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The effect of maps permutation on the global attractor of a periodic Beverton-Holt model

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Abstract

Consider a p-periodic difference equation $x_{n+1} = f_n(x_n)$ with a global attractor. How does a permutation $[f_{\sigma(p-1)}, \ldots, f_{\sigma(1)}, f_{\sigma(0)}]$ of the maps affect the global attractor? In this paper, we limit this general question to the Beverton-Holt model with p-periodic harvesting. We fix a set of harvesting quotas and give ourselves the liberty to permute them. The total harvesting yield is unchanged by the permutation, but the population geometric-mean may fluctuate. We investigate this notion and characterize the cases in which a permutation of the harvesting quotas has no effect or tangible effect on the population geometric-mean. In particular, as long as persistence is assured, all permutations within the dihedral group give same population geometric-mean. Other permutations may change the population geometric-mean. A characterization theorem has been obtained based on block reflections in the harvesting quotas. Finally, we associate directed graphs to the various permutations, then give the complete characterization when the periodicity of the system is four or five.

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1 Introduction

The Beverton-Holt model [6] is given by the first order difference equation

$$x_{n+1} = x_n f(x_n) := \frac{\mu k x_n}{k + (\mu - 1)x_n}, \quad n \in \mathbb{N} := \mathbb{Z}^+ \cup \{0\}, \tag{1.1}$$

where $x_0 \geq 0$ is the initial density of the population, k is the population carrying capacity and μ is the population growth rate. The Beverton-Holt model is a simple single-species model that

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assumes no lag on the effect of the environment on vital rates [29], and from the dynamics point of view, it is the discrete analog of the well-known continuous logistic model $x'(t) = rx(t)\left(1 - \frac{x(t)}{k}\right)$. In a periodically fluctuating environment, Cuching and Henson [15, 16] considered the Beverton-Holt model and raised the notion of Resonance and Attenuance, i.e., whether a periodic environment enhances or weakens population growth. This notion lead to a burst of research on the subject [18, 19, 24, 23, 25]. It has been found that although the Beverton-Holt model shows Attenuance when only the carrying capacity fluctuates [19, 23], the result in general is model dependent [21].

The effect of harvesting strategies on the dynamics of population models has been a topic of growing research interest [5, 4, 27, 8, 9, 10, 11, 12, 13, 30]. Various harvesting strategies on the Beverton-Holt model have been investigated by AlSharawi and Rhouma in [5, 4]. For sufficiently large initial populations, it has been found that constant harvest is more beneficial to both the population and the maximum sustainable yield. On the other hand, periodic harvest has a short term advantage when the initial population is low, and conditional harvest has the advantage of lowering the risk of extinction. In this paper, we force p-periodic harvesting on Eq. (1.1), i.e., we consider

$$x_{n+1} = x_n f(x_n) - \tilde{h}_n, \quad \tilde{h}_{n+p} = \tilde{h}_n \quad \text{for all} \quad n \in \mathbb{N}, \tag{1.2}$$

where $\{\tilde{h}_0, \ldots, \tilde{h}_{p-1}\}$ is a p-periodic sequence of harvesting quotas. Our motivation in considering this problem stems from mathematical and biological factors. The mathematical factor comes out of the interest in understanding the structure of cycles and attractors in periodic discrete systems [19, 1, 3, 20, 26, 7]. The combinatorial structure of cycles in one dimensional maps is an interesting branch of iteration theory [2] that lead to the characterization of periodic structures including Sharkovsky's Theorem and Chaos Theory. When a parameter is forced to be periodic as in Eq. (1.2), a sequence of maps need to be iterated rather than one single map, which adds a factor of complexity to the existences of cycles and their combinatorial structures. For more information on the subject, we refer the interested reader to [3, 1] and the references therein.

On the other hand, the biological factor is related to the paramount significance of harvesting strategies and their impact on the yield and sustainability. Mathematical analysis of population models with harvesting aids scientists and strategic catch regulators in designing an optimal strategy that addresses the market demand without compromising the sustainability of species abundance [14].

It is our belief that understanding the impact of the combinatorial arrangements of the harvesting sequence $[\tilde{h}_0, \ldots, \tilde{h}_{p-1}]$ on the population stock size will go along way in serving the aforementioned two factors. For instance, what happens to the global attract and the stock size of species represented by Eq. (1.2) when two terms (say $\tilde{h}_j, \tilde{h}_{j+1}$) in the harvesting sequence $[\tilde{h}_0, \ldots, \tilde{h}_j, \tilde{h}_{j+1}, \ldots, \tilde{h}_{p-1}]$ are interchanged? Addressing such a question can provide decision makers with a quantitative evaluation of the consequences of alternative actions. Although market demands can influence the harvesting sequence, species abundance may dictate another. This emphasizes the need to consider the combinatorial alternatives of the harvesting sequence in Eq. (1.2). A different narrative can be to consider one harvesting sequence with perturbation; however, this takes our problem to another direction in which the analysis becomes extremely challenging.

This paper is organized as follows: In section two, we give a preliminary that makes our paper self contained. In section three, we focus on block reflection of the harvesting quotas, which enables us to investigate their effect on the population geometric-mean. In section four, we focus on the concrete cases $p = 4$ and 5, and we associate directed graphs to the various options of block reflections, which give an illustration of our results. Finally, we close this paper by a conclusion and some discussion that opens the gate for more open questions about this subject.

2 Preliminaries

Consider the p-periodic harvesting sequence in Eq. (1.2) with $\tilde{h}_j \geq 0$ and $\mu > 1$. Always the periodicity is used to indicate the minimal period (or prime period), and therefore, we must have $\sum_{j=0}^{p-1} \tilde{h}_j > 0$. Let $\alpha = \frac{k}{\mu-1}$, $x_n = \alpha y_n$ and $h_n = \frac{\tilde{h}_n}{\alpha}$. Eq. (1.2) becomes

$$\text{(DE1)} \quad y_{n+1} = \frac{\mu y_n}{1+y_n} - h_n =: f_n(y_n), \quad \mu > 1, \;\; f_{n+p} = f_n \text{ and } h_n \geq 0 \;\text{ for all }\; n \in \mathbb{N}. \tag{2.1}$$

Thus, an orbit of Eq. (2.1) is given by the iterates of the maps f_n, i.e.,

$$\mathcal{O}^+(y_0) := \{y_0, f_0(y_0), f_1(f_0(y_0)), f_2(f_1(f_0(y_0))), \ldots\}. \tag{2.2}$$

A solution $\{y_n\}_{n=0}^{\infty}$ of (DE1) is called persistent if $y_n > 0$ for all $n \geq 0$. The set of all initial conditions that give persistent solutions is called the persistent set. When no persistent solutions exist in (DE1), the species modeled by (DE1) goes extinct. Thus, it is necessary to distinguish between the extinction of species (total collapse) and the non-persistence of certain populations at low densities. In the Beverton-Holt model, whether deterministic or periodic, it is straightforward to determine the threshold level between persistence and non-persistence, which we clarify here before we embark on our analysis. Note that in the case all maps are identical, we obtain the constant harvesting case, and Eq. (DE1) reduces to the autonomous equation $y_{n+1} = \frac{\mu y_n}{1+y_n} - h$. This equation has two equilibrium solutions when $0 < h < (\sqrt{\mu}-1)^2$, say $\bar{y}_1$ and $\bar{y}_2$. The large equilibrium solution (which we denote $\bar{y}_2$) is a global attractor with respect to the interior of the persistent set, i.e., $(\bar{y}_1, \infty)$. On the other hand, the small equilibrium solution $\bar{y}_1$ is a repeller and forms the floor of the persistent set. When the constant h is perturbed into a p-periodic sequence $\{h_j\}$, the attracting equilibrium $\bar{y}_2$ bifurcates into an attracting p-cycle

$$\bar{C}_2 := \left\{\overline{y}_{0,2}, \overline{y}_{1,2}, \cdots, \overline{y}_{p-1,2}\right\} \tag{2.3}$$

and the repeller $\bar{y}_1$ bifurcates into a repelling p-cycle

$$\bar{C}_1 := \left\{\overline{y}_{0,1}, \overline{y}_{1,1}, \cdots, \overline{y}_{p-1,1}\right\}. \tag{2.4}$$

Furthermore, $\bar{C}_1$ and $\bar{C}_2$ are ordered sets in the sense that they are written based on the initial time $n = 0$. If the initial time changes into $n = n_0$, then the elements of $\bar{C}_1$ and $\bar{C}_2$ have to be arranged accordingly. These facts are becoming classical by now; however, for more information, we refer the reader to [19, 4, 5, 3, 1]. Note that $h \leq (\sqrt{\mu}-1)^2$ is necessary and sufficient for assuring persistence in the autonomous equation, and $h_n \leq (\sqrt{\mu}-1)^2$ for all n is sufficient (but not necessary) for assuring persistence in the non-autonomous case. In fact, the persistent set

will be $[\overline{y}_{0,1}, \infty)$. Reference [4] gives further details on this topic. From the orbit in Eq. (2.2), we can fold the p-maps f_j into one map, namely $F_0 := f_{p-1} \circ f_{p-2} \circ \cdots \circ f_0$, and the iteration of this individual map can be used as a crude indicator of the behaviour of Eq. (2.1). AlSharawi *et. al.* [3] developed this notion into the so called "folding and unfolding", then used it to obtain valuable information about the length as well as the structure of cycles. The map F_0 has two fixed points if and only if Eq. (2.1) has two cycles. Furthermore, we must have $F_0(\overline{y}_{0,2}) = \overline{y}_{0,2}$ and $F_0(\overline{y}_{0,1}) = \overline{y}_{0,1}$, where $\overline{y}_{0,1}$ and $\overline{y}_{0,2}$ are the starting points of the cycles $\bar{C}_1$ and $\bar{C}_2$, respectively.

