly decreasing on $[0, \infty)$ and $-\frac{1}{x} \leq f'(x) \leq 0$ for all $x > 0$.
(C3) $xf(x)$is an increasing concave down function and $xf(x) \leq M$ for some positive constant M.

Observe that without condition (C1), there is no long term survival for any population regardless of the h value. Also, condition (C3) implies that $\lim_{x\to\infty} f(x) = 0$.

The effect of harvesting on the dynamics of populations governed by differential equations has been studied extensively by several authors [30, 31, 32, 33, 34, 35, 36, 37]. In contrast, the effect of harvesting on populations governed by discrete models has received less attention. Much of the literature regarding discrete model involves, game-theoretic models in fishery, see [15] for a nice brief literature survey starting with the work of Levhari and Mirman [23] in the early 80's. For example, in [6, 7], it was shown that harvesting in the presence of a reserve area leads to higher levels of sustainability in exploiting fish stocks. In [14, 15] dynamical models of commercial exploitations of renewable resources were studied assuming that two agents behave adaptively. In particular, they obtained results about the existence and stability of the of the positive equilibrium characterizing the sustainable use of the renewable resource. Recently, Al-Sharawi and Rhouma have studied the effect of both constant and periodic harvesting for the BH equation in constant and fluctuating capacity environment and have shown that constant rate harvesting in constant capacity environment can be the optimal strategy [1, 2]. Under certain conditions, harvesting was also shown to lead to coexistence in an exclusive competitive environment with species governed by a discrete model [3].

This paper is structured as follows: We investigated local stability, boundedness and persistence of solutions in sections two and three. In section four, we investigate the global behavior of solutions. As a particular case of Eq. (1.3), we study the effect of constant harvesting on Pielou's equation with delay one in section five, i.e., $x_{n+1} = \frac{bx_n}{1+x_{n-1}} - h$. We characterize the persistent set, and use a 8-periodic solution of the Lyness type equation to construct an invariant region for Pielou's equation when $h < 1$.

2 Equilibrium solutions and local stability

Without harvesting ($h = 0$), Eq. (1.3) has two equilibrium solutions, namely, the origin $\bar{x}_{1,0} = 0$ and another positive one, say $\bar{x}_{2,0} = c = f^{-1}(1)$. A positive h shifts $\bar{x}_{1,0}$ upward and $\bar{x}_{2,0}$ downward till they collide and disappear when $h > h_{\max}$, where

$$h_{max} := \max\{x(f(x) - 1) : \ x \in [0, c]\}. \tag{2.1}$$

The equilibrium points are determined by the intersections between the curve $y = x(f(x)-1)$ and the line $y = h$. Let $0 \leq h \leq h_{\max}$. If we assume that we have more than two equilibrium points, then we can use the Mean Value Theorem to obtain a contradiction with concavity assumption in (C3). To stress the role of h in the existence of the equilibrium points, we denote them by $\bar{x}_{1,h}$ and $\bar{x}_{2,h}$, where $\bar{x}_{1,h} \leq \bar{x}_{2,h}$. Thus, we have the following:

Proposition 2.1. *Consider $h_{\max}$ as define in Eq. (2.1), and let $0 \leq h < h_{\max}$. Eq. (1.3) has two equilibrium solutions. The two equilibrium solutions are equal when $h = h_{\max}$.*

The linearized equation associated with Eq. (1.3) at a fixed point $\bar{x}$ is given by

$$y_{n+1} - f(\bar{x})y_n - \bar{x}f'(\bar{x})y_{n-1} = 0.$$

Let $p := f(\bar{x})$ and $q := \bar{x}f'(\bar{x})$, where $\bar{x} \in \{\bar{x}_{1,h}, \bar{x}_{2,h}\}$. The linearized stability theorem [17] gives the stability of each fixed point depending on the values of p and q. Notice that because $\bar{x}f(\bar{x}) = \bar{x} + h$, then $p = 1 + h/\bar{x}$ and thus

$$1 \leq p \leq b.$$

Also, the fact that f is decreasing makes q negative which combined with assumption (C2) leads to the inequality

$$-1 \leq q \leq 0.$$

Condition (C3) implies that $p + q > 0$ for both fixed points $\bar{x}_{1,h}$ and $\bar{x}_{2,h}$. However, since the graph of $xf(x)$ crosses the line $y = x$ with a slope greater than 1 at $\bar{x}_{1,h}$, then $p + q > 1$ and thus $\bar{x}_{1,h}$ is a saddle. On the other hand the graph of $xf(x)$ crosses the line $y = x$ with a slope less than 1 at $\bar{x}_{2,h}$, giving $p + q < 1$ and thus $\bar{x}_{2,h}$ is locally asymptotically stable when $0 < h < h_{max}$. We summarize these facts in the following proposition.

Proposition 2.2. *Consider h_{max} as defined in Eq. (2.1), and let $p = f(\bar{x}), q = \bar{x}f'(\bar{x})$. Each of the following holds true for Eq. (1.3).*

(i) $\bar{x}_{1,h}$ is a saddle and $\bar{x}_{2,h}$ is locally asymptotically stable for all $0 \leq h < h_{\max}$.
(ii) If $h = h_{max}$, then $\bar{x}_{1,h_{max}} = \bar{x}_{2,h_{max}}$ is nonhyperbolic (one of the eigenvalues is equal to 1.)
(iii) $\bar{x}_{2,h}$ starts as stable at $h = 0$, then as h varies, it can go to a repeller or a nonhyperbolic before it disappears.

3 Persistence and boundedness

Persistence in population models is of paramount importance in general, and harvesting adds another factor of significance to its characterization in our equation. One can find several definitions pertaining to persistence (persistence, permanence, uniform persistence, strong persistence,...etc) in the literature (see for instance [16, 38]); however, we give ourselves the liberty to use the following definition throughout this paper.

Definition 3.1. *A solution of Eq. (1.3) is called persistent if the corresponding population survives indefinitely. We call a persistent solution strongly persistent if* $\liminf x_n > 0$. *A set* $\mathcal{D} := \{(x, y) : (x, y) \in {\mathbb{R}^+}^2\}$ *is persistent if each solution of Eq. (1.3) with* $(x_{-1}, x_0) \in \mathcal{D}$ *is persistent. It is called the persistence set if it is the largest persisting set. We use* $\mathcal{D}_h$ *for the persistence set at a harvesting level* h.

If $\{x_n\}$ is a persistent solution of Eq. (1.3), then we have

$$x_{n+2} = x_{n+1}f(x_n) - h = x_n f(x_n)f(x_{n-1}) - hf(x_n) - h \leq Mb - hf(x_n) - h < Mb - h.$$

Also,

$$x_n = \frac{(x_{n+1} + h)}{f(x_{n-1})} > \frac{1}{b}(x_{n+1} + h) > \frac{h}{b}.$$

Thus, the following result becomes clear.

Lemma 3.1. *Persistent solutions of Eq. (1.3) are bounded and strongly persistent.*

The next results shows that the harvesting quota cannot exceed h_{max}.

Theorem 3.1. *If* $h > h_{max}$, *then* $\mathcal{D}_h$ *is empty.*

Proof. Consider $h > h_{max}$ and start with an initial condition (x_{-1}, x_0) such that $x_0 \leq x_{-1}$, then

$$x_1 = x_0 f(x_{-1}) - h \leq x_0 f(x_0) - h.$$

Since $h > h_{max}$, then the function $F(t) = tf(t) - h$ has no fixed points. In particular, $x_1 = x_0 f(x_0) - h < x_0$ and $x_1 < x_0 \leq x_{-1}$. and thus by induction, we obtain a decreasing sequence. This decreasing sequence either becomes negative and thus is non persistent or it converges to a positive fixed point, which is a contradiction. On the other hand, if $x_0 > x_{-1}$, then either we obtain an increasing sequence or $x_m \leq x_{m-1}$ for some positive integer m. If the sequence $\{x_n\}$ is increasing, then it is bounded because $xf(x) \leq M$ by assumption and thus must converge to a positive fixed point which is a contradiction. If $x_m \leq x_{m-1}$, we obtain that $x_{m+1} \leq x_m \leq x_{m-1}$ and again induction leads to a scenario similar to the first case, and hence, no population persists. □

The next result shows that if a population starting above $\bar{x}_{1,h}$ goes below $\bar{x}_{1,h}$ at a certain time, then it is doomed to extinction.

Proposition 3.1. *Let* $0 < h < h_{\max}$, *and define the set*

$$\mathcal{S}_0 := \{(x, y) \in {\mathbb{R}^+}^2 : y < \bar{x}_{1,h} < x\} \cup \{(x, y) \in {\mathbb{R}^+}^2 : y < \frac{h}{f(x) - 1}, x \leq \bar{x}_{1,h}\}.$$

Then $\mathcal{S}_0 \cap \mathcal{D}_h = \phi$.

PROOF: Let $(x_{-1}, x_0) \in \mathcal{S}_0$, we show that $(x_{-1}, x_0) \notin \mathcal{D}_h$. If $x_0 < x_{-1}$, then

$$x_1 = x_0 f(x_{-1}) - h < x_0 f(x_0) - h < x_0.$$

By induction, we obtain a strictly decreasing sequence less than $\bar{x}_1$, which implies that $(x_{-1}, x_0) \notin \mathcal{D}_h$. If $x_{-1} < x_0$, then

$$x_0 < \frac{h}{f(x_{-1}) - 1} \quad \text{implies} \quad x_1 = x_0 f(x_{-1}) - h < x_0.$$

Thus, $x_1 < x_0$ and we apply the first scenario on (x_0, x_1). Hence, the proof is complete. □

With the convention that empty sum is set to 0 and empty product is set to 1, we use induction on Eq. (1.3) to obtain

$$x_n = \prod_{j=-1}^{n-2} f(x_j) x_0 - h \left[\prod_{j=0}^{n-2} f(x_j) + \prod_{j=1}^{n-2} f(x_j) + \cdots + \prod_{j=n-3}^{n-2} f(x_j) + f(x_{n-2}) + 1 \right]. \tag{3.1}$$

Now, factor the coefficient of x_0 and use the summation notation to obtain

$$x_n = \prod_{j=-1}^{n-2} f(x_j) \left[x_0 - h \sum_{k=0}^{n-1} \prod_{j=-1}^{k-1} (f(x_j))^{-1} \right]. \tag{3.2}$$

for all $n \in \mathbb{N}$. Consider $\{x_n\}$ to be a persistent solution, then it is bounded, and the partial sums of the series in the numerator form an increasing bounded sequence, then

$$\sum_{k=-1}^{\infty} \prod_{j=-1}^{k} (f(x_j))^{-1}$$

must converge. Thus, the limit of the n^{th} term is zero, and consequently

$$\frac{x_0}{h} = \sum_{k=-1}^{\infty} \prod_{j=-1}^{k} (f(x_j))^{-1}. \tag{3.3}$$

We close this section by giving a result concerning the case when the two fixed points collide to form a nonhyperbolic fixed point.

