ON SOME BOUNDS FOR FREQUENCIES OF MEMBRANES AND PLATES

by

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Declaration of Authorship

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Dedication

First, I want to dedicate this thesis to my parents, strong and gentle souls, who taught me to trust in Allah, believe in hard work and that so much can be done with little. And to my siblings for picking me up when I fall down and for always pushing me to do my best.

I would also like to dedicate this work to my husband, who has been a great source of support and encouragement. Thank you for believing in me and empowering me.

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Abstract

The problem of optimizing a domain dependent functional, while keeping a domain's measure (its volume, perimeter, etc.) fixed, is called an isoperimetric problem. The isoperimetric inequalities have a long history in mathematics dating back to the Greeks and Dido's problem, when the first classical isoperimetric inequality appeared in the Euclidean geometry. With the introduction of the calculus of variations in the 17th century, the isoperimetric inequalities found their way into mathematical physics. Among the isoperimetric problems, here we propose the investigation of those linking the shape of a membrane to the sequence of its frequencies. The starting point in this research field is the Faber-Krahn inequality, which states that among all fixed membranes of given area, the first frequency is minimal for the circular membrane. As for the second frequency of fixed membranes of given area, we know that it is minimized by the disjoint union of two identical circular membranes (Krahn's inequality). For other types of membranes several results are known, but a lot of questions remain open. In this thesis we are going to present some classical isoperimetric inequalities, as well as some universal bounds, which are not isoperimetric, but in some cases they represent the best possible bounds obtained at their time. Finally, we will present some new universal bounds we have obtained for the frequencies of clamped and buckled plates.

Keywords: ***Isoperimetric inequalities; clamped plate; buckled plate; universal bounds; eigenvalues; frequencies of membranes and plates.***

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Chapter 1. Introduction

In this chapter, we are going to give an overview of the main concepts and types of problems we are dealing with in this thesis. First, we clarify the notion of the *shape optimization problem*, whose solutions can always be given in the form of some isoperimetric inequalities. The definition will be followed by some very suggestive examples of geometric nature. Next, we will introduce some classical eigenvalue problems which appear as models for frequencies of different types of membranes. Finally, we compute these frequencies for some special shapes and present the main properties of these frequencies, in general.

1.1 Shape optimization problems. Isoperimetry

The word *optimum* is Latin and means *the ultimate ideal*, while the Latin word *optimus* means *the best*. An *optimization problem* is thus the problem of finding the best solution from all feasible solutions. In this chapter we are interested in a special type of optimization problems, namely *the shape optimization problems*, where we look for the best possible shape of an object which maximize or minimize a certain quantity related to it. More precisely:

Definition 1.1.1. A *shape optimization problem* is a problem of the following type: Find (the shape) $\Omega^* \in \mathcal{F}$, solution to the following optimization problem:

$$F(\Omega^*) = \underset{\Omega\in\mathcal{F}}{min}/\underset{\Omega\in\mathcal{F}}{max} F(\Omega), \tag{1.1}$$

where $\mathcal{F} \subseteq P(\mathbb{R}^n)$ is the class of all admissible domains from $\mathbb{R}^n$ and $F : \mathcal{F} \to \mathbb{R}$ is a domain-dependent functional.

Obviously, these shape optimization problems can be expressed in terms of so-called *isoperimetric inequalities*, as shown in the following examples.

Example 1.1. Among all rectangles of given perimeter P, the square has the largest area A. This statement can be expressed as an isoperimetric inequality, as follows:

$$A \leq \frac{1}{16}P^2, \tag{1.2}$$

where the equality holds for a square.

Example 1.2. Among all triangles of given perimeter P, the equilateral triangle has the largest area A. This statement can be expressed as an isoperimetric inequality, as follows:

$$A \leq \frac{1}{12\sqrt{3}} P^2, \tag{1.3}$$

where the equality holds for an equilateral triangle.

Example 1.3. Among all closed planar curves of given perimeter P, the circle has the largest area A (Queen Dido of Cartage). This statement can be expressed as an isoperimetric inequality, as follows:

$$A \leq \frac{1}{4\pi} P^2, \tag{1.4}$$

where the equality holds for a circle.