An interesting approach to tackle Eq. (2.1) is a special approach that uses matrix notation [4, 5]. Define $z_{n+1} := 1 + y_n$ to obtain the system

$$\begin{cases} z_{n+1} = & 1 + y_n, \\ y_{n+1} z_{n+1} = & -h_n + (\mu - h_n)\, y_n, \end{cases}$$

which can be written in vector form as

$$z_{n+1} \begin{bmatrix} 1 \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -h_n & \mu - h_n \end{bmatrix} \begin{bmatrix} 1 \\ y_n \end{bmatrix} =: A_n \begin{bmatrix} 1 \\ y_n \end{bmatrix}. \tag{2.5}$$

Now, an iteration process in Eq. (2.5) leads to

$$\left(\prod_{j=1}^{n} z_j \right) \begin{bmatrix} 1 \\ y_n \end{bmatrix} = A_{n-1} A_{n-2} \cdots A_0 \begin{bmatrix} 1 \\ y_0 \end{bmatrix}. \tag{2.6}$$

To avoid confusion, we agree to use the product notation in matrices as

$$\prod_{j=k}^{q} A_j = \begin{cases} A_q A_{q-1} \cdots A_k & \text{if} \quad q \geq k \\ I & \text{if} \quad q < k. \end{cases}$$

In the sequel of this paper, we use the matrix approach in Eq (2.5) and the "folding-unfolding" approach to achieve our task in investigating the effect of permuting the harvesting quotas on the geometric-mean of population cycles. We formalize the relationship between the eigenvalues of $A := A_{p-1} A_{p-1} \ldots A_0$ and the cycles of Eq. (2.1) in Lemma 2.1. The characteristic equation of A is $\lambda^2 - Tr(A)\lambda + \det(A) = 0$, and its solution is given by

$$\lambda_i = \frac{Tr(A) + (-1)^i \sqrt{(Tr(A))^2 - 4\det(A)}}{2} = \frac{Tr(A) + (-1)^i \sqrt{(Tr(A))^2 - 4\mu^p}}{2}, \tag{2.7}$$

where $i = 1, 2$. Obviously, $Tr(A) \geq 2\mu^{\frac{p}{2}}$ is necessary and sufficient for the existence of real eigenvalues; however, we replace it by the sufficient condition $h_j \leq (\sqrt{\mu} - 1)^2$ for all $j = 0, \ldots, p-1$ since this condition gives us more freedom to permute the harvesting quotas. We formalize this discussion in the following two results.

Lemma 2.1. *Let $A = A_{p-1} A_{p-2} \cdots A_0$ and assume $0 \leq h_j < (\sqrt{\mu} - 1)^2$ for all $j = 0, 1, \ldots, p-1$. Then Eq. (2.1) has a non-attracting p-cycle $\bar{C}_1$ and an attracting p-cycle $\bar{C}_2$ as given in Eqs. (2.4) and (2.3). Furthermore, the small and large eigenvalues (λ_1 and λ_2, respectively) of A are positive, satisfy*

$$\lambda_i = \prod_{j=0}^{p-1} (1 + \overline{y}_{j,i}), \ i = 1, 2, \tag{2.8}$$

and $\overline{y}_{j,i} > 0$ for all $i = 1, 2$, $j = 0, \ldots, p-1$.

Proof. The condition $0 \le h_j < (\sqrt{\mu}-1)^2$ assures the existence of two nonnegative equilibria of f_j for all $j = 0, 1, \cdots, p-1$. Define $x_0 := \max\{x : \ f_j(x) = x, \ j = 0, 1, \ldots, p-1\}$ and $F_0 := f_{p-1} \circ f_{p-2} \circ \cdots \circ f_0$, then the orbit of x_0 under F_0 persists. Since the folded map F_0 is of the same type as each f_j, i.e., increasing and asymptotic to a positive constant, we obtain two nonnegative equilibria of F_0 as well. On the other hand, from Eq. (2.6), we obtain

$$\lambda \begin{bmatrix} 1 \\ \overline{y} \end{bmatrix} = A_{p-1}A_{p-2}\cdots A_0 \begin{bmatrix} 1 \\ \overline{y} \end{bmatrix} = A \begin{bmatrix} 1 \\ \overline{y} \end{bmatrix}, \quad \text{and} \quad \lambda = \left(\prod_{j=0}^{p-1} z_j \right).$$

The eigenvector $\begin{bmatrix} 1 & \bar{y} \end{bmatrix}^t$ is in fact coming from the fixed points of F_0, which have been denoted by $\bar{y}_{0,1}$ and $\bar{y}_{0,2}$. Thus, $\bar{y}_{0,1}$ and $\bar{y}_{0,2}$ give rise to the non-attracting p-cycle $\bar{C}_1$ and the attracting p-cycle $\bar{C}_2$, respectively. Furthermore, the elements of the cycles are the fixed points of the folded maps $F_k = f_{p-1+k} \circ \cdots \circ f_{k+1} \circ f_k$, which are positive using the same argument. Thus, $\overline{y}_{j,i} > 0$ for all $i = 1, 2$, $j = 0, \ldots, p-1$. Finally,

$$\lambda_i = \prod_{j=0}^{p-1} \left(1 + \overline{y}_{j,i}\right) > 0 \quad \text{for} \quad i = 1, 2.$$

□

Proposition 2.1. *Consider Eq. (2.1) and its vector form in Eq. (2.5). Let* $F_0 = f_{p-1} \circ f_{p-2} \circ \ldots \circ f_0$ *and* $A = A_{p-1}A_{p-1}\ldots A_0$. *Then the following statements are equivalent:*

(i) A has positive eigenvalues
(ii) $Tr(A) \ge 2\mu^{\frac{p}{2}}$
(iii) F_0 *has a positive fixed point*
(iv) The species of Eq. (2.1) does not go extinct.

Proof. (i) ⇒ (ii): The product matrix A has positive eigenvalues, namely, λ_1 and λ_2 as given in Eq. (2.8) of Lemma 2.1. Because the eigenvalues are real, we obtain $(Tr(A))^2 \ge 4\mu^p$, and because they are positive, we obtain $Tr(A) \ge 2\mu^{\frac{p}{2}}$.
(ii) ⇒ (iii): It follows directly from lemma 2.1.
(iii) ⇒ (iv): Since each individual map f_j is monotonic and asymptotic to $\mu - h_j$, $h_j < \mu$, the folded map F_0 is monotonic and bounded. Thus, if F_0 has a positive fixed point, then all populations above the positive fixed point are within the persistent set. Hence the species of Eq. (2.1) does not go extinct.
(iv) ⇒ (i): The population survival and the monotonicity of F_0 assure the existence of a positive fixed point for the map F_0, and consequently, the matrix A must have positive eigenvalues. □

We end this section by the following illustrative example.

Example 2.1. *Consider the period* $p = 5$ *and* $\mu = 16$. *Since we are assuming* $0 \le h_j \le (\sqrt{\mu}-1)^2 = 9$, *we consider* $h_j = j$ *for* $j = 0, 1, 2, 3$ *and* $h_4 = h$, *where* $0 \le h \le 9$. *Based on the notations of Proposition 2.1, we have*

$$A = \begin{bmatrix} -224 & 45856 \\ 224h - 3328 & 676608 - 45856h \end{bmatrix}$$

with $det(A) = 16^5$ *and* $Tr(A) = 676384 - 45856h$. *For* $0 \le h \le 9$, *we have* $263680 \le Tr(A) \le 676384$. *Therefore, A has positive eigenvalues and* $Tr(A) \ge 2\mu^{\frac{p}{2}} = 2048$.