Theorem 3.2. *Consider Eq. (1.3) with $h = h_{\max}$. The nonhyperbolic fixed point $\bar{x}_{1,h} = \bar{x}_{2,h}$ attracts all persistent solutions.*

PROOF: Consider $h = h_{\max}$ and suppose that $\{x_n\}_{n=-1}^{\infty}$ is a persistent solution. Define $A_n = f(x_{n-1})(x_n + h)$, then

$$\frac{A_{n+1}}{A_n} = g(x_n), \quad \text{where} \quad g(t) = \frac{t}{t+h} f(t).$$

Observe that $g(t) \le 1$ for all $t \in [0, \infty]$. Indeed, if $t > c$, then

$$g(t) = \frac{t}{t+h} f(t) < f(t) < f(c) = 1.$$

However, if $t \in [0, c]$, then

$$g(t) = \frac{1}{t+h}\, tf(t) = \frac{1}{t+h}\left[t\left(f(t)-1\right)+t\right] \leq \frac{1}{t+h}\left[h_{\max}+t\right] = 1.$$

That said, we obtain

$$\frac{A_{n+1}}{A_n} = g(x_n) < 1 \quad \text{whenever } x_n \text{ is not an equilibrium}$$

Thus, A_n is decreasing. Since any persistent solution $\{x_n\}$ is bounded away from zero, the sequence A_n is also bounded and must converge to a positive value, say $\gamma_0 > 0$. Now,

$$A_{n+1} = A_n g(x_n)$$

implies

$$\lim g(x_n) = 1,$$

and consequently x_n is attracted to the nonhyperbolic equilibrium. □

4 A comparison principle

In this section, we develop a useful comparison principle. Consider the difference equations

$$x_{n+1} = F(x_n, x_{n-1}) \tag{4.1}$$

and

$$x_{n+2} = F(x_n, x_{n-1}), \tag{4.2}$$

where $F(x, y)$ is increasing in x and decreasing in y. The next result gives a comparison principle for equations of the form (4.1) and (4.2) in a non-autonomous settings.

Theorem 4.1. *Suppose there exist two sequences α_n and β_n such that $\alpha_n \leq x_n \leq \beta_n$ for all $n \geq -1$, and define $f_{n+1}(x) = F(x, \alpha_n)$ and $g_{n+1}(x) = F(x, \beta_n)$, then each of the following holds true:*

(i) *The solution x_n of Eq. (4.1) satisfies*

$$g_n g_{n-1} \cdots g_0(x_0) \leq x_{n+1} \leq f_n f_{n-1} \cdots f_0(x_0)$$

(ii) *The solution x_n of Eq. (4.2) satisfies*

$$g_{2k} g_{2k-2} \cdots g_0(x_0) \leq x_{2k} \leq f_{2k-2} f_{2k-4} \cdots f_0(x_0)$$

and

$$g_{2k+1} g_{2k-1} \cdots g_1(x_1) \leq x_{2k+1} \leq f_{2k-1} f_{2k-3} \cdots f_1(x_1)$$

for all $k \geq 1$.

PROOF: We use mathematical induction on n to prove part (i). For $n = 0$,

$$g_0(x_0) = F(x_0, \beta_{-1}) \leq x_1 = F(x_0, x_{-1}) \leq F(x_0, \alpha_{-1}) = f_0(x_0).$$

Assume the statement is true for $n = k - 1$, i.e.,

$$g_{k-1}g_{k-2}\cdots g_0(x_0) \le x_k \le f_{k-1}f_{k-2}\cdots f_0(x_0),$$

then for $n = k$, we obtain

$$x_{k+1} = F(x_k, x_{k-1}) \le F(x_k, \alpha_{k-1}) = f_k(x_k) \le f_k f_{k-1}\cdots f_0(x_0)$$

and

$$x_{k+1} = F(x_k, x_{k-1}) \ge F(x_k, \beta_{k-1}) = g_k(x_k) \ge g_k g_{k-1}\cdots g_0 g_{-1}(x_0).$$

The proof of (ii) is similar and omitted. □

Lemma 4.1. *Consider* $t_{n+1} = \frac{(t_n+h)}{f(t_n)}$ *such that* $t_0 > 0$. *The sequence* $\{t_n\}$ *converges to* $\bar{x}_{1,h}$ *in an increasing fashion if* $0 < t_0 < \bar{x}_{1,h}$ *and on a decreasing fashion if* $\bar{x}_{1,h} < t_0 < \bar{x}_{2,h}$.

Proof. Define $G(t) := \frac{(t+h)}{f(t)}$, then $G(t)$ is increasing with the same fixed points as Eq. (1.3). Since $G(0) = \frac{h}{b}$, then $G(t) > t$ for $0 < t < \bar{x}_{1,h}$, $G(t) < t$ for $\bar{x}_{1,h} < t < \bar{x}_{2,h}$ and $G(t) > t$ for $\bar{t} > \bar{x}_{2,h}$. Now, the fact that $t_n = G^n(t_0)$ and a simple stair step diagram completes the proof. □

Theorem 4.2. *If* $t_0 = \frac{h}{b}$ *in Lemma 4.1 , then any persistent solution* $\{x_n\}_{-1}^{\infty}$ *of Eq. (1.3) satisfies* $x_n \ge t_n$ *for all* $n \in \mathbb{N}$.

Proof. By definition, "if $\{x_n\}_{n=-1}^{\infty}$ is a persistent solution, then $(x_{n-1}, x_n) \in D_h$ for all $n \ge 0$." Indeed, if $\{x_n\}_{n=-1}^{\infty}$ is a persistent solution, then by Proposition 3.1 $(x_{n-1}, x_n) \notin S_0$ for all $n \ge 0$. So, for all $n \ge 0$, either $\left(x_n \ge \bar{x}_1 \text{ and } x_n \ge \frac{h}{f(x_{n-1})-1}\right)$, $(x_n \ge \bar{x}_1 \text{ and } x_{n-1} > \bar{x}_1)$, or $\left(x_{n-1} \le \bar{x}_1 \text{ and } x_n \ge \frac{h}{f(x_{n-1})-1}\right)$. As such, for all $n \ge 0$,

$$x_n \ge \min\left\{\bar{x}_1, \frac{h}{f(x_{n-1}) - 1}\right\}.$$

But $h/(f(x) - 1)$ is increasing with $f(0) = b$ and $f(\bar{x}_1) = \bar{x}_1$. Therefore, $x_n \ge h/(b-1)$ for all $n \ge 0$. In other words, we established the fact that $x_n \ge h/(b-1) > h/b = t_0$ for all $n \ge 0$.

Now, we assume $x_n \ge t_k$ for all $n \ge k$, and use induction on k to show that $x_n \ge t_{k+1}$ for all $n \ge k+1$. Indeed,

$$t_k \le x_{k+2} = x_{k+1}f(x_k) - h \le x_{k+1}f(t_k) - h \quad \Rightarrow \quad x_{k+1} \ge \frac{1}{f(t_k)}(t_k + h) = t_{k+1},$$

$$t_k \le x_{k+3} = x_{k+2}f(x_{k+1}) - h \le x_{k+2}f(t_{k+1}) - h \Rightarrow x_{k+2} \ge \frac{(t_k+h)}{f(t_{k+1})} \ge \frac{(t_k+h)}{f(t_k)} = t_{k+1}$$

and similarly $x_{k+j} \ge t_{k+1}$ for all $j = 3, 4, \ldots$. Thus, $x_n \ge t_{k+1}$ for all $n \ge k+1$ and the proof is complete. □

Proposition 3.1 shows that it is necessary for a persistent solution $\{x_n\}$ to have $x_0 > \frac{h}{b-1}$; however, it is not necessarily sufficient. Consider $t_0 = \frac{h}{b}$, and use Lemma 4.1 to obtain a sequence $\{t_k\}$. Now, in conjunction with Eq. (3.3), define

$$s_n := h\sum_{i=1}^{n}\prod_{j=0}^{i-1}(f(t_j))^{-1}.$$

The sequence $\{s_n\}$ is a strictly increasing sequence bounded above by $\frac{h}{f(\bar{x}_{1,h})-1} = \bar{x}_{1,h}$. Thus, we define

$$\gamma := \lim_{n\to\infty} s_n$$

and give the following result.

Proposition 4.1. *Define* $\mathcal{S}_1 := \{(x,y) : y < (\gamma + h)(f(x))^{-1}\}$. $\mathcal{S}_1 \cap \mathcal{D}_h = \phi$.

PROOF: Suppose $(x_{-1}, x_0) \in \mathcal{S}_1 \cap \mathcal{D}_h$, and let $\{x_n\}_{n=-1}^{\infty}$ be the associated persistent solution. We use the comparison principle of Theorem 4.1 to obtain a contradiction. Consider the sequence α_n to be the sequence t_n obtained in Lemma 4.1 with $t_0 = \frac{h}{b}$, and define $f_{n+1}(x) = f(\alpha_n)x - h$. By part (i) of Theorem 4.1, $x_{n+1} < f_n f_{n-1}\cdots f_0(x_0)$. However, when $(x_{-1}, x_0) \in \mathcal{S}_1, x_1 = \frac{bx_0}{1+x_{-1}} - h < \gamma$ and $f_n f_{n-1}\cdots f_0(x_0)$ goes negative for sufficiently large n. This contradicts our assumption that $\{x_n\}$ is persistent. □

Theorem 4.2 and Proposition 3.1 show that persistent solutions with initial conditions below $\bar{x}_{1,h}$ (if any) either converge to $\bar{x}_{1,h}$ or go up and stay above $\bar{x}_{1,h}$. Now, we proceed to establish certain bounds on the asymptotic behavior of persisting solutions.