1.2 Model Problems: Frequencies of Membranes

In this section we are going to present some model problems for two different types of vibrating membranes, whose shapes are always represented by a bounded open set $\Omega \subseteq R^N$, with Lipschitz boundary. In what follows we will see that the frequencies of such membranes are directly related to the eigenvalues of the Laplacian operator on Ω.

1.2.1 Frequencies of a fixed membrane. The model problem for the frequencies of a fixed membrane is given by the following **eigenvalue problem for the Dirichlet-Laplacian**:

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

This problem has a real and purely discrete spectrum

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq ... \leq \lambda_n(\Omega) \to +\infty. \tag{1.6}$$

The normal modes and proper frequencies that characterize the vibrations of a fixed planar membrane of the given shape Ω are determined by the solutions of this problem in the case $N = 2$. More precisely, in this case, $\sqrt{\lambda_n}$, with $n \in \mathbb{N}$, represent *the* *frequencies of the fixed membrane of given shape* Ω. These eigenvalues are explicitly computable only for special domains (rectangles, disks). Since we cannot compute the eigenvalues of an arbitrary domain Ω, then we want at least to get optimal estimates of them in terms of geometric quantities related to the underlying domain Ω. For instance, a first conjecture on a possible isoperimetric bound for the first frequency was stated by Lord Rayleigh [10], and it says the following: *Among all fixed membranes of given area, the circular one gives the lowest first frequency*. A proof of this conjecture will be presented in Chapter 2.

1.2.2 Frequencies of a free membrane. The model problem for the frequencies of a free membrane is given by the following **eigenvalue problem for the Neumann-Laplacian**:

$$\begin{cases} \Delta u + \mu u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

This problem also has a real and purely discrete spectrum

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq ... \to \infty. \tag{1.8}$$

The normal modes and the proper frequencies that characterize the vibrations of a free membrane of given shape Ω are determined by the solutions of this problem, when $N = 2$. In such a case, $\sqrt{\mu_n}$, with $n \in \mathbb{N}$, represent *the frequencies of the free membrane of given shape* Ω. These eigenvalues are again explicitly computable only for special domains (rectangles, disks). And since we cannot compute the eigenvalues of an arbitrary domain Ω, then we want at least to get optimal estimates of them in terms of geometric quantities related to the underlying domain Ω. For instance, a first conjecture on a possible isoperimetric inequality for the first frequency was stated by E.T. Kornhauser and I. Stackgold [8] and it says the following: *Among all free membranes of given area, the circular one gives the largest first frequency*. Two proofs of this conjecture will be presented in Chapter 3.

1.3 The Special Case of Rectangles and Disks

In this section we are going to show how one may obtain explicitly the eigenvalues and the eigenfunctions of the Laplacian in the case of some simple shapes, for which we can do that. To this end, let's first remind the one dimensional case of the problem given in equation (1.5), since it will often appear in our computations for the two dimensional case, when the method of separation of variables is used. More precisely, if we consider the one dimensional case of the problem given in equation (1.5), that is the following eigenvalue problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then one can easily notice that the only non-trivial solutions are

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \qquad u_n(x) = sin(\frac{n\pi x}{L}), \quad n \geq 1. \tag{1.9}$$

Next, we will use the method of separation of variables to clarify the situation for rectangles.

1.3.1 The case of rectangles. Let $\Omega = (0, L) \times (0, l)$ be a planar rectangle. We then have:

Proposition 1.3.1. The only non-trivial solutions of the problem given in equation (1.5) for the planar rectangle Ω are:

$$\begin{aligned} \lambda_{m,n} &= \pi^2(\frac{m^2}{L^2} + \frac{n^2}{l^2}) \\ u_{m,n}(x, y) &= \frac{2}{\sqrt{Ll}} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{l}\right) \end{aligned} \quad , m, n \in \mathbb{N}^*. \tag{1.10}$$