Next, F_0 *is given by*

$$F_0(y) = \frac{(21144 - 1433h)y + 7h - 104}{1433y - 7},$$

which has two positive fixed points for all $0 \le h \le 9$. *In particular, if we fix* $h = \frac{75214}{8591} \approx 8.75$, *then the fixed points of* F_0 *are*

$$\bar{y}_{0,1} = \frac{61161}{12310903} \approx 0.005 \quad \textit{and} \quad , \bar{y}_{0,2} = 6.$$

Furthermore, the repelling 5*-cycle is given by*

$$\bar{C}_1 := \left\{ \frac{61161}{12310903}, \frac{61161}{773254}, \frac{144161}{834415}, \frac{21839}{61161}, \frac{12553}{10375} \right\}$$

and the attractor is given by the 5*-cycle*

$$\bar{C}_2 := \left\{ 6, \frac{96}{7}, \frac{1433}{103}, \frac{1241}{96}, \frac{15845}{1337} \right\}.$$

3 Permuting the harvesting quotas and population cycles

In this section, we focus on Eq. (2.1) in which the harvesting quotas $\{h_0, h_1, \ldots, h_{p-1}\}$ are fixed and satisfy $0 \le h_j < (\sqrt{\mu} - 1)^2$ for all $j = 0, \ldots, p-1$. Let $[\hat{h}_0, \hat{h}_1, \ldots, \hat{h}_{p-1}]$ be a permutation of $[h_0, h_1, \ldots, h_{p-1}]$. Eq. (2.1) becomes

$$\text{(DE2)} \qquad y_{n+1} = \frac{\mu y_n}{1 + y_n} - \hat{h}_n =: \hat{f}_n(y_n), \quad \hat{f}_{n+p} = \hat{f}_n. \tag{3.1}$$

Obviously, we have $p!$ choices for the sequence $\{\hat{h}_j\}$. However, since a time shift ($n = n_0$ rather than $n = 0$) has no effect on the dynamics of a non-autonomous difference equation, we are left with $(p-1)!$ choices. Theorem 3.1, which is a modification of a result obtained in [5] narrows down the choices into $\frac{1}{2}(p-1)!$ when $p \ge 3$.

To proceed, we recall the concept of similarity in matrices since it is used in the sequel. Here, we are interested in real matrices. Two matrices $A, B \in \mathbb{R}^{n \times n}$ are called similar if there exists an invertible matrix S such that $AS = SB$. It is well known [22] that every complex matrix is similar to its Jordan matrix, and every Jordan block is permutation similar to its transpose. Thus, a complex matrix is similar to its transpose. In fact, more can be said: a matrix $A \in \mathbb{R}^{n \times n}$ is similar to its transpose through a similarity matrix S that is symmetric [28], i.e., $A = SA^tS^{-1}$ and $S^t = S$. We call two sequences of matrices $\{A_j\}$ and $\{B_j\}$ simultaneously similar if there exists an invertible matrix S such that $A_jS = SB_j$ for all values of j. Observe that S is the same for all elements of the sequence. For instance, the sequence of matrices $\{A_j\}$ as given in Eq. (2.5) is simultaneously similar to the sequence of their corresponding transpose $\{A_j^t\}$. It is an elementary task to check that $A_jS = SA_j^t$, where a similarity matrix S can be taken to be the symmetric matrix

$$S = \begin{bmatrix} -1 & 1 \\ 1 & \mu - 1 \end{bmatrix}. \tag{3.2}$$

Now, we give the following lemma, which is used in our consequent theorem.

Lemma 3.1. *Let $S_k = \{A_k\}$ be a sequence of matrices in $\mathbb{R}^{n\times n}$ that is simultaneously similar to its transpose S_k^t. For each $j \geq i$, $A_jA_{j-1}\cdots A_i$ and $A_iA_{i+1}\cdots A_j$ are similar.*

Proof. Because S_k is simultaneously similar to its transpose, then there exists an invertible matrix S such that $A_kS = SA_k^t$ for all k. Also, the fact that a matrix and its transpose are similar assures the existence of an invertible matrix P such that

$$A_jA_{j-1}\cdots A_iP = P(A_jA_{j-1}\cdots A_i)^t = PA_i^tA_{i+1}^t\cdots A_j^t.$$

Thus, we obtain

$$A_jA_{j-1}\cdots A_iP = PS^{-1}A_iA_{i+1}\cdots A_jS,$$

and consequently, $A_jA_{j-1}\cdots A_i$ and $A_iA_{i+1}\cdots A_j$ are similar with a similarity matrix PS^{-1}. □

Theorem 3.1. *Consider Eq. (3.1) in which $[\hat{h}_0, \hat{h}_1, \ldots, \hat{h}_{p-1}]$ is a permutation of $[h_0, h_1, \ldots, h_{p-1}]$. Assume $0 \leq h_j < (\mu-1)^2$ for all j. All permutations in the dihedral group $\mathcal{D}_p$ give same population geometric-mean.*

Proof. No matter which permutation we consider, the conditions $0 \leq h_j < (\mu-1)^2$ assure the existence of an attracting cycle. Now, the population geometric-mean is determined by the geometric mean of the attracting cycle, which is

$$\left(\prod_{j=0}^{p-1}(1+\hat{\overline{y}}_{j,2})\right)^{\frac{1}{p}} = (\lambda_2)^{\frac{1}{p}} = \left(\frac{1}{2}\left(Tr(A) + \sqrt{(Tr(A))^2 - 4\mu^p}\right)\right)^{\frac{1}{p}}, \tag{3.3}$$

where $A = A_{p-1}A_{p-2}\cdots A_0$ and for each j, $\overline{y}_{j,2}$ is an element of the attracting cycle related to the folded map $f_{p-1+j}\circ f_{p-2+j}\circ\ldots\circ f_j$. Note that the indexes are taken mod p all the time and the matrix A_j is associated with the map f_j, which belongs to h_j. Now, the elements of the dihedral group are rotations and reflections. Obviously, rotations give the same geometric-mean because we can consider time shift in the difference equation, i.e., we start at $n = n_0$ rather than $n = 0$. Another simple justifications is to use the cyclic property of the trace $Tr(UV) = Tr(VU)$. Thus, it remains to clarify the reflections. This can be done by proving that $Tr(A_{p-1}A_{p-2}\cdots A_0) = Tr(A_0A_1\cdots A_{p-1})$, which follows from Lemma 3.1 since similar matrices have same eigenvalues. □

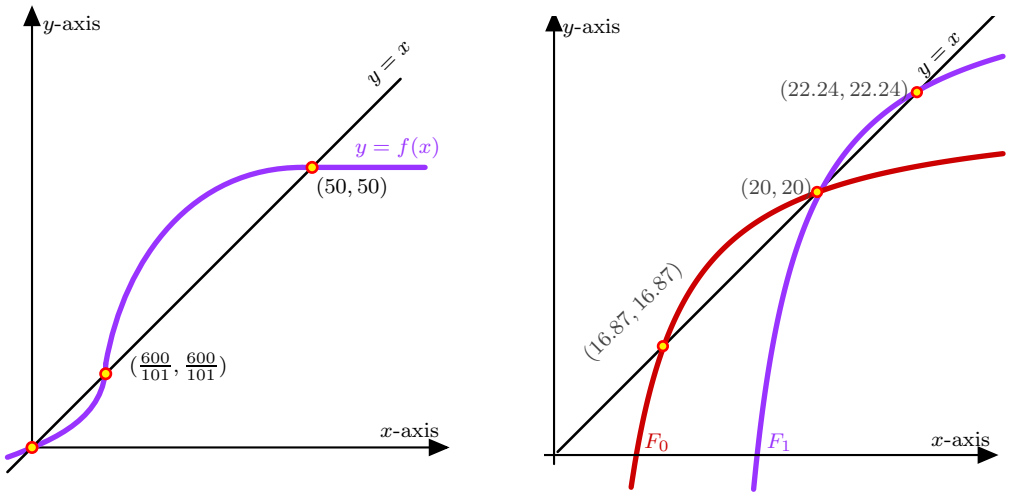
An immediate consequence of Theorem 3.1 is the following: If the difference equation has period $p = 2$ or $p = 3$, then permuting the harvesting quotas has no effect on the population geometric-mean. At this point, it is interesting to know whether or not the result of Theorem 3.1 is part of a universal phenomenon. We address this issue in the following example by considering monotonic maps f_j with Allee effect.