Lemma 4.2. *Let* $0 \le h < h_{max}$. *The function* $g(t) = h\frac{f(t)+1}{f(t)f(\bar{x}_{1,h})-1}$ *is increasing and has exactly two fixed points in the interval* $[0, f^{-1}(1/f(\bar{x}_{1,h}))$. *One of the fixed points is* $\bar{x}_{1,h}$ *and the other one (say* $x^*_{2,h}$*) is larger than* $\bar{x}_{2,h}$.

Proof. It is clear by observing that $g'(t) > 0, g(\bar{x}_{1,h}) = \bar{x}_{1,h},\ g(\bar{x}_{2,h}) < \bar{x}_{2,h}$ and $g(t) \to \infty$ as t approaches $f^{-1}(1/f(\bar{x}_{1,h}))$ from the left. □

Theorem 4.3. *Let* $0 \le h < h_{max}$. *and* $x^*_{2,h}$ *as in Lemma 4.2. A persistent solution* $\{x_n\}_{n=-1}^{\infty}$ *of Eq. (1.3) satisfies*

$$\bar{x}_{1,h} \le \liminf x_n \le \limsup x_n \le x^*_{2,h} < f^{-1}\left(\frac{1}{f(\bar{x}_{1,h})}\right).$$

Proof. From Theorem 4.2 and Lemma 4.1, we obtain

$$\liminf x_n \ge \bar{x}_{1,h}.$$

Next, write Eq. (1.3) in the form of Eq. (4.1) by taking

$$F(x_n, x_{n-1}) = x_n f(x_n) f(x_{n-1}) - hf(x_n) - h,$$

then again, consider α_n to be the sequence t_n with $t_0 = \frac{h}{b}$ as given in Lemma 4.1. Now, we take $f_n(x) = f(\alpha_n)f(x)x - hf(x) - h$ and use part (ii) of Theorem 4.1 to obtain

$$x_{2k} \le f_{2k-2}f_{2k-4}\cdots f_0(x_0) \le f_{2k-2}f_{2k-4}\cdots f_0(x_0)$$

and

$$x_{2k+1} \le f_{2k-1}f_{2k-3}\cdots f_1(x_1) \le f_{2k-1}f_{2k-3}\cdots f_1(x_0)$$

for all $k \ge 1$. In both cases, the right hand side is eventually smaller than the larger fixed point of $f_\infty(x)$, which is given by $x^*_{2,h}$ as clarified in Lemma 4.2. Thus, we obtain

$$\limsup x_n \le x^*_{2,h} < f^{-1}\left(\frac{1}{f(\bar{x}_{1,h})}\right)$$

and the proof is complete. □

Theorem 4.3 shows that a persistent solution is either converging to $\bar{x}_{1,h}$ monotonically or eventually larger than $\bar{x}_{1,h}$. Since $\bar{x}_{1,h}$ is a saddle all the time, then orbits that converge to $\bar{x}_{1,h}$ are the ones assured by the stable manifold theorem. Those orbits show that harvesting can alter the oscillatory behavior of populations; however, those populations make no significant contribution to the persistent set, and therefore, we are more concerned about populations that are larger than $\bar{x}_{1,h}$. Thus, without further mention, we limit or attention to solutions that satisfy $x_n \geq \bar{x}_{1,h}$ for all n.

5 Oscillation of solutions

In this section, we discuss the oscillatory character of solutions of Eq. (1.3). If a persistent solution $\{x_n\}$ of Eq. (1.3) is neither eventually less than nor larger than $\bar{x}_{2,h}$, then we call it oscillatory about $\bar{x}_{2,h}$, or oscillatory for short. Notice that because of Theorem 4.2, we cannot have solutions oscillating about $\bar{x}_{1,h}$. We call a solution $\{x_n\}$ of Eq. (1.3) oscillatory about a curve $H(x, y) = 0$ if $X_n = (x_n, x_{n+1})$ does not eventually stay on one side of the curve. It is easy to observe that persistent solutions that are non-monotonic must oscillate about the curves of $y - x = 0$ and $y - xf(x) + h = 0$. Otherwise, if $x_{n+1} < x_n f(x_n) - h$ for all n, then $x_n f(x_{n-1}) - h < x_n f(x_n) - h$ for all n and, consequently, $x_{n-1} > x_n$ for all n, i.e., $\{x_n\}$ is monotonic which is a contradiction. Other cases can be handled similarly.

Proposition 5.1. *A persistent solution of Eq. (1.3) either converges to an equilibrium point or oscillates about the curve* $y = \frac{(x+h)}{f(x)}$.

Proof. Let $\{x_n\}$ be a persistent solution. If x_n eventually stays on one side of the curve, say $x_{n+1} < \frac{(x_n+h)}{f(x_n)}$ for all $n \geq n_0$, then Eq. (1.3) gives

$$x_{n+2} = x_{n+1} f(x_n) - h < x_n$$

for all $n \geq n_0$. This leads to convergence to a periodic solution of period 2. If this is the case then there exists $\beta > \alpha > 0$ such that

$$\beta = \alpha f(\beta) - h \qquad \text{and} \qquad \alpha = \beta f(\alpha) - h.$$

In particular we have

$$f(\alpha) = \frac{\alpha + h}{\beta} < \frac{\beta + h}{\alpha} = f(\beta)$$

which contradicts the assumption that f is decreasing. Since Eq. (1.3) has no periodic solutions of minimal period 2, then the convergence must be to an equilibrium solution. The other side is similar and omitted. □

We associate orbits of Eq. (1.3) in the positive quadrant by the map

$$T(x, y) = (y, yf(x) - h). \tag{5.1}$$

It is straightforward to verify that T is one to one on the positive quadrant (the x axis is not included), and therefore, the action of Eq. (1.3) on a region can be visualized by testing the action of T on the boundary of that region. For a fixed constant $c > 0$, points above $y = c$ are mapped to points on the right of $x = c$. Also, since $T(c, t) = (t, tf(c) - h)$ and $x_1 = x_0 f(x_{-1}) - h < 0$ if and only if $x_0 < h/f(x_{-1})$, then one can square the persistent set inside the box $\mathbb{S}_c = \{(x,y) : \ 0 \le x < c, 0 < y < c\}$ provided that $f(x) < h/(xf(c) - h)$. It is worth stressing again that Theorem 4.2 and Lemma 3.1 limit our interest to the region $\{(x,y) : x, y \ge \bar{x}_{1,h}\}$. We define the following subregions:

$$
\begin{aligned}
&\mathcal{R}_1 := \{(x,y) \in \mathbb{R}^2 : \ \bar{x}_{1,h} < x < \bar{x}_{2,h}, y \ge \bar{x}_{2,h}\}\\
&\mathcal{R}_2 := \{(x,y) \in \mathbb{R}^2 : \ x \ge \bar{x}_{2,h}, y > \bar{x}_{2,h}\}\\
&\mathcal{R}_{2i} := \{(x,y) \in \mathcal{R}_2 : y > x\}\\
&\mathcal{R}_{2ii} := \{(x,y) \in \mathcal{R}_2 : y \le x\}\\
&\mathcal{R}_3 := \{(x,y) \in \mathbb{R}^2 : \ \bar{x}_{1,h} < y \le \bar{x}_{2,h}, x > \bar{x}_{2,h}\}\\
&\mathcal{R}_4 := \{(x,y) \in \mathbb{R}^2 : \ \bar{x}_{1,h} < x \le \bar{x}_{2,h}, \bar{x}_{1,h} < y < \bar{x}_{2,h}\}\\
&\mathcal{R}_{4i} := \{(x,y) \in \mathcal{R}_4 : y < x\}\\
&\mathcal{R}_{4ii} := \{(x,y) \in \mathcal{R}_4 : y \ge x\}.
\end{aligned}
$$

For our convenience, if (x,y) is in one of the above regions but $yf(x) - h \le \bar{x}_{1,h}$, then the population with initial conditions $(x_{-1}, x_0) = (x,y)$ goes extinct (see Proposition 3.1) , and therefore, we say $T(x,y) \in \{0\}$. Now, by tracing the action of T on the defined boundaries of these regions, we conclude the following result:

Proposition 5.2. *Let $0 < h < h_{max}$. Each of the following holds true:*

(i) $T(\mathcal{R}_1) \subset \mathcal{R}_{2i}$.
(ii) $T(\mathcal{R}_{2i}) \subset \mathcal{R}_{2ii}$ *if* $h = 0$ *and* $T(\mathcal{R}_{2i}) \subset \mathcal{R}_{2i} \cup \mathcal{R}_{2ii}$.
(iii) $T(\mathcal{R}_3) \subset \mathcal{R}_{4i} \cup \{0\}$.
(iv) $T^2(\mathcal{R}_{2i}) \subset \mathcal{R}_{2ii} \cup \mathcal{R}_3 \cup \{0\}$.
(v) $T(\mathcal{R}_{4ii}) \subset \mathcal{R}_{4ii} \cup \mathcal{R}_1$.
(vi) $T(\mathcal{R}_{4i}) \subset \mathcal{R}_{4i} \cup \mathcal{R}_{4ii} \cup \{0\}$.

Recall [20] that given a fixed point $\bar{x}$ for a sequence $\{x_n\}$, a positive semi-cycle is subsequence of consecutive terms that are all greater or equal to $\bar{x}$. A negative semi-cycle is subsequence of consecutive terms that are all less or equal than $\bar{x}$.

The next result becomes straightforward.