Proof: For the proof we are using the separation of variables method, which means that we are looking for a solution of the form

$$u(x, y) = f(x)g(y). \tag{1.11}$$

If we plug (1.11) into (1.5), we get

$$f^{''}(x)g(y) + f(x)g^{''}(y) + \lambda f(x)g(y) = 0. \tag{1.12}$$

Dividing (1.12) by fg and rearranging the terms, we can rewrite equation (1.12) as

$$\frac{f''(x)}{f(x)} = -\left(\frac{g^{''}(y)}{g(y)} + \lambda\right) = \text{constant} = k. \tag{1.13}$$

Now, taking into account the boundary condition from equation (1.5), one can easily notice that f and g satisfy the following boundary value problems

$$\begin{cases} f''(x) - kf(x) = 0, \\ f(0) = f(L) = 0, \end{cases} \tag{1.14}$$

respectively,

$$\begin{cases} g''(y) + (\lambda + k)g(y) = 0, \\ g(0) = g(l) = 0. \end{cases} \tag{1.15}$$

Clearly from what is known from the one dimensional case, the problem given in equation (1.14) will have the following non-trivial solutions, considering $a = -k > 0$,

$$a = \pi^2 \frac{n^2}{L^2} \quad \text{and} \quad f(x) = c \sin\left(\frac{n\pi}{L} x\right). \tag{1.16}$$

Next, with the value of k found in (1.16), we get

$$\begin{cases} g''(y) + \left(\lambda - \pi^2 \frac{n^2}{L^2}\right) g(y) = 0, \\ g(0) = g(l) = 0. \end{cases} \tag{1.17}$$

Again, from what is known from the one dimensional case, the problem given in equation (1.17) will have the following non-trivial solutions, considering $b = \lambda - \pi^2 \frac{n^2}{L^2}$,

$$b = \pi^2 \frac{m^2}{l^2} \quad \text{and} \quad g(y) = \bar{c} \sin\left(\frac{m\pi}{l} y\right). \tag{1.18}$$

In conclusion, the non-trivial solutions for the problem given in equation (1.5) are given as in (1.10). □

Next, using the same idea we can obtain similar results for the free membrane problem.

Proposition 1.3.2. The only non-trivial solutions of the problem given in equation (1.7) for the planar rectangle Ω are:

$$\begin{aligned} \mu_{m,n} &= \pi^2 \left(\frac{m^2}{L^2} + \frac{n^2}{l^2} \right) \\ u_{m,n}(x, y) &= \frac{2}{\sqrt{Ll}} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi y}{l}\right) \end{aligned} , m, n \in \mathbb{N}^*. \tag{1.19}$$

1.3.2 The case of disks. Let $\Omega = B_R$ be the disk of radius R centered at the origin. Using again the separation of variables method, one may obtain the following.

Proposition 1.3.3. The non-trivial solutions of problem (1.5) on the disk Ω are:

$$\lambda_{0,k} = \frac{j_{0,k}^2}{R^2}, \qquad u_{0,k}(r,\theta) = \sqrt{\frac{1}{\pi}}\frac{1}{R|J_0'(j_{0,k})|}J_0(j_{0,k}\frac{r}{R}), \qquad k \geq 1, \tag{1.20}$$

$$\lambda_{n,k} = \frac{j_{n,k}^2}{R^2}, \qquad u_{n,k}(r,\theta) = \begin{matrix} \sqrt{\frac{2}{\pi}}\frac{1}{R|J_n'(j_{n,k})|}J_n(j_{n,k}\frac{r}{R})\cos(n\theta) \\ \sqrt{\frac{2}{\pi}}\frac{1}{R|J_n'(j_{n,k})|}J_n(j_{n,k}\frac{r}{R})\cos(n\theta) \end{matrix}, \qquad n,k \geq 1, \tag{1.21}$$

where $j_{n,k}$ is the $k-th$ zero of *the Bessel function* J_n.