Example 3.1. *Consider $x_{n+1} = f(x_n)$, where $f(x) = \frac{x(\alpha x+\beta)}{(x^2+x+\gamma)}$ and $\alpha, \beta, \gamma > 0$. If we consider $\alpha = k+a+1$, $\beta = 0$ and $\gamma = ak$, $0 < a < k$, then we obtain a Beverton-Holt model with an Allee effect [17]. In this case, $\bar{x} = a$ is the Allee equilibrium and $\bar{x} = k$ is the attracting equilibrium with basin of attraction (a, ∞). Also, observe that $f(x)$ is monotonic and asymptotic to $\alpha = k+a+1$. Now, we force periodic harvesting of period three, i.e., we consider $f_n(x) = f(x) - h_n$, where*

$h_n \in \{h_0, h_1, h_2\}$. *We are interested in comparing the population geometric mean for the* 3*-periodic systems* $[f_0, f_1, f_2]$ *and* $[f_1, f_0, f_2]$. *We articulate our choice of the parameters as follows: Fix*

$$k = 50,\ a = \frac{600}{101},\ h_0 = \frac{30}{17},\ h_1 = \frac{200}{17} \quad \textit{and} \quad h_2 = \frac{370}{17}.$$

Since $f_0(20) = 30,\ f_1(30) = 30$ *and* $f_2(30) = 20$, *the system* $[f_0, f_1, f_2]$ *has the 3-cycle* $C_2 := \{20, 30, 30\}$. *Indeed, it is the attracting cycle since the Allee cycle is given by (with rounding to six decimal places)* $C_1 := \{16.866494, f_0(16.866494), f_1(f_0(16.866494))\}$. *On the other hand, since* $f_1(20) = 20,\ f_0(20) = 30$ *and* $f_2(30) = 20$, *the system* $[f_1, f_0, f_2]$ *has the* 3*-cycle* $\hat{C}_1 := \{20, 20, 30\}$. *This cycle is the Allee cycle since the attracting cycle is given by (with rounding to six decimal places)* $\hat{C}_2 := \{22.236357, f_1(22.236357), f_0(f_1(22.236357))\}$. *The product of the elements of* C_2 *is* 18000 *while the product of the elements of* $\hat{C}_2$ *is* 17023.995281. *Fig. 3 gives an illustration of our maps.*



(a) (b)

Figure 1: The map in Figure (a) represents a Beverton-Holt model with an Allee effect. The maps in Fig. (b) represent the folded maps $F_0(x) = f_2(f_1(f_0(x)))$ and $F_1(x) = f_2(f_0(f_1(x)))$ in the Beverton-Holt model with an Allee effect and 3-periodic harvesting.

Up to this end, we addressed the issue of reflecting all harvesting quotas, and that lead us to Theorem 3.1. However, a more general issue to address is the reflection of a block (not necessarily the whole sequence) of harvesting quotas, and see the effect on the population geometric mean. In particular, we focus on comparing the geometric mean of the populations in (DE1) and (DE2) when a block is reflected as follows:

$$[f_{p-1}, \ldots, f_{j+1}, \boxed{f_j, f_{j-1}, \ldots, f_i}, f_{i-1}, \ldots, f_0] \tag{3.4}$$

versus

$$[f_{p-1}, \ldots, f_{j+1}, \boxed{f_i, f_{i+1}, \ldots, f_j}, f_{i-1}, \ldots, f_0]. \tag{3.5}$$

To avoid the trivial block of length 1, we consider $j > i$. On the other hand, if the reflected block is of length p or $p-1$, then the first case has been addressed by Theorem 3.1, while the latter case is obvious due to the use of the trace cyclic property and Theorem 3.1. Thus, we proceed to tackle the case when the reflected block is of length $j-i+1 \neq 1, p-1$ or p, i.e., $0 < j-i < p-2$. This narrows down to investigating the sign of the expression

$$\begin{aligned} T :=& Tr\left(A_{p-1}A_{p-2}\cdots A_0\right) - Tr\left(A_{p-1}\cdots A_{j+1}A_iA_{i+1}\cdots A_jA_{i-1}\cdots A_0\right) \\ =& Tr\left[A_{p-1}\cdots A_{j+1}\left(A_jA_{j-1}\cdots A_i - A_iA_{i+1}\cdots A_j\right)A_{i-1}A_{i-2}\cdots A_0\right] \\ =& Tr\left[A_{p-1}\cdots A_{j+1}RA_{i-1}A_{i-2}\cdots A_0\right], \end{aligned}$$

where

$$R := \left(A_jA_{j-1}\cdots A_i - A_iA_{i+1}\cdots A_j\right). \tag{3.6}$$

We give a characterization of R in the next general lemma.

Lemma 3.2. *Let $S_k = \{A_k\}$ be a sequence of matrices in $\mathbb{R}^{n\times n}$ that is simultaneously similar to its transpose S_k^t through a similarity matrix S . For each $j \geq i$, if μ is an eigenvalue of the matrix*

$$R_{i,j} = A_jA_{j-1}\cdots A_i - A_iA_{i+1}\cdots A_j,$$

then $-\mu$ is also an eigenvalue. Furthermore, when S is symmetric, the matrices $S^{-1}R_{i,j}$ are skew symmetric with eigenvalues zero or pure imaginary.

Proof. We have

$$\begin{aligned} R_{i,j}^t =& \left(A_jA_{j-1}\cdots A_i - A_iA_{i+1}\cdots A_j\right)^t \\ =& S^{-1}A_iA_{i+1}\cdots A_jS - S^{-1}A_jA_{j-1}\cdots A_iS \\ =& -S^{-1}R_{i,j}S. \end{aligned}$$

Since we are in the field $\mathbb{R}$, $R_{i,j}$ is necessarily singular when n is odd. Next, suppose λ is a nonzero eigenvalue of $R_{i,j}$, then

$$R_{i,j}X = \lambda X \quad \Leftrightarrow S^{-1}R_{i,j}X = \lambda S^{-1}X \quad \Leftrightarrow \quad (S^{-1}R_{i,j} - \lambda S^{-1})X = 0.$$

Replace $S^{-1}R_{i,j}$ by $-R_{i,j}^tS^{-1}$ to obtain $(R_{i,j}^t + \lambda I)S^{-1}X = 0$. Therefore, $-\lambda$ is an eigenvalue of $R_{i,j}^t$, and consequently, an eigenvalue of $R_{i,j}$. Hence, the eigenvalues of $R_{i,j}$ are either zeros or come in couples of the form $\pm\mu$.

To verify the second part of the assertion, assume the invertible matrix S is symmetric. The fact that $R_{i,j}^t = -S^{-1}R_{i,j}S$ implies

$$S^{-1}R_{i,j} = -R_{i,j}^tS^{-1} = -(S^{-1}R_{i,j})^t.$$

Thus, $S^{-1}R_{i,j}$ is skew symmetric. It is well-known that the eigenvalue of a skew symmetric matrix are zero or pure imaginary. □

Based on Lemma 3.2, and back to the particular 2×2 matrices that we have, write

$$T = Tr\left[A_{p-1}\cdots A_{j+1}S(S^{-1}R)A_{i-1}A_{i-2}\cdots A_0\right]. \tag{3.7}$$

Since $S^{-1}R$ is skew symmetric, we write $S^{-1}R = \alpha S^*$ for some scalar $\alpha := \alpha(h_i, \ldots, h_j)$ and S^* is the orthogonal matrix

$$S^* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, the expression of T in Eq. (3.7) becomes

$$T = \alpha(h_i, \ldots, h_j) Tr\left[A_{p-1} \cdots A_{j+1} C_1 A_{i-1} A_{i-2} \cdots A_0\right], \tag{3.8}$$

$$\text{where} \quad C_1 = SS^* = \begin{bmatrix} 1 & 1 \\ \mu - 1 & -1 \end{bmatrix}.$$

Next, we give an interesting representation of the expression $\alpha(h_i, \ldots, h_j)$.