Proposition 5.3. *Consider Eq. (1.3) with $0 \le h < h_{max}$. Each of the following holds true:*

(i) Non-equilibrium persistent solutions that do not converge to $\bar{x}_{1,h}$ monotonically must oscillate about $\bar{x}_{2,h}$.
(ii) When h is zero, a positive semi-cycle has more than two terms (except possibly the first one) with a maximum in the second term, and decreasing afterwards.
(iii) When h is zero, a negative semi-cycle has at least two terms (except possibly the first one) with a minimum in the second term, and increasing afterwards.

For $h > 0$ and by checking the effect of T, T^2 on the regions $\mathcal{R}_1$ and $\mathcal{R}_3$, one can observe that an increase in the harvesting quota increases the length of semi-cycles. Therefore, solutions overshoot or undershoot the stable equilibrium when they oscillate, and consequently, the risk of extinction for populations governed by Eq. (1.3) becomes high. However, obtaining exact values of h that do not lead to population extinction remains an extremely challenging task. Nevertheless, one can use the extreme values of semi-cycles and the approach used in [17, 20, 9] to prove that certain persistent solutions are attracted to $\bar{x}_{2,h}$ under some mild conditions on h. Define $g_1(t) = t(f(t)-1)$ and $g_2(t) = (h/f(\bar{x}_{2,h}))f(t)$, then $g_1 = g_2$ at $t = \bar{x}_{2,h}$. Furthermore, for small values of h, they intersect at a second positive point smaller than $\bar{x}_{2,h}$. Thus, $g_1(t) \le g_2(t)$

for all $t \geq \bar{x}_{2,h}$ and for sufficiently small values of h. As we increase h, we continue to obtain $g_1(t) \leq g_2(t)$ for all $t \geq \bar{x}_{2,h}$ as long as $\bar{x}_{2,h}$ is the largest solution of $g_1 = g_2$. This fact gives us a constraint on h, which we use to obtain the following result:

Lemma 5.1. *Assume that $tf(t)$ is increasing. Let $0 \leq h < h_{\max}$ such that $\bar{x}_{2,h}$ is the largest solution of $g_1 = g_2$. A positive semi-cycle of a persistent and oscillatory solution of Eq. (1.3) has at least three terms (except possibly the first one) with a maximum in the second or third term, and decreasing afterwards.*

Proof. We trace the boundary of $\mathcal{R}_1$. Since

$$T^2(\bar{x}_{2,h}, t) = T(t, tf(\bar{x}_{2,h}) - h) = (tf(\bar{x}_{2,h}) - h, (tf(\bar{x}_{2,h}) - h)f(t) - h).$$

For $t > \bar{x}_{2,h}$, $tf(\bar{x}_{2,h}) - h > t$ and $T(t,t) = (t, tf(t) - h)$ implies $tf(t) - h > \bar{x}_{2,h}$. Thus, a positive semi-cycle has at least three terms. On the other hand,

$$tf(\bar{x}_{2,h}) - h > (tf(\bar{x}_{2,h}) - h)f(t) - h \ \Leftrightarrow g_1(t) < g_2(t),$$

and therefore, the condition on h makes the maximum takes place at the second or third term. □

Next, define the functions

$$f_0(t) = \bar{x}_{2,h}f(t) - h,\ f_1(t) = f(\bar{x}_{2,h})t - h,\ G_0(t) = f_1(f_0(t)),\ \text{and}\ G_1(t) = f_1(G_0(t)).$$

The motivation behind taking these functions will be clear in the proof of Lemma 5.2 and Theorem 5.1; however, it is easy to observe that G_0 and G_1 are both decreasing, and $G_j(\bar{x}_{2,h}) = \bar{x}_{2,h}$. For $j = 0, \ldots, 3$, let $C_{r+j} := \{x_{k_j+1}, x_{k_j+2}, \ldots, x_{k_{j+1}}\}$ be four consecutive semi-cycles starting with the negative semi-cycle C_r. Also, let c_{r+j}, $j = 0, \ldots, 3$ be the extreme values of the semi-cycles respectively. Next, assume that $xf(x)$ is increasing. If c_{r+3} and c_{r+2} lie in the second term, then

$$\begin{aligned} c_{r+3} = x_{k_3+2} =& x_{k_3}f(x_{k_3})f(x_{k_3-1}) - hf(x_{k_3}) - h \\ \leq& \bar{x}_{2,h}f(\bar{x}_{2,h})f(x_{k_3-1}) - hf(\bar{x}_{2,h}) - h \\ \leq& G_0(c_{r+2}) \end{aligned}$$

and

$$\begin{aligned} c_{r+2} = x_{k_2+2} =& x_{k_2}f(x_{k_2})f(x_{k_2-1}) - hf(x_{k_2}) - h \\ \geq& \bar{x}_{2,h}f(\bar{x}_{2,h})f(x_{k_2-1}) - hf(\bar{x}_{2,h}) - h \\ \geq& G_0(c_{r+1}). \end{aligned}$$

Which implies

$$c_{r+3} \leq G_0(c_{r+2}) \leq G_0(G_0(c_{r+1})) = G_0^2(c_{r+1}).$$

If c_{r+3} and c_{r+2} lie in the third term, then

$$c_{r+3} = x_{k_3+3} = x_{k_3+2}f(x_{k_3+1}) - h \leq f_1(x_{k_3+2}) \leq f_1\left(G_0(c_{r+2})\right)$$

and

$$c_{r+2} = x_{k_2+3} = x_{k_2+2}f(x_{k_2+1}) - h \geq f_1(x_{k_2+2}) \geq f_1\left(G_0(c_{r+1})\right).$$

Which implies

$$c_{r+3} \leq G_1(c_{r+2}) \leq G_1(G_1(c_{r+1})) = G_1^2(c_{r+1}).$$

Similarly, if c_{r+2} and c_{r+3} lie in the third and second terms respectively, then we obtain

$$c_{r+3} \leq G_0(c_{r+2}) \leq G_0(G_1(c_{r+1})),$$

and if c_{r+2} and c_{r+3} lie in the second and third terms respectively, then we obtain

$$c_{r+3} \leq G_1(c_{r+2}) \leq G_1(G_0(c_{r+1})).$$

Now, we can summarize in the following result:

Lemma 5.2. *Suppose that $xf(x)$ is increasing. Let $c_{r+j}, j = 1, 2, 3$ be the extreme values of three consecutive semi-cycles starting with a positive one. Each of the following holds true:*

(i) If c_{r+2} and c_{r+3} lie in the second term, then $c_{r+3} \leq G_0^2(c_{r+1})$.
(ii) If c_{r+2} and c_{r+3} lie in the third term, then $c_{r+3} \leq G_1^2(c_{r+1})$.
(iii) If c_{r+2} and c_{r+3} lie in the second and third terms respectively, then $c_{r+3} \leq G_1(G_0((c_{r+1})))$.
(iv) If c_{r+2} and c_{r+3} lie in the third and second terms respectively, then $c_{r+3} \leq G_0(G_1((c_{r+1})))$.

The next result is a classical one, and the proof is straightforward.

Lemma 5.3. *Let $g \in \mathcal{C}([0, \gamma])$ be decreasing such that $g(\gamma) = 0$ and $0 < g(0) < \gamma$. The map g has a unique fixed point in $(0, c)$. Moreover, the fixed point is a global attractor for the interval $[0, \gamma]$ if and only if g has no periodic points of period two.*

Now, we give the main result of this section.

Theorem 5.1. *Suppose that $tf(t)$ is increasing, and let $0 \leq h < h_{\max}$ such that $\bar{x}_{2,h}$ is the largest solution of $g_1 = g_2$. If $G_1^2(0) > 0$ and G_1 has no periodic points of period two, then for any persistent solution $\{x_n\}$ of Eq. (1.3), we have $\limsup x_n \leq \bar{x}_{2,h}$.*

Proof. From Lemma 5.1, the extreme values of positive semi-cycles take place at the second or third term. When $t > \bar{x}_{2,h}$, we have

$$G_1(t) = f_1(G_0(t)) < G_0(t) \quad \text{implies} \quad G_1^2(t) > G_1G_0(t) \text{ and } G_0G_1(t) > G_0^2(t).$$

Also, since $f_1(t) > t$ for $t > \bar{x}_{2,h}$, then $G_1^2(t) = f_1G_0G_1(t) > G_0G_1(t)$. Thus, Lemma 5.2 narrows down to $c_{r+3} < G_1^2(c_{r+1})$. Now, for a persistent and oscillatory solution $\{x_n\}_{n=-1}^{\infty}$ take $\{c_m\}_{m=0}^{\infty}$ to be the sequence of extreme values of the positive semi-cycles (ordered as they appear in $\{x_n\}$). Then we have $c_m < G_1^{2m}(c_0)$ and the conditions on G_1 together with Lemma 5.3 give us the conclusion. □

Corollary 5.1. *Suppose the hypotheses of Theorem 5.1 are satisfied, then persistent and oscillatory solutions are attracted to $\bar{x}_{2,h}$.*

We close this section by the following remark:

Remark 5.1. *In Theorem 5.1, the conditions on G_1 could be stronger than the condition on $\bar{x}_{2,h}$ to be the largest solution of $g_1 = g_2$, and therefore, the latter one could be redundant. However, keeping the two conditions saved us some deep technicalities in the proof.*

6 Pielou's equation with harvesting

In this section, we focus on a particular case of Eq. (1.3) and consider Pielou's Equation with harvesting

$$x_{n+1} = \frac{bx_n}{1+x_{n-1}} - h;\ b > 1. \tag{6.1}$$

In this equation, we have $f(x) = \frac{b}{1+x}$. Furthermore, it is straightforward to check that conditions (C1)-(C3) are satisfied if $b \leq 4$. Now, we have $h_{max} := (\sqrt{b}-1)^2$, and when $0 \leq h < h_{max}$, the two positive equilibria are given by

$$\bar{x}_{i,h} = \frac{b-1-h+(-1)^i\sqrt{(b-1-h)^2-4h}}{2}, \quad i = 1, 2. \tag{6.2}$$

It is worth mentioning that Eq. (6.1) was studied when $h = 0$ by Kuruklis and Ladas in [21]. In Section 2, we discussed the local stability of these equilibria in general; however, we can be more specific here. If $h < 1$, then $\bar{x}_{2,h}$ is stable. At $h = 1$, $\bar{x}_{2,h}$ is a nonhyperbolic fixed point and Eq. (6.1) reduces to a Lyness type equation. If $h > 1$, then $\bar{x}_{2,h}$ is a repeller. $\bar{x}_{1,h}$ is a saddle for all $h < h_{max}$. At $h = h_{max}$, $\bar{x}_{1,h} = \bar{x}_{2,h} = \sqrt{b} - 1$ is a nonhyperbolic point. It is stable when $b < 4$ and unstable when $b > 4$.