Theorem 3.2. *The value of α in Eq. (3.8) is given by the expression*

$$\alpha(h_i, \ldots, h_j) = \frac{1}{\mu} Tr\left(C_1 A_j \ldots A_i\right).$$

Proof. Observe that

$$\begin{aligned} Tr\left(A_j \cdots A_i C_1\right) =& Tr\left(\left(A_j \cdots A_i C_1\right)^t\right) \\ =& Tr\left(C_1^t A_i^t \cdots A_j^t\right) \\ =& Tr\left(S^{*t} A_i \cdots A_j S\right) \\ =& - Tr\left(S^* A_i \cdots A_j S\right) \\ =& - Tr\left(A_i \cdots A_j C_1\right). \end{aligned}$$

Now, we have $R = \alpha S S^* = \alpha C_1$. Thus, $\alpha I = R C_1^{-1} = \frac{1}{\mu} R C_1$, and consequently,

$$\begin{aligned} 2\alpha =& Tr(\alpha I) = \frac{1}{\mu} Tr(R C_1) \\ =& \frac{1}{\mu} \left(Tr\left(A_j \cdots A_i C_1 - A_i \cdots A_j C_1\right)\right) \\ =& \frac{2}{\mu} Tr\left(A_j \cdots A_i C_1\right). \end{aligned}$$

Hence, we obtain

$$\alpha(h_i, \ldots, h_j) = \frac{1}{\mu} Tr\left(A_j \cdots A_i C_1\right) = -\frac{1}{\mu} Tr\left(A_i \cdots A_j C_1\right),$$

and the proof is complete. □

Notice that Eq. (3.8) and Theorem 3.2 are becoming the core results of our paper. The expression of $Tr(C_1 A_j \cdots A_i)$ can be gigantic and not simple to handle computationally when the length of the reflected block is large. We give the following illustrative example when the difference between i and j is $1, 2$ or 3.

Example 3.2. *Each of the following holds true:*

(i) If the reflected block of maps is $[f_{i+1}, f_i]$, *then*

$$\alpha(h_i, h_{i+1}) = \frac{1}{\mu} Tr(C_1 A_{i+1} A_i) = (h_{i+1} - h_i). \tag{3.9}$$

(ii) If the reflected block of maps is $[f_{i+2}, f_{i+1}, f_i]$, *then*

$$\alpha(h_i, h_{i+1}, h_{i+2}) = \frac{1}{\mu} Tr(C_1 A_{i+2} A_{i+1} A_i) = (h_{i+2} - h_i)(1 + \mu - h_{i+1}). \tag{3.10}$$

(iii) If the reflected block of maps is $[f_{i+3}, f_{i+2}, f_{i+1}, f_i]$, *then*

$$\begin{aligned} \alpha =& \alpha(h_i, h_{i+1}, h_{i+2}, h_{i+3}) = \frac{1}{\mu} Tr(C_1 A_{i+3} A_{i+2} A_{i+1} A_i) \\ &= \left[(h_{i+3} - h_i) \left(\mu^2 + (\mu + 1)(1 - h_{i+1} - h_{i+2}) + h_{i+1} h_{i+2} \right) + \mu(h_{i+2} - h_{i+1}) \right]. \end{aligned}$$

Next, we proceed to investigate further the sign of the expression obtained by Theorem 3.2. Certainly, we need to decipher the signs of the two expressions

$$Tr\left[C_1 A_j \cdots A_i\right] \qquad \text{and} \qquad Tr\left[A_{p-1} \cdots A_{j+1} C_1 A_{i-1} A_{i-2} \cdots A_0\right].$$

We decompose the matrix A_j using the sum of two constant matrices as

$$A_j = B - h_j K, \quad \text{where} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & \mu \end{bmatrix} \text{ and } K = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \tag{3.11}$$

then depend on the fact that $A_q C_1 A_k = \mu C_1 + \mu(h_k - h_q)K$ to establish the following result.

Lemma 3.3. *Let* $p \geq 4$, $p - 1 \geq j > i \geq 0$, *and consider the matrices* K *and* C_1 *as defined in Eq. (3.11) and Eq. (3.8), respectively. Each of the following holds true:*

(i) *We have*

$$Tr\left(\left(\prod_{k=j}^{p-1} A_k \right) K \left(\prod_{q=0}^{i} A_q \right) \right) > 0.$$

(ii) *Let* s *be the floor integer* $\lfloor \frac{j-i+1}{2} \rfloor$. *If* $h_{j-t} \geq h_{i+t}$ *for all* $t = 0, \cdots, s-1$, *then*

$$Tr\left(C_1 \left(\prod_{q=i}^{j} A_q \right) \right) \geq 0,$$

and reversing the inequalities in the harvesting quotas will have the opposite effect on the expression.

Proof. **(i)** If the products on the left and right of K are empty, i.e., the identity matrix, then we obtain $Tr(K) = 1 > 0$. If the product on the right is not empty while the one on the left is empty, then we have

$$Tr\left(K \left(\prod_{q=0}^{i} A_q \right) \right),$$

which is the sum of the coefficients of x in $f_i \circ f_{i-1} \circ \cdots \circ f_0(x)$. This must be positive for all values of i because of the monotonicity of the composite map and it is asymptotic behaviour above $(\sqrt{\mu}-1)$ due to the conditions $h_n \leq (\sqrt{\mu}-1)^2$. If the product on the left is not empty, then we can use the trace cyclic property and rotate the matrices from the left of the expression to its right. Now, as before, the trace will be the sum of the coefficients of x in

$$f_i \circ f_{i-1} \circ \cdots \circ f_0 \circ f_{p-1} \circ f_{p-2} \circ \cdots \circ f_j(x),$$

which is again positive by the nature of the composition. Hence, in all cases, the trace is positive and the proof of Part (i) is complete.

(ii) Write

$$\begin{aligned} Tr\left(C_1\left(\prod_{q=i}^{j} A_q\right)\right) &= Tr\left(A_i C_1 A_j \left(\prod_{q=i+1}^{j-1} A_q\right)\right) \\ &= Tr\left[(\mu C_1 + \mu(h_j - h_i)K)A_{j-1}\cdots A_{i+1}\right] \\ &= \mu Tr(C_1 A_{j-1}\cdots A_{i+1}) + \mu(h_j - h_i)Tr(KA_{j-1}\cdots A_{i+1}). \end{aligned}$$

Continue to expand by induction, we obtain

$$Tr\left(C_1\left(\prod_{q=i}^{j} A_q\right)\right) = \mu^s Tr\left(H\right) + \sum_{t=1}^{s} \mu^t (h_{j+1-t} - h_{i+t-1}) Tr\left(K \prod_{q=i+t}^{j-t} A_q\right),$$

where $s = \lfloor \frac{j-i+1}{2} \rfloor$, and the matrix $H \in \{C_1, C_1 A_{s+1}\}$. Observe that $Tr(C_1) = Tr(A_q C_1) = Tr(C_1 A_q) = 0$ for all $q = i, \ldots, j$. Thus, the term $\mu^s Tr\left(H\right) = 0$, and based on Part (i), the proof of Part (ii) is complete.

□

Next, we need to tackle the expression $Tr\left[A_{p-1}\cdots A_{j+1} C_1 A_{i-1} A_{i-2}\cdots A_0\right]$. Observe that if $A_{p-1}\cdots A_{j+1}$ or $A_{i-1}A_{i-2}\cdots A_0$ is empty (i.e., the identity matrix), then Part (ii) of Lemma 3.3 addresses the issue, and Corollary 3.1 gives the conclusion. Otherwise, we explore the expression in the following result.

Theorem 3.3. *Consider the expression $Tr\left[A_{p-1}\cdots A_{j+1} C_1 A_{i-1} A_{i-2}\cdots A_0\right]$, in which neither $A_{p-1}\cdots A_{j+1}$ nor $A_{i-1}A_{i-2}\cdots A_0$ is empty. Each of the following holds true:*

(i) Let $i+j \in \{p, p-1, p-2\}$. If $h_{i-t} \geq h_{j+t}$ for all $t = 1, \ldots, i$, then $Tr(A_{p-1}\cdots A_{j+1} C_1 A_{i-1}\cdots A_0) \geq 0$, and reversing the inequalities in the harvesting quotas will have the opposite effect on the expression.

(ii) Let $i + j < p - 2$ and consider $s = \lfloor \frac{p-j-i-1}{2} \rfloor$. If $h_{i-t} \geq h_{j+t}$ for all $t = 1, \ldots, i$ and $h_{p-1-t} \geq h_{i+j+1+t}$ for all $t = 0, \ldots, s$, then

$$Tr\left(A_{p-1}\cdots A_{j+1} C_1 A_{i-1}\cdots A_0\right) \geq 0.$$

Reversing the inequalities in the harvesting quotas will have the opposite effect on the expression.

(iii) Let $i+j > p$ and consider $s = \lfloor \frac{i+j+1-p}{2} \rfloor$. If $h_{i-t} \geq h_{j+t}$ for all $t = 1, \ldots, p-1-j$ and $h_{i+j-p-t} \geq h_t$ for all $t = 0, \ldots, s$, then

$$Tr\left(A_{p-1} \cdots A_{j+1} C_1 A_{i-1} \cdots A_0\right) \geq 0.$$

Reversing the inequalities in the harvesting quotas will have the opposite effect on the expression.