In the remaining of this section, we first study the special case $h = 1$ which, as mentioned earlier, reduces equation (6.1) to Lyness equation allowing us to use previous results to establish both the persistence set and the general behavior of the solutions. Then, in subsection (6.2) we find a range of parameters (b, h) where solutions converge to the larger equilibrium $\bar{x}_{2,h}$. The last subsection is dedicated to narrowing down the persistence set D_h. While this is a challenging task requiring tedious calculations, we feel that it is necessary to attempt to describe the set of initial populations that will survive a certain level of harvesting h.

6.1 A connection with a Lyness type equation

When $h = 1$, Eq. (6.1) reduces to the well-known Lyness equation (or its conjugate form) [24, 25, 26]. Indeed, we can use the change of variables $by_n = 1 + x_n$, to obtain

$$y_{n+1} = \frac{y_n - \frac{1}{b}}{y_{n-1}},\ n \in \mathbb{N}. \tag{6.3}$$

Eq. (6.3) has been extensively studied [4, 12, 11, 18]. Notice that when $h = 1$ and $1 < b < 4$, then Theorem 3.1 gives an empty persistent set. The case $h = 1$ and $b = 4$ has been discussed in [11] where certain solutions can be written in explicit form. The case $h = 1$ is of particular interest to us, since we use one of the 8-periodic solutions of Eq. (6.3) to introduce a trapping region for Eq. (6.1).

Theorem 6.1. *Consider Eq. (6.1) with $h = 1$ and $b \geq 4$. Each of the following holds true.*

(i) The curves given by

$$\mathcal{I}_b(x, y) := \left(1 + \frac{b}{1+x}\right)\left(1 + \frac{b}{1+y}\right)(1 + x + y) = C,$$

define the invariants of Eq. (6.1), where C is a constant given by $\mathcal{I}_b(x_{-1}, x_0)$.

(ii) A population persists if and only if the constant C in (i) is bounded by

$$C_1 := 2 + (b+1)^2 - \overline{x}_2(b-4) \leq C \leq C_2 := 2 + (b+1)^2 - \overline{x}_1(b-4).$$

(iii) The invariant $\mathcal{I}_b(x,y) = C_2$ defines the boundary of the persistence set $\mathcal{D}_1$.

(iv) $\overline{x}_2$ is a center and the positive solutions inside $\mathcal{D}_1$ oscillate about $\overline{x}_2$ such that every semi-cycle (except possibly the first one) has at least three elements.

PROOF: (i) and (ii) can be extracted from [18] and [12]. Since $b\bar{x} = (1+\bar{x})^2$, by direct computations, which can be checked easily using a symbolic computational language such as MAPLE,

$$C_1 = I_b(\bar{x}_2, \bar{x}_2) = 2 + (b+1)^2 - \overline{x}_2(b-4)$$

and

$$C_2 = I_b(\bar{x}_1, \bar{x}_1) = 2 + (b+1)^2 - \overline{x}_1(b-4).$$

To prove (iii), use (ii) to obtain the persistent set

$$\mathcal{D}_1 = \{(x,y) : \; C_1 \leq I_b(x,y) \leq C_2\},$$

and the boundary of this region is given by the invariant $\mathcal{I}_b(x,y) = C_2$. Finally, (iv) follows from (i) and Proposition 5.2. □

The map in (5.1) rotates the invariants $\mathcal{I}_b(x,y) = C, \; C_1 < C < C_2$ clockwise such that each orbit along the invariant is periodic of the same period or dense in the invariant. For instance, the orbits along the invariants

$$\mathcal{I}_{2(b_1-2)}(x,y) = 18b_1 - 24, \mathcal{I}_{b_1}(x,y) = 9(b_1-1), \; \mathcal{I}_5(x,y) = 36,$$

where $b_1 := 4 + \sqrt{7}$ are 7-periodic, 7-periodic, 8-periodic respectively. It is worth mentioning that it is possible for a fixed $b > 4$ to have two invariants with different periods. For instance, if we fix $b = 5$, then $\mathcal{I}_5(x,y) = 36$ and $\mathcal{I}_5(x,y) = \frac{75}{2}$ are 8-periodic and 10-periodic respectively.

6.2 Convergence to the large equilibrium

In this subsection, we apply the results of Section 5 on Pielou's equation.

For Eq. (6.1), the condition $g_1(t) \leq g_2(t)$ in Lemma 5.1 simplifies to

$$t^2 + (1-b)t + h(1+\bar{x}_{2,h}) \geq 0,$$

and that is satisfied as long as

$$h \leq \frac{(b-1)^2}{2(b+1)}. \tag{6.4}$$

This constraint on the harvesting level guarantees that the extreme values of positive semi-cycles in an oscillatory and persistent solution take place at the second or third term. Next, we test the conditions of Theorem 5.1. The condition $G_1^2(0) > 0$ implies $G_1(0) < f_0^{-1} f_1^{-1} f_1^{-1}(0)$, or equivalently

$$f_1(f_1(b\bar{x}_{2,h} - h)) < f_0^{-1}\left(f_1^{-1}\left(\frac{h}{(1+\bar{x}_{1,h})}\right)\right).$$

Keep in mind that $f(x) = \frac{b}{1+x}$, which implies $f(\bar{x}_{2,h}) = 1 + \bar{x}_{1,h}$ and $\bar{x}_{1,h}\bar{x}_{2,h} = h$. Further computations give us,

$$f_1(b\bar{x}_{2,h} - h\bar{x}_{1,h} - 2h + hb) < f_0^{-1}\left(\frac{h(2+\bar{x}_{1,h})}{(1+\bar{x}_{1,h})^2}\right),$$

and consequently

$$(1+\bar{x}_{1,h})(b\bar{x}_{2,h} - h\bar{x}_{1,h} - 2h + hb) - h < \frac{b(1+\bar{x}_{1,h})^2}{\bar{x}_{1,h}(2+\bar{x}_{1,h}+(1+\bar{x}_{1,h})^2)} - 1. \tag{6.5}$$

Again, we rewrite this inequality as

$$(b - h^2 + 2h)\bar{x}_{2,h} + h(-h^2 + 2h + bh - 1) + 1 < \frac{b^3}{((b+2)h + (b^2+b+1)\bar{x}_{1,h} + h\bar{x}_{2,h})}$$

or

$$\bar{x}_{2,h} > \frac{b^2h^4 - b^2(3+2b)h^3 + (2b^3 + 3b^2 - 1 + b^4)h^2 + (b+2)h - 1}{-b^2h^3 + (3b^2 + b^3 + 1)h^2 - (2 + 2b + 3b^2)h + b^2 + b + 1}. \tag{6.6}$$

Graph 1 shows the solution of this inequality in the (b, h)-plane.

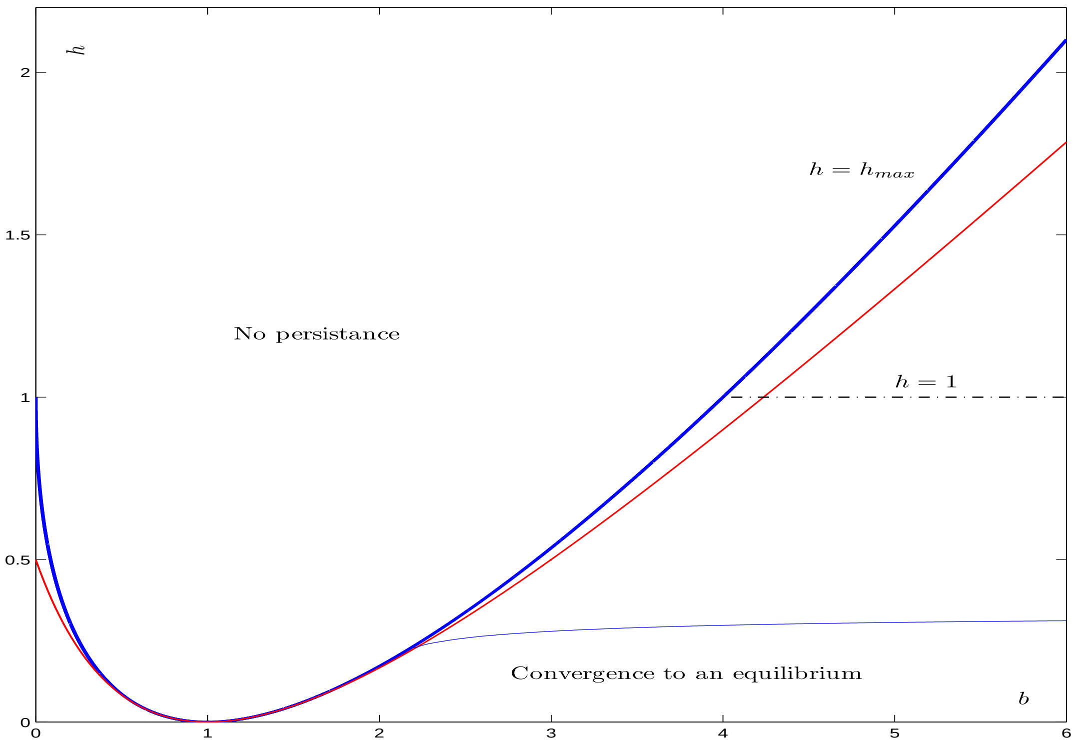


Figure 1: This figure shows the curves of $h_{max} = (\sqrt{b} - 1)^2, h = \frac{(b-1)^2}{2(b+1)}, h = 1$. Also, the red dashed curve gives the upper boundary of the region given by Inequality (6.6)

6.3 Characterizing the persistent set

Finding the exact persistent set when $h \neq 1$ is a non-easy task; however, in this section, we characterize the persistent set through bounds, trapping regions and numerical simulations. We start by the case $h = (\sqrt{b}-1)^2$.

Theorem 6.2. *Consider $h = (\sqrt{b}-1)^2$. Each of the following holds true:*

(i) If $b > 4$, then

$$\mathcal{D}_h \subset \{(x,y) : 0 \leq x \leq \sqrt{b}-1,\ (\sqrt{b}-1)(x+2-\sqrt{b}) \leq y \leq \sqrt{b}-1\}.$$

(ii) If $b = 4$, then

$$\mathcal{D}_h \subset \{(x,y) :\ y = \frac{-3+16x-x^2+\sqrt{49-148x+150x^2-52x^3+x^4}}{2(x+5)},\ 0 \leq x \leq 1\}.$$

PROOF: From the proof of Theorem 3.2, $A_n = \frac{h+x_n}{1+x_{n-1}}$ converges to $\bar{A} = \sqrt{b}-1$ on a decreasing fashion. Thus, $h + x_{n+1} > (\sqrt{b}-1)(1+x_n)$, which implies $x_{n+1} \geq (\sqrt{b}-1)(x_n + 2-\sqrt{b})$, and we obtain (i). (ii) can be obtained from the previous section. In particular, when $h = 1$ and $b = 4$, we obtain one invariant curve given by $\mathcal{I}_4(x,y) = 27$. If $(x_{n-1}, x_n) \in \mathcal{I}_4$ where $x_n < x_{n-1}$, then

$$x_{n+1} = \frac{4x_n}{1+x_{n-1}} - 1 < \frac{4x_n}{1+x_n} - 1 < x_n.$$

Thus the population decreases to extinction along the invariant curve. So, the lower branch of the invariant curve does not belong to the persistent set. If $(x_{n-1}, x_n) \in \mathcal{I}_4$ where $x_n > x_{n-1}$, then

$$x_{n+1} = \frac{4x_n}{1+x_{n-1}} - 1 > \frac{4x_n}{1+x_n} - 1 > x_n.$$

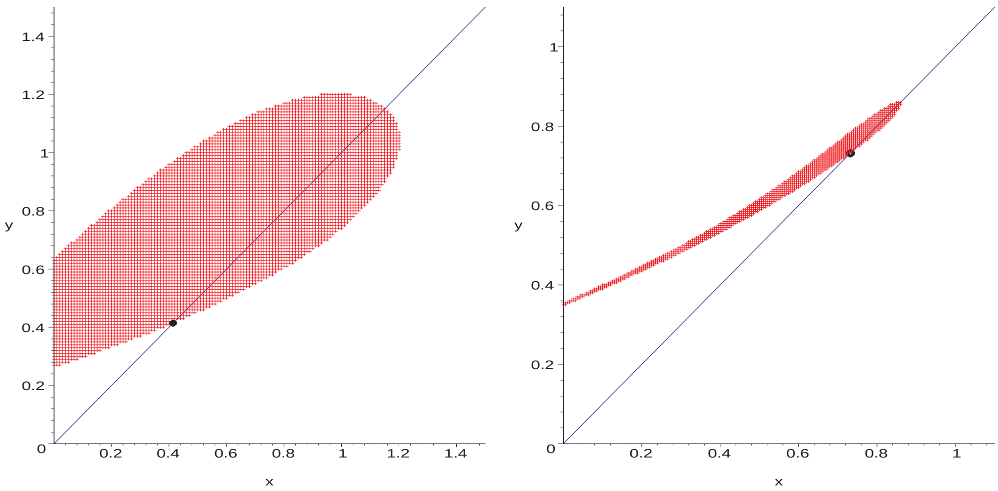
This inequality is obvious due to the fact that $x_n > \frac{1+x_{n-1}}{4}$. Hence, the persistent set is the upper branch of $\mathcal{I}_4$, which is what we have in (ii). □

Theorem 6.3. *Consider $\mathcal{S}_0$ as defined in Proposition 3.1. If $h > \frac{1}{3}(-b-2+2\sqrt{b^2+b+1})$, then*

$$\mathcal{D}_h \subset \{(x,y) : 0 < x < b-1,\ 0 < y < b-1\} \setminus \mathcal{S}_0.$$

PROOF: Since few iterates of the map T in (5.1) are sufficient to take initial conditions (x_{-1}, x_0) with $x_0 \geq b-1$ to (x_{n_0-1}, x_{n_0}) with $x_{n_0} < b-1$ and $x_{n_0-1} \geq b-1$, then we start with (x_{n_0-1}, x_{n_0}) and show that we have no persistence. We have

$$\begin{aligned}
x_{n_0+1} &= \frac{bx_{n_0}}{1+x_{n_0-1}} - h \leq x_{n_0} - h \\
x_{n_0+2} &= \frac{bx_{n_0+1}}{1+x_{n_0}} - h \\
&= \frac{b}{1+x_{n_0}}\left(\frac{bx_{n_0}}{1+x_{n_0-1}} - h\right) - h \\
&\leq \frac{b(x_{n_0}-h)}{(1+x_{n_0})} - h.
\end{aligned}$$



(a) The set of initial conditions that survived 1000 iterations at $b = 2.0$ and $h = (\sqrt{2.0} - 1)^2$, where $x = x_{n-1}$ and $y = x_n$.

(b) The set of initial conditions that survived 1000 iterations at $b = 3.0$ and $h = (\sqrt{3.0} - 1)^2$, where $x = x_{n-1}$ and $y = x_n$.

Figure 2: These figures show numerical simulations of the persistent set at the nonhyperbolic fixed point. The fixed point is indicated by a bold black dot.

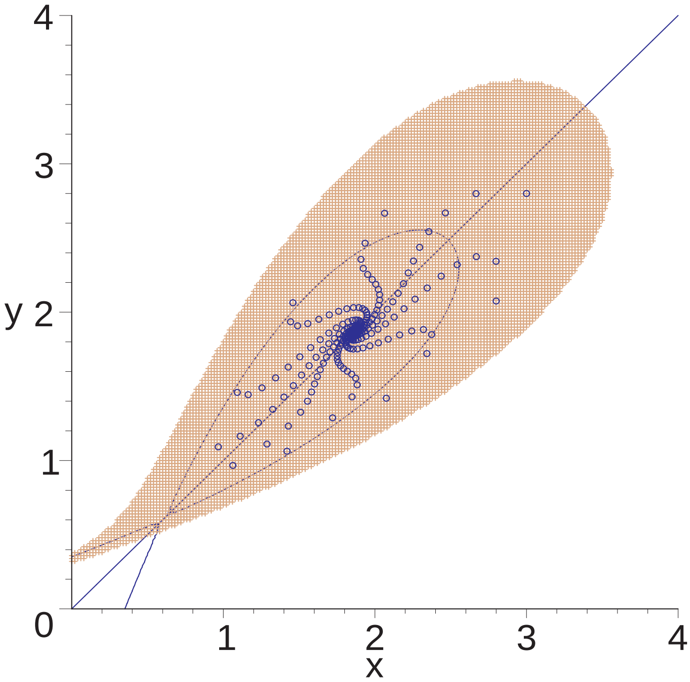


Figure 3: The shaded region in this figure shows the set of initial conditions that survived 500 iterations at $h = 0.90$ and $b = 4.25$, where $x = x_{n-1}$ and $y = x_n$. The blue circles show an orbit oscillating and converging to $\bar{x}_2$. The fading loop shows the largest invariant curve at $h = 1, b = 4.25$.

Now, the given condition on h implies

$$\frac{b(x_{n_0} - h)}{(1 + x_{n_0})} - h < \frac{h}{b}(1 + x_{n_0}),$$

and Proposition 4.1 shows that there is no persistence. □

In the rest of this section, we use an 8-periodic solution of the case $h = 1$, to define a trapping region when $h < 1$. To achieve this task and avoid complicated computations, we transform Eq. (6.1) to a new form where the origin becomes the small equilibrium. Take $z_n = x_n - \bar{x}_{1,h}$ to obtain

$$z_{n+1} = \frac{\frac{b}{1+\bar{x}_{1,h}} z_n - \frac{h+\bar{x}_{1,h}}{1+\bar{x}_{1,h}} z_{n-1}}{1 + \frac{z_{n-1}}{1+\bar{x}_{1,h}}}. \tag{6.7}$$

Now, define

$$y_n := \frac{z_n}{1+\bar{x}_{1,h}}, \; \alpha := \frac{b}{1+\bar{x}_{1,h}}, \quad \text{and} \quad \beta := \frac{h+\bar{x}_{1,h}}{1+\bar{x}_{1,h}}$$

to obtain

$$y_{n+1} = \frac{\alpha y_n - \beta y_{n-1}}{1 + y_{n-1}}, \; n \in \mathbb{N}. \tag{6.8}$$

The positive equilibrium of Eq. (6.8) is given by $\bar{y} = \alpha - \beta - 1$ and that should not be confused with $\bar{x}_{2,h}$.