Proof. **(i)** The idea is similar to Part (ii) of Lemma 3.3. Expand $Tr(A_{p-1} \cdots A_{j+1} C_1 A_{i-1} \cdots A_0)$ as

$$\mu Tr(A_{p-1} \cdots A_{j+2} C_1 A_{i-2} \cdots A_0) + \mu(h_{i-1} - h_{j+1}) Tr(A_{p-1} \cdots A_{j+2} K A_{i-2} \cdots A_0),$$

then continue the expansion on the left term to obtain $Tr(A_{p-1} \cdots A_{j+1} C_1 A_{i-1} \cdots A_0) =$

$$\mu^i Tr(H) + \sum_{t=1}^{i} \mu^t (h_{i-t+1} - h_{j+t-1}) Tr\left(\left(\prod_{k=j+t}^{p-1} A_k\right) K \left(\prod_{q=0}^{i-t} A_q\right)\right),$$

Where $H \in \{C_1, C_1 A_0, A_{p-1} C_1\}$. Since $Tr(H) = 0$, the conclusion of Part (i) becomes clear.

(ii) Since $i + j < p-2$, the expansion of the expression $Tr(A_{p-1} \cdots A_{j+1} C_1 A_{i-1} \cdots A_0)$ takes the form

$$\mu^i Tr(A_{p-1} \cdots A_{i+j+1} C_1) + \sum_{t=1}^{i} \mu^t (h_{i-t} - h_{j+t}) Tr\left(\left(\prod_{k=j+t+1}^{p-1} A_k\right) K \left(\prod_{q=0}^{i-(t+1)} A_q\right)\right).$$

Invoke Part (ii) of Lemma 3.3 on $Tr(A_{p-1} \cdots A_{i+j+1} C_1)$ to obtain the conclusion.

(iii) Since $i + j > p$, the expansion of the expression $Tr(A_{p-1} \cdots A_{j+1} C_1 A_{i-1} \cdots A_0)$ takes the form

$$\mu^{p-1-j} Tr(C_1 A_{i+j-p} \cdots A_0) + \sum_{t=1}^{p-1-j} \mu^t (h_{i-t} - h_{j+t}) Tr\left(\left(\prod_{k=j+t+1}^{p-1} A_k\right) K \left(\prod_{q=0}^{i-(t+1)} A_q\right)\right).$$

Now, invoke Part (ii) of Lemma 3.3 on $Tr(C_1 A_{i+j-p} \cdots A_0)$ to obtain the conclusion.

□

Next, we utilize our previous results to summarise the effect of reflecting a block of harvesting quotas on the population geometric mean in the following two corollaries.

Corollary 3.1. *Let $p \geq 4$, and assume $0 < h_j \leq (\sqrt{\mu} - 1)^2$ for all $j = 0, \ldots, p-1$. Suppose the block of maps $[f_j, \cdots, f_i]$ in DE1 is reflected as $[f_i, \cdots, f_j]$ to give DE2. Consider $s_1 = \lfloor \frac{j-i+1}{2} \rfloor$, $s_2 = \lfloor \frac{i}{2} \rfloor$ and $s_3 = \lfloor \frac{p-1-j}{2} \rfloor$. Each of the following holds true:*

(i) If $(i, j) = (0, p-1), (1, p-1)$ or $(0, p-2)$, then the geometric-mean of the species in DE2 equals the geometric-mean of the species in DE1.

(ii) Let $2 \leq i < j = p-1$. If

(a) $h_{j-t} \geq h_{i+t}$ for all $t = 0, \cdots, s_1 - 1$, and

(b) $h_{i-1-t} \geq h_t$ *for all* $t = 0, \ldots, s_2 - 1$,

then the geometric-mean of the species in DE2 is smaller than the geometric-mean of the species in DE1. Reversing all the inequalities in (a) and (b) have the same effect, while reversing the inequalities in one of them only have the opposite effect.

(iii) Let $0 = i < j \leq p - 3$. *If*

(a) $h_{j-t} \geq h_{i+t}$ *for all* $t = 0, \cdots, s_1 - 1$, *and*

(b) $h_{p-1-t} \geq h_{j+1+t}$ *for all* $t = 0, \ldots, s_3 - 1$,

then the geometric-mean of the species in DE2 is smaller than the geometric-mean of the species in DE1. Reversing all the inequalities in (a) and (b) have the same effect, while reversing the inequalities in one of them only have the opposite effect.

Proof. Part (i) is the conclusion of Theorem 3.1. To prove (ii), observe that the condition $2 \leq i < j = p - 1$ avoids the discrepancy with part (i) and makes $A_{p-1} \cdots A_{j+1} = I$ in $Tr(A_{p-1} \cdots A_{j+1} C_1 A_{i-1} \cdots A_0)$. Now, invoke Part (ii) of Lemma 3.3 on both $Tr(C_1 A_j \cdots A_i)$ and $Tr(C_1 A_{i-1} \cdots A_0)$ to obtain the result. For part (iii), the condition $0 = i < j \leq p - 3$. avoids the discrepancy with part (i) and makes $A_{i-1} \cdots A_0 = I$ in $Tr(A_{p-1} \cdots A_{j+1} C_1 A_{i-1} \cdots A_0)$. Now, invoke Part (ii) of Lemma 3.3 on both $Tr(C_1 A_j \cdots A_i)$ and $Tr(C_1 A_{p-1} \cdots A_{j+1})$ to obtain the result. □

Corollary 3.2. *Let* $p \geq 4, i \neq 0, j \neq p-1$, *and assume* $0 < h_j \leq (\sqrt{\mu} - 1)^2$ *for all* $j = 0, \ldots, p-1$. *Suppose the block of maps* $[f_j, \cdots, f_i]$ *in DE1 is reflected as* $[f_i, \cdots, f_j]$ *to give DE2. Consider* $s_1 = \lfloor \frac{j-i+1}{2} \rfloor$. *Each of the following holds true:*

(i) Let $i + j \in \{p, p-1, p-2\}$. *If*

(a) $h_{j-t} \geq h_{i+t}$ *for all* $t = 0, \cdots, s_1 - 1$, *and*

(b) $h_{i-t} \geq h_{j+t}$ *for all* $t = 1, 2, \ldots, i$,

then the geometric-mean of the species in DE2 is smaller than the geometric-mean of the species in DE1. Reversing all the inequalities in (a) and (b) have the same effect, while reversing the inequalities in one of them only have the opposite effect.

(ii) Let $i + j < p - 2$ *and consider* $s_2 = \lfloor \frac{p-j-i-1}{2} \rfloor$. *If*

(a) $h_{j-t} \geq h_{i+t}$ *for all* $t = 0, \cdots, s_1 - 1$,

(b) $h_{i-t} \geq h_{j+t}$ *for all* $t = 1, \ldots, i$ *and*

(c) $h_{p-1-t} \geq h_{i+j+1+t}$ *for all* $t = 0, \ldots, s_2$

are all satisfied, then the geometric-mean of the species in DE2 is smaller than the geometric-mean of the species in DE1. Reversing all the inequalities in (a), (b) and (c) have the same effect. Reversing the inequalities only in (a) (or only in (b) & (c)) have the opposite effect.

(iii) Let $i + j > p$ *and consider* $s_3 = \lfloor \frac{i+j+1-p}{2} \rfloor$. *If*

(a) $h_{j-t} \geq h_{i+t}$ *for all* $t = 0, \cdots, s_1 - 1$,

(b) $h_{i-t} \geq h_{j+t}$ *for all* $t = 1, \ldots, p-1-j$ *and*

(c) $h_{i+j-p-t} \geq h_t$ *for all* $t = 0, \ldots, s_3$

are all satisfied, then the geometric-mean of the species in DE2 is smaller than the geometric-mean of the species in DE1. Reversing all the inequalities in (a), (b) and (c) have the same effect. Reversing the inequalities only in (a) (or only in (b) & (c)) have the opposite effect.

Proof. Part (i) is the conclusion of Part (i) in Theorem 3.3. Part (ii) is the conclusion of Part (ii) in Lemma 3.3 and Part (ii) in Theorem 3.3. Finally, Part (iii) is the conclusion of Part (ii) in Lemma 3.3 and Part (iii) in Theorem 3.3. □

It is possible to explore the relationship between various permutations through a sequence of small-block interchanges such as blocks of length two or three. In this case, Corollary 3.2 can be shortened. We explore this option further in the next section when $p = 4$ or 5.