Recall the map T introduced in (5.1), which takes the form $T(x, y) = (y, \frac{\alpha y - \beta x}{1+x})$. Next, define

$$\begin{aligned} A :=& \; \alpha\sqrt{(\alpha^4 - 2\alpha^3 + 4\alpha\beta - 4\beta^2 + \alpha^2 + 4\beta^2\alpha - 2\beta\alpha^2 - 2\alpha^3\beta + \beta^2\alpha^2)} \\ s^* :=& \; \frac{1}{4\beta(\alpha-1)}(\alpha^3 - \alpha^2\beta - 2\alpha\beta - \alpha^2 + 4\beta + A) \end{aligned}$$

and

$$\begin{aligned} f_1(t) :=& \frac{(\alpha-\beta)t}{1+t}, & f_2(t) :=& \frac{\beta t}{\alpha - 1 - t} \\ g_1(t) :=& \frac{t(\alpha^2 - \alpha\beta - \beta - \alpha t)}{\alpha - \beta}, & g_2(t) :=& \frac{t(\alpha^2 - \alpha - \beta - \alpha t)}{\beta(\alpha - 1)}. \end{aligned}$$

To assure that A is real, we need to impose the following constraint on α and β :

$$\alpha \geq \alpha_0 := \frac{1}{2}(\beta + 1) + \frac{1}{2}\sqrt{\beta^2 + (10 + 8\sqrt{2})\beta + 1}. \tag{6.9}$$

To keep things in perspective of the previous section, we have $\alpha = 1 + \bar{x}_{2,h}, \frac{\alpha}{\beta} = \frac{\bar{x}_{2,h}}{h}$ and Inequality (6.9) is equivalent to

$$\bar{x}_{2,h} \geq \frac{2(2+\sqrt{2})h}{(b-h-1)} \quad \Leftrightarrow \quad b \geq h + 1 + 2\sqrt{2h}. \tag{6.10}$$

Here, computations are becoming tedious. So, we keep our writing concise and use the MAPLE computer algebra system to perform computations. Condition (6.9) assures that A is real, and to assure that s^* is larger than the positive equilibrium $\bar{y} = \alpha - \beta - 1$, we impose the following constraint:

$$\begin{cases} h < b + 3 - 2\sqrt{2(b+1)}, & 1 < b \leq 2(1+\sqrt{2}) \\ h < \frac{b(b-2)}{2(b+2)}, & b > 2(1+\sqrt{2}). \end{cases} \tag{6.11}$$

Proposition 6.1. *Each of the following holds true:*

(i) If Condition (6.10) is satisfied, then A is real.
(ii) Condition (6.11) assures that $s^* > \bar{y} = \alpha - \beta - 1$.
(iii) The choice of s^* assures that $T^4(s^*, s^*)$ lies along the diagonal $y = x$.
(iv) If $h < 1$ and Condition (6.11) is satisfied, then $T^j(s^*, s^*), j = 1, 2, 3, 4$ stay in the positive quadrant.

Proof. (i) Write $\alpha = 1 + \bar{x}_{2,h}$ and

$$\beta = \frac{(h + \bar{x}_{1,h})\bar{x}_{2,h}}{(1 + \bar{x}_{1,h})\bar{x}_{2,h}} = \frac{h(\bar{x}_{2,h} + 1)}{(\bar{x}_{2,h} + h)},$$

then we obtain

$$A = \frac{(1 + \bar{x}_{2,h})^2}{(h + \bar{x}_{2,h})}\sqrt{(\bar{x}_{2,h}^4 - 6h\bar{x}_{2,h}^2 + h^2)} = \frac{\bar{x}_{2,h}(1 + \bar{x}_{2,h})^2}{(h + \bar{x}_{2,h})}\sqrt{(b - h - 1)^2 - 8h}$$

and Condition (6.10) makes the radicand positive. (ii) Write s^* in terms of $\bar{x}_{1,h}, \bar{x}_{2,h}$ and h as

$$s^* = \frac{1}{4h\bar{x}_{2,h}}\left(h - 3h\bar{x}_{2,h} + \bar{x}_{2,h}^2 + \bar{x}_{2,h}^3 + (1 + \bar{x}_{2,h})\sqrt{(\bar{x}_{2,h}^4 - 6h\bar{x}_{2,h}^2 + h^2)}\right) \quad (6.12)$$

$$= \frac{1}{4h}\left(\bar{x}_{1,h} - 3h + \bar{x}_{2,h} + \bar{x}_{2,h}^2 + (1 + \bar{x}_{2,h})\sqrt{(\bar{x}_{2,h}^2 + \bar{x}_{1,h}^2 - 6h)}\right). \quad (6.13)$$

$s^* > \alpha - \beta - 1$ if and only if

$$\sqrt{(\bar{x}_{2,h}^4 - 6h\bar{x}_{2,h}^2 + h^2)} > \frac{(3h - \bar{x}_{2,h})\bar{x}_{2,h}^2}{(h + \bar{x}_{2,h})} - h,$$

which can be written as

$$\sqrt{(b - h - 1)^2 - 8h} > \frac{4h}{b}(1 + \bar{x}_{2,h}) - (b - h - 1). \quad (6.14)$$

The solution of Inequality (6.14) is given by Condition (6.11). (iii) We show that $T^4(s^*, s^*)$ lies along the diagonal $y = x$. Computations show that

$$\{(x, y) : x = g_2(y)\} \overset{T}{\to} \{(x, y) : y = f_2(x)\} \overset{T}{\to} \{(x, y) : y = x\}$$

and

$$\{(x, y) : y = x\} \overset{T}{\to} \{(x, y) : y = f_1(x)\} \overset{T}{\to} \{(x, y) : y = g_1(x)\}.$$

In particular,

$$(g_2 f_2^{-1}(s^*), f_2^{-1}(s^*)) \overset{T}{\to} (f_2^{-1}(s^*), s^*) \overset{T}{\to} (s^*, s^*) \overset{T}{\to} (s^*, f_1(s^*))$$

and

$$(s^*, f_1(s^*)) \overset{T}{\to} (f_1(s^*), g_1 f_1(s^*)) \overset{T}{\to} (g_1 f_1(s^*), f_2 g_1 f_1(s^*)) \overset{T}{\to} (f_2 g_1 f_1(s^*), f_2 g_1 f_1(s^*)).$$

We alert the reader that computations here were tedious; however, if you take $y_{-1} = y_0 = s^*$ as given in Eq. (6.12) and take x_2 for $\bar{x}_{2,h}$, then use the MAPLE commands

```
> y[-1]:=s*; y[0]:=s*; alpha:=x[2]+1; beta:=h*(x[2]+1)/(h+x[2]);
```

```
> For j from 0 to 3 do
> y[j+1]:=factor(simplify(expand(rationalize(
(alpha*y[j]-beta*y[j-1])/(1+y[j-1]))))):
> end do;
```

to obtain

$$
\begin{aligned}
y_1 &= \frac{x_2^2 + 2x_2 - h + \sqrt{h^2 + x_2^4 - 6hx_2^2}}{2(h + x_2)} \\
y_2 &= \frac{x_2^2 + 2x_2 h - h - \sqrt{h^2 + x_2^4 - 6hx_2^2}}{2(h + x_2)} \\
y_3 &= \frac{(1 + x_2)(x_2^3 + hx_2^2 - 3x_2 h + h^2 - (x_2 + h)\sqrt{h^2 + x_2^4 - 6hx_2^2})}{4(h + x_2)x_2} \\
y_4 &= y_3.
\end{aligned}
$$

Thus $T^4(s^*, s^*)$ lies along the diagonal $y = x$. Finally, to prove (iv), assume conditions (6.10) and (6.11) are satisfied. Because $h = \bar{x}_{1,h}\bar{x}_{2,h}$, then it is obvious that $y_1 > 0$. Also, since

$$2\bar{x}_{1,h} < b \Leftrightarrow 2h < b\bar{x}_{2,h} \Leftrightarrow (\bar{x}_{2,h} + 1)(h + \bar{x}_{2,h}) \Leftrightarrow (\bar{x}_{2,h}^2 + (1 + h)\bar{x}_{2,h} - h) > 0$$

and $(\bar{x}_{2,h}^2 + (1 + h)\bar{x}_{2,h} - h) > 0$ is sufficient condition for $y_2, y_3 > 0$, then the proof is complete. □

The next result shows that our choice of s^* is based on an 8-periodic solution of the case $h = 1$.

Theorem 6.4. *Let $h = 1$ and assume that $b > 2(1 + \sqrt{2})$. The initial conditions $(y_{-1}, y_0) = (s^*, s^*)$ define an 8-periodic solution of Eq. (6.8).*

PROOF: The proof is computational. When $h = 1$, we obtain $\beta = 1$,

$$A = \alpha\sqrt{(\alpha^2 - 2)(\alpha^2 - 4\alpha + 2)} \quad \text{and} \quad s^* = \frac{1}{4(\alpha - 1)}(\alpha^3 + 4 - 2\alpha^2 - 2\alpha + A),$$

and consequently

$$y_1 = \frac{1}{2\alpha^2}(\alpha^3 - 2\alpha + A), y_2 = y_1^{conj}, y_3 = y_4 = y_0^{conj}, y_5 = y_2, y_6 = y_1,$$

where "*conj*" is used to mean that we replace A with $-A$. Finally, the strict inequality $b > 2(1 + \sqrt{2})$ makes the radicand in A positive and the period a minimal period. □

Consider $(x_{-1}, x_0) = (g_2 f_2^{-1}(s^*), f_2^{-1}(s^*))$ and define Γ_j to be the line segment connecting the point (x_{j-1}, x_j) with (x_j, x_{j+1}) for all $j = 0, 1, \ldots, 7$. For $j = 8$ and $\beta < 1$, we define Γ_8 to be the line segment connecting (x_7, x_8) with (x_{-1}, x_0). Observe that when $\beta = 1$, $(x_7, x_8) = (x_{-1}, x_0)$, and consequently Γ_8 shrinks to a point. Define

$$s_1 := \alpha^2 - \alpha - \beta \quad \text{and} \quad s_2 = \frac{(\alpha^2 - \beta)}{\beta} - \alpha \tag{6.15}$$

$$q_1(t) = \frac{\beta(1 - \beta)t(t - s_2)}{(\beta + t)^2} + (\alpha - 1)\frac{(1 + t)^2}{(\beta + t)^2} \tag{6.16}$$

$$q_2(t) = \frac{(\alpha - \beta)\beta^2(t - s_2)}{(t - s_1)}. \tag{6.17}$$

The next lemma is used in the proof of Theorem 6.5.

Lemma 6.1. *Assume that Condition (6.11) is satisfied, $h \leq 1$ and $s^* < s_1$. Then $q_2(s^*) \geq q_1(s^*)$.*

Proof. First observe that $s_1 - s_2 = \frac{1}{\beta}(1-\beta)(\beta - \alpha^2) < 0$ for all $h < 1$. Thus, $\bar{y} < s^* < s_1 < s_2$. Also,

$$q_1(\bar{y}) = q_2(\bar{y}) = \frac{\beta(\alpha-\beta)^3}{(\alpha-1)^2}.$$
$$q_2(0) - q_1(0) = \frac{\bar{x}_{2,h}}{h^2}\left(\frac{\bar{x}_{2,h}+h}{\bar{x}_{2,h}+1}\right)^2 > 0.$$

Next, we write

$$q_1(t) - q_2(t) = \frac{(\beta-1)(t-\bar{y})(At^2+Bt+C)}{(t+\beta)^2(s_1-t)},$$

where

$$\begin{aligned} A :=&(\beta+1)\bar{y} + 2\beta \\ C :=&\bar{y}\left[\beta(\alpha+1)(\beta+1)+\alpha\right] + \beta^2(\alpha+2) + \alpha\beta \\ B :=&\alpha\left[2(\alpha+1)(\beta+1) - \alpha^2 - 3\right] - \beta^2(\alpha+\beta+1) - \beta - 1. \end{aligned}$$

The task will be achieved if we show that $p_2(t) := At^2 + Bt + C$ is positive at $t = s^*$. Write A, B and C in terms of $\bar{x}_{2,h}$ and h, then substitute s^* from Eq. (6.13) and factor $p(s^*)$. Neglect the positive factors, then substitute $\bar{x}_{1,h}\bar{x}_{2,h}$ in place of h and factor again. Neglect the positive factors and collect the terms with a radical to obtain

$$\begin{aligned} &\left[(1-h)b^3 + 2\left(h^2+2h-3\right)b^2 + (1-h)(h^2-6h+1)b + 2h(1-h)^3\right]\sqrt{(b-h-1)^2-8h} + \\ &(1-h)b^4 + (3h^2+4h-3)b^3 + (-3h^3+5h^2-13h+3)b^2 - \\ &(1-h)(7h^3+11h^2-11h+1)b + 2h(3h-1)(1-h)^3, \end{aligned} \tag{6.18}$$

which must be positive. A graphing utility shows that it is indeed positive when $h < 1$ and Condition (6.11) is satisfied. □

Theorem 6.5. *Let $h \leq 1$ and assume that Condition (6.11) is satisfied. The region bounded by the closed curve defined by the line segments Γ_j, $j = 0, 1 \ldots, 8$ (Γ_8 is a point when $h = 1$) defines an invariant region for Eq. (6.8). Moreover, when $\beta < 1$, removing the segment Γ_8 makes the region a trapping region where orbits can enter through the removed segment and do not exit.*

Proof. Since the map T is one-to-one, then it is sufficient to show that T does not map the boundary of the closed curve outside. Since

$$\Gamma_0 \overset{T}{\to} \Gamma_1 \overset{T}{\to} \Gamma_2 \overset{T}{\to} \Gamma_3 \qquad \text{and} \qquad \Gamma_4 \overset{T}{\to} \Gamma_5 \overset{T}{\to} \Gamma_6 \overset{T}{\to} \Gamma_7,$$

then we need to check $T(\Gamma_3)$ and $T(\Gamma_8)$. Since

$$\Gamma_3 := \left\{(x,y):\ y = \frac{a}{1+s^*}x - \frac{bs^*}{1+s^*},\ f_1(s^*) \leq x \leq s^*\right\},$$

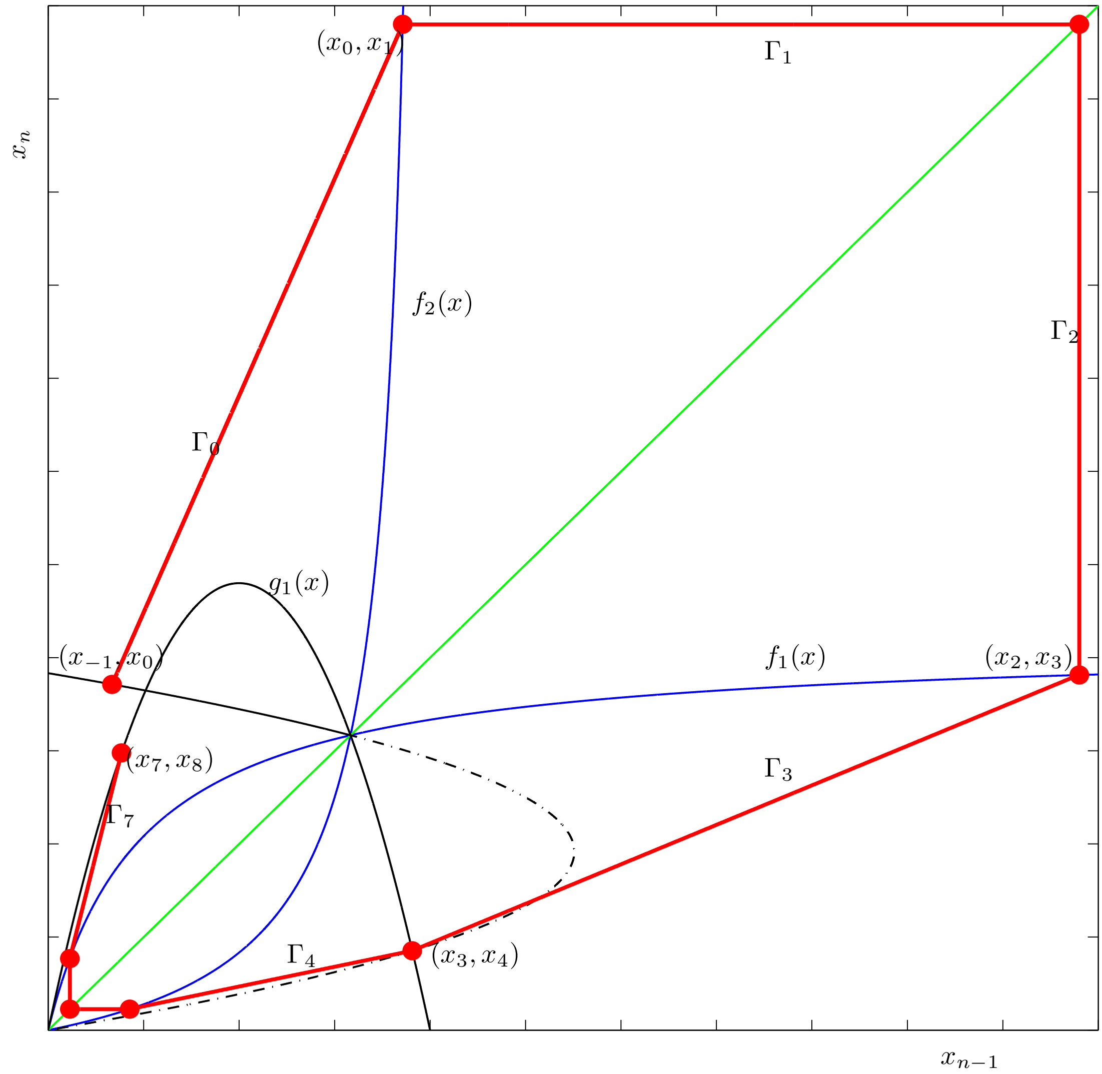


Figure 4: This figure shows the trapping region inside the red bold curve when $h < 1$.

then

$$T(\Gamma_3) = \left\{ (x, y) : y = \frac{(\alpha^2 - \beta(s^* + 1))x - \beta^2 s^*}{\alpha + \beta s^* + (s^* + 1)x}, \; \frac{\alpha f_1(s^*) - \beta s^*}{1 + s^*} \le x \le f_1(s^*) \right\}.$$

Since $T(\Gamma_3)$ defines a concave function, then it is above Γ_4. Also, since $T(x, y)$ is below the diagonal for all points (x, y) such that $y < x$ and $x \ge \bar{x}_2$, then $T(\Gamma_3)$ is within the region bounded by the curves $\Gamma_j, \; j = 0, \ldots, 8$. Next, we check $T(\Gamma_8)$. From the one-to-one property of T and Proposition 5.2, it is sufficient to show that $x_{-1} \le x_{-7}$ for all $h \le 1$. Observe that

$$\begin{aligned} x_{-1} \le x_7 \quad &\Leftrightarrow \quad g_2(f_2^{-1}(s^*)) \le \frac{(\alpha-\beta) f_2(g_1(f_1(s^*)))}{1 + f_2(g_1(f_1(s^*)))} \\ &\Leftrightarrow \quad [(\alpha-1) + (\beta-1) g_1(f_1(s^*))] g_2(f_2^{-1}(s^*)) \le (\alpha-\beta)\beta g_1(f_1(s^*)) \\ &\Leftrightarrow \quad [(\alpha-1) + (\beta-1) g_1(f_1(s^*))] \frac{s^*(s_1 - s^*)}{(\beta+s^*)^2} \le (\alpha-\beta)\beta g_1(f_1(s^*)), \end{aligned}$$

where s_1 as defined in (6.15). If $s^* \ge s_1$, then it is an obvious case since the intersection of the region bounded by the nine-gone defined by Γ_j with the positive quadrant forms an invariance. So, we proceed with $\bar{y} < s^* < s_1$ and obtain

$$[(1+s^*)^2(\alpha-1) + (\beta-1)\beta s^*(s_2 - s^*)] \frac{(s_1 - s^*)}{(\beta+s^*)^2} \le (\alpha-\beta)\beta^2(s_2 - s^*) \tag{6.19}$$

$$\frac{(\beta-1)\beta s^*(s_2 - s) + (\alpha-1)(1+s^*)^2}{(\beta+s^*)^2} \le \frac{(\alpha-\beta)\beta^2(s_2 - s^*)}{(s_1 - s^*)}, \tag{6.20}$$

where s_2 as defined in (6.15). Now, Lemma 6.1 completes the proof.

□

Finally, we remark that when Condition (6.6) is satisfied, then all solutions inside the trapping region as defined above are attracted to the positive equilibrium $\alpha - \beta - 1$. However, when Condition 6.6 is not satisfied but $h < 1$, then we conjecture that the same scenario happens.

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