4 Directed graph representation and the cases $\mathbf{p = 4, 5}$

To facilitate the permutation comparison process, we assume $0 \leq h_0 < h_1 < \cdots < h_{p-1}$. We have a total of $p!$ permutations. Since members of the dihedral group give same population geometric-mean, we divide the permutations into equivalence classes. For instance, when $p = 4$, $[f_3, f_2, f_1, f_0]$ and $[f_0, f_1, f_2, f_3]$ belong to the same equivalence class. We consider the equivalence classes to be vertices and introduce a directed graph $G = (V, E)$. The set of directed edges E is defined as follows: let v_i and $v_j \in V$, v_i is connected to v_j ($v_i \to v_j$) if for all values of $\mu > 1$, the permutations in v_i give a larger population geometric-mean compared to the ones in v_j. Corollaries 3.1 and 3.2 are the handy tools to help us in this task. Note that when p is large, obtaining the full directed graph will be cumbersome, and therefore, we can talk about directed subgraphs based on reflecting certain blocks. It is worth stressing here that reflecting a block is the same as reflecting its complement block.

Proposition 4.1. *Reflecting the block* $[f_j, f_{j-1}, \ldots, f_i]$ *in* $[f_{p-1}, \ldots, f_j, \ldots, f_i, \ldots, f_0]$ *has the same effect as reflecting its complement block.*

Proof. When we reflect $[f_j, f_{j-1}, \ldots, f_i]$ in $[f_{p-1}, \ldots, f_j, \ldots, f_i, \ldots, f_0]$, we obtain

$$[f_{p-1}, \ldots, f_{j+1}, f_i, \ldots, f_j, f_{i-1} \ldots, f_0] \equiv [f_i, f_{i+1} \ldots, f_j, f_{i-1} \ldots, f_0, f_{p-1}, \ldots, f_{j+1}].$$

On the other hand,

$$[f_{p-1}, \ldots, f_j, \ldots, f_i, \ldots, f_0] \equiv [f_j, \ldots, f_i, f_{i-1}, \ldots, f_0, f_{p-1}, f_{p-2}, \ldots, f_{j+1}].$$

So, the complement block is $[f_{i-1}, \ldots, f_0, f_{p-1}, f_{p-2}, \ldots, f_{j+1}]$, and when reflected, we obtain the same result based on Theorem 3.1. □

In directed graphs, we can talk about the degree of each vertex v_i as $DIn(v_i)$ (the number of edges that get into v_i), $DOut(v_i)$ (the number of edges that get out of v_i) and $TD(v_i)$ (the total degree of v_i, which equals $DIn(v_i) + DOut(v_i)$. When considering a directed subgraph that belongs to reflecting blocks of certain length, $TD(v_i) \leq p$ for all i. Proving the exact values of $DIn(v_i)$ and $DOut(v_i)$ will be interesting. In fact, characterizing the directed graphs defined in this section will be an interesting topic; however, we leave it for those who may have interest in this topic. Here, we just focus on the cases $p = 4$ and $p = 5$.

4.1 The case p = 4

Let $0 \leq h_0 < h_1 < h_2 < h_3$ in Eq. DE1, and assume that $h_j < (\sqrt{\mu} - 1)^2$ for all $j = 0, \ldots, 3$. To move to Eq. DE2, we have 24 permutations in three distinct equivalence classes. For the readers convenience, we give the following chart:

Map notation	Harvesting notation	Matrix notation	Vertex
$[f_3, f_2, f_1, f_0]$	$[h_0, h_1, h_2, h_3]$	$A_3A_2A_1A_0$	v_1
$[f_2, f_3, f_1, f_0]$	$[h_0, h_1, h_3, h_2]$	$A_2A_3A_1A_0$	v_2
$[f_3, f_1, f_2, f_0]$	$[h_0, h_2, h_1, h_3]$	$A_3A_1A_2A_0$	v_3

Table 1: Note that $[f_3, f_2, f_1, f_0]$ is just a representative of the eight permutations in the equivalence class v_1. The other seven ones are the shifts and reflections. Similarly for all others.

Observe that v_2 can be obtained from v_1 by reflecting the block $[f_3, f_2]$. Since, $p = 4, j = 3$ and $i = 2$, we are within the scope of Part (ii) in Corollary 3.1. We have $h_3 > h_2$ and $h_1 > h_0$. Hence, $v_1 \rightarrow v_2$. On the other hand, v_3 can be obtained from v_1 by reflecting the block $[f_2, f_1]$. Since, $p = 4, j = 2$ and $i = 1$, we are within the scope of Part (i) in Corollary 3.2. We have $h_2 > h_1$ and $h_3 > h_0$. Hence, $v_3 \rightarrow v_1$. Finally, $v_3 \rightarrow v_2$ through a sequence of two block reflections (i.e., the transitivity of $v_3 \rightarrow v_1$ and $v_1 \rightarrow v_2$). The directed graph is given in Fig. 2. Hence, considering a permutation of the harvesting quotas within the equivalence class v_3 is more advantageous in terms of the population geometric-mean.

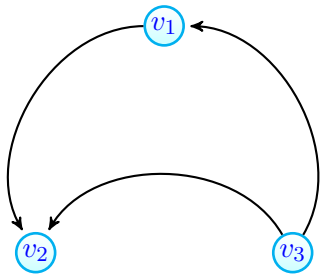


Figure 2: This figure shows the directed graph representation when $p = 4$.

4.2 The case p = 5

Let $0 \leq h_0 < h_1 < h_2 < h_3 < h_4$ in Eq. (DE1), and assume that $h_j < (\sqrt{\mu} - 1)^2$ for all $j = 0, \ldots, 4$. To move to Eq. DE2, there are 120 permutations in 12 distinct equivalence classes that form the vertices of the directed graph. For ease of reference, we give the chart in Table 2. Unlike the directed graph when $p = 4$, the directed graph of $p = 5$ will not be simple. However, we follow the notion of block reflection. Note that reflecting blocks of length one, four or five will give a trivial subgraph. Now, based on Proposition 4.1, reflecting blocks of length three has the same effect as reflecting blocks of length two.

Map notation	Harvesting notation	Matrix notation	Vertex
$[f_4, f_3, f_2, f_1, f_0]$	$[h_0, h_1, h_2, h_3, h_4]$	$A_4A_3A_2A_1A_0$	v_1
$[f_3, f_4, f_2, f_1, f_0]$	$[h_0, h_1, h_2, h_4, h_3]$	$A_3A_4A_2A_1A_0$	v_2
$[f_4, f_2, f_3, f_1, f_0]$	$[h_0, h_1, h_3, h_2, h_4]$	$A_4A_2A_3A_1A_0$	v_3
$[f_2, f_4, f_3, f_1, f_0]$	$[h_0, h_1, h_3, h_4, h_2]$	$A_2A_4A_3A_1A_0$	v_4
$[f_3, f_2, f_4, f_1, f_0]$	$[h_0, h_1, h_4, h_2, h_3]$	$A_3A_2A_4A_1A_0$	v_5
$[f_2, f_3, f_4, f_1, f_0]$	$[h_0, h_1, h_4, h_3, h_2]$	$A_2A_3A_4A_1A_0$	v_6
$[f_4, f_3, f_1, f_2, f_0]$	$[h_0, h_2, h_1, h_3, h_4]$	$A_4A_3A_1A_2A_0$	v_7
$[f_3, f_4, f_1, f_2, f_0]$	$[h_0, h_2, h_1, h_4, h_3]$	$A_3A_4A_1A_2A_0$	v_8
$[f_4, f_1, f_3, f_2, f_0]$	$[h_0, h_2, h_3, h_1, h_4]$	$A_4A_1A_3A_2A_0$	v_9
$[f_3, f_1, f_4, f_2, f_0]$	$[h_0, h_2, h_4, h_1, h_3]$	$A_3A_1A_4A_2A_0$	v_{10}
$[f_4, f_2, f_1, f_3, f_0]$	$[h_0, h_3, h_1, h_2, h_4]$	$A_4A_2A_1A_3A_0$	v_{11}
$[f_4, f_1, f_2, f_3, f_0]$	$[h_0, h_3, h_2, h_1, h_4]$	$A_4A_1A_2A_3A_0$	v_{12}

Table 2: Again here, we give the equivalence classes v_j, $j = 1, \ldots, 12$ and a representative of each equivalence class. The other nine elements of the equivalence class are shifts and reflections.

Let us start by the vertex $v_1 : [f_4, f_3, f_2, f_1, f_0]$ and reflect all blocks of length three in this circular permutation. We give the technical details in this case, but we skip them for the other vertices. We have

$$\underset{v_6}{[\mathbf{f_2}, \mathbf{f_3}, \mathbf{f_4}, f_1, f_0]} \quad \underset{v_{12}}{[f_4, \mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}, f_0]} \quad \underset{v_2}{[f_4, f_3, \mathbf{f_0}, \mathbf{f_1}, \mathbf{f_2}]} \quad \underset{v_3}{[f_3, f_2, \mathbf{f_4}, \mathbf{f_0}, \mathbf{f_1}]} \quad \underset{v_7}{[f_2, f_1, \mathbf{f_3}, \mathbf{f_4}, \mathbf{f_0}]}$$

and summarize the rest of the work in Table 3. Note that in v_3 and v_7, you can relabel the maps. For instance $v_1 = [f_3, f_2, f_1, f_0, f_4] = [\hat{f}_4, \hat{f}_3, \hat{f}_2, \hat{f}_1, \hat{f}_0]$. In this case, $v_3 = [\hat{f}_4, \hat{f}_3, \hat{f}_0, \hat{f}_1, \hat{f}_2]$ and applying Part (iii) of Corollary 3.1 becomes obvious.

v_1 in DE1	DE2	Corollaries 3.1 and 3.2	Obtained edge
$[\mathbf{f_4}, \mathbf{f_3}, \mathbf{f_2}, f_1, f_0]$	v_6	Part (ii) of Corollary 3.1	$v_1 \rightarrow v_6$
$[f_4, \mathbf{f_3}, \mathbf{f_2}, \mathbf{f_1}, f_0]$	v_{12}	Part (i) of Corollary 3.2	$v_1 \leftarrow v_{12}$
$[f_4, f_3, \mathbf{f_2}, \mathbf{f_1}, \mathbf{f_0}]$	v_2	Part (iii) of Corollary 3.1	$v_1 \rightarrow v_2$
$[f_3, f_2, \mathbf{f_1}, \mathbf{f_0}, \mathbf{f_4}]$	v_3	Part (iii) of Corollary 3.1	$v_1 \leftarrow v_3$
$[f_2, f_1, \mathbf{f_0}, \mathbf{f_4}, \mathbf{f_3}]$	v_7	Part (iii) of Corollary 3.1	$v_1 \leftarrow v_7$

Table 3: Reflecting blocks of length three in v_1 and the obtained directed edges based on corollaries 3.1 and 3.2.

Similarly for all the other vertices. In this case, the directed graph is given in Fig. 3.

We revisit Example 2.1, and illustrative the comparison between the attractors. Table 4 shows the possible 12 equivalence classes together with the attractor in each case and its geometric-mean.

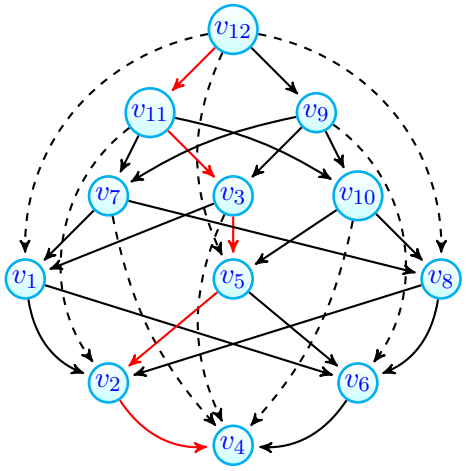


Figure 3: This graph illustrates corollaries 3.1 and 3.2 applied when $p = 5$ and blocks of length three are reflected (the same graph must be obtained when reflecting blocks of length two). When a block of harvesting quotas is reflected in a vertex v_i, we obtain a vertex v_j. A directed edge from v_i to v_j ($v_i \rightarrow v_j$) means the geometric-mean of the population that belongs to v_j is smaller than the geometric-mean of the population that belongs to v_i. A dashed edge is meant to highlight a short path, which can be replaced by a longer path through transitivity. The red edges show a path that takes us from the largest population geometric-mean to the smallest population geometric-mean.

Equivalence Class	Vertex	Attractor	G-Mean
$[f_4, f_3, f_2, f_1, f_0]$	v_1	$\{6, \frac{96}{7}, \frac{1433}{103}, \frac{1241}{96}, \frac{15845}{1337}\}$	11.189
$[f_3, f_4, f_2, f_1, f_0]$	v_2	$\{10.745, 14.638, 13.977, 12.932, 6.097\}$	11.163
$[f_4, f_2, f_3, f_1, f_0]$	v_3	$\{6.082, 13.741, 13.915, 11.927, 12.762\}$	11.210
$[f_2, f_4, f_3, f_1, f_0]$	v_4	$\{11.717, 14.742, 13.984, 11.932, 6.008\}$	11.160
$[f_3, f_2, f_4, f_1, f_0]$	v_5	$\{11.747, 14.745, 13.984, 6.177, 11.771\}$	11.198
$[f_2, f_3, f_4, f_1, f_0]$	v_6	$\{12.641, 14.827, 13.989, 6.178, 10.771\}$	11.177
$[f_4, f_3, f_1, f_2, f_0]$	v_7	$\{6.007, 13.717, 12.913, 13.850, 11.923\}$	11.193
$[f_3, f_4, f_1, f_2, f_0]$	v_8	$\{10.768, 14.640, 12.977, 13.855, 6.168\}$	11.182
$[f_4, f_1, f_3, f_2, f_0]$	v_9	$\{6.161, 13.766, 12.916, 11.850, 13.755\}$	11.229
$[f_3, f_1, f_4, f_2, f_0]$	v_{10}	$\{11.836, 14.754, 12.984, 6.101, 12.747\}$	11.201
$[f_4, f_2, f_1, f_3, f_0]$	v_{11}	$\{6.095, 13.745, 11.915, 13.761, 12.916\}$	11.215
$[f_4, f_1, f_2, f_3, f_0]$	v_{12}	$\{6.167, 13.767, 11.917, 12.761, 13.837\}$	11.231

Table 4: Consider $p = 5, \mu = 16$ and $h_0 = 0, h_1 = 1, h_2 = 2, h_3 = 3, h_4 = \frac{75214}{8591}$. This table shows the 12 possible equivalence classes together with the attractor in each case and its geometric-mean.

It is worth mentioning that the numerical data in Table 4 can be used to give a complete

graph, while the graph in Fig. 3 is a subgraph of the complete graph.

5 Conclusion and discussion

In this paper, we considered the Beverton-Holt model with p-periodic harvesting

$$x_{n+1} = \frac{\mu k x_n}{k + (\mu - 1)x_n} - h_n = f_n(x_n), \quad \mu > 1, n \in \mathbb{N} := \mathbb{Z}^+ \cup \{0\}, \tag{5.1}$$

and gave ourselves the liberty to permute the harvesting quotas $[h_0, h_1, \ldots, h_{p-1}]$. This notion keeps the total harvesting yield unchanged, while it may benefit the species in terms of its geometric mean. In other words, we consider the combinatorial effect of changing the fixed points of the individual maps f_n on the global attractor of the periodic system. Therefore, this process has to be done under the assumption that we have a globally asymptotical stable p-cycle all the time. This is guaranteed under the simple generic condition $0 \leq h_j \leq (\sqrt{\mu} - 1)^2$ for all $j = 0, \ldots, p-1$. It has been found that reflections and rotations of $[h_0, h_1, \ldots, h_{p-1}]$ have no effect on the population geometric mean. The effect of other permutations has been characterized based on block reflections of the harvesting quotas.

In a p-periodic system, we have a total of $p!$ permutations that can be divided (based on Theorem 3.1 into $\frac{1}{2}(p-1)!$ equivalence classes. We defined the equivalence classes to be the vertices of a directed subgraph. A directed edge from v_i to v_j ($v_i \rightarrow v_j$) means the geometric-mean of the population that belongs to v_j is smaller than the geometric-mean of the population that belongs to v_i. Based on block reflections certain directed subgraphs can be obtained. The directed graphs that belong to $p = 4$ and $p = 5$ are given, which made the effect of the permutations on the population geometric-mean interestingly visual.

Although several interesting results that address certain characteristics of our problem have been obtained, this paper opened the gate for several interesting questions that are worth further research. We give a few of them here. So, consider Eq. (5.1) with $\mu > 1$ and $0 \leq h_0 < h_1 < \cdots < h_{p-1}$.

(i) In general, which permutation $[f_{\sigma(p-1)}, f_{\sigma(p-2)}, \ldots, f_{\sigma(0)}]$ will give largest population geometric-mean? Similarly, for the smallest.

(ii) Are the permutations in the dihedral group the only ones that give same geometric-mean for all values of $\mu > 1$?

(iii) Characterize the paths of the directed graphs introduced in this paper. Similarly, for the degrees of the vertices.

(iv) Are the directed graphs introduced based on blocks reflection always connected?

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