Proof: In polar coordinates problem (1.5) takes the following form

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \lambda u = 0 \\ u(R,\theta) = 0 \end{cases} , r \in (0,R) \text{ and } \theta \in [0,2\pi]. \tag{1.22}$$

Now, we are going to use the separation of variables method, which means that we are looking for a solution of the form

$$u(r,\theta) = f(r)g(\theta). \tag{1.23}$$

By substituting (1.23) into (1.22) and performing some operations we get

$$\frac{r^2(f''(r) + \frac{1}{r}f'(r) + \lambda f(r))}{f(r)} = -\frac{g''(\theta)}{g(\theta)} = constant = k. \tag{1.24}$$

Since θ is a cyclic coordinate, we must require that u is periodic with respect to θ, of period 2π. Therefore, we can take $k = p^2$. In such a case, from what is known from the one dimensional case, we can easily find that

$$g(\theta) = a\cos p\theta + b\sin p\theta. \tag{1.25}$$

Next, we introduce a new variable

$$x := \sqrt{\lambda}r, \tag{1.26}$$

and define a new function

$$y(x) := f(\frac{x}{\sqrt{\lambda}}) = f(r). \tag{1.27}$$

We then notice that $y(x)$ satisfies the following Bessel Equation of order p

$$y^{''}(x) + \frac{1}{x}y^{'}(x) + (1 - \frac{p^2}{x^2})y(x) = 0, \tag{1.28}$$

with the condition on the boundary

$$y(\sqrt{\lambda}R) = 0. \tag{1.29}$$

In what follows, we will try to solve equation (1.28) by looking for a solution given as a power series

$$y(x) = x^r \sum_{n\geq 0} C_n x^n, \qquad x \in \mathbb{R}, \tag{1.30}$$

where the exponent r and the constants C_k are to be determined. We assume that the series $\sum_{n\geq 0} C_n x^n$ has radius of convergence $\rho > 0$. Then, substituting this power series into equation (1.28), we obtain

$$\sum_{n\geq 0}[(r+n)(r+n-1)+(r+n)]C_n x^n + (x^2 - p^2)\sum_{n\geq 0} C_n x^n = 0, \tag{1.31}$$

or, equivalently,

$$\sum_{n\geq 0}[(r+n)^2 - p^2]C_n x^n = -\sum_{n\geq 0} C_n x^{n+2}. \tag{1.32}$$

Identifying the coefficients in (1.32) we get

$$(r^2 - p^2)C_0 = 0, \qquad\qquad [(r+1)^2 - p^2]C_1 = 0, \tag{1.33}$$

$$[(r+k)^2 - p^2]C_k = -C_{k-2}, \qquad k = 2, 3, 4, \ldots . \tag{1.34}$$

Without loss of generality, we take $C_0 \neq 0$. Otherwise the series (1.32) starts with the term $C_1 x^{r+1}$ and with change of notation $r+1 = r_1$ and $n = m+1$ we will have:

$$y(x) = x^{r_1} \sum_{m\geq 0} C_m x^n, \tag{1.35}$$

that is a series of the same form as (1.30). Then, using the first equation from (1.33), we obtain the following algebraic equation

$$r^2 - p^2 = 0, \tag{1.36}$$

which has the roots p and -p. For $r = p$, the second equation from (1.33) gives

$$(2p+1)C_1 = 0 \Rightarrow C_1 = 0. \tag{1.37}$$

On the other hand, equation (1.34) leads to the following relation of recurrence:

$$C_k = -\frac{C_{k-2}}{k(2p+k)}, \qquad k = 2, 3, 4, \dots . \tag{1.38}$$

Consequently, all the coefficients C_n of odd index are equal to 0. Also, by (1.38), every coefficient of even index can be expressed by the following relation of recurrence

$$C_{2m} = -\frac{C_{2m-2}}{2^2 m(p+m)}, \qquad m = 2, 3, 4, \dots . \tag{1.39}$$

The successive application of this recurrent formula allows us to express C_{2m} in terms of C_0, as follows

$$C_{2m} = (-1)^m \frac{C_0}{2^m m!(p+1)(p+2)...(p+m)}. \tag{1.40}$$

Using the fact that

$$(p+1)(p+2)...(p+m) = \frac{\Gamma(p+m+1)}{\Gamma(p+1)}, \tag{1.41}$$

we then obtain

$$C_{2m} = \frac{(-1)^m \Gamma(p+1) C_0}{m! 2^m \Gamma(p+m+1)}, \qquad m = 1, 2, 3, \dots . \tag{1.42}$$

In conclusion

$$y(x) = x^p \sum_{m \geq 0} C_{2m} x^{2m} = \Gamma(p+1) C_0 \sum_{m \geq 0} \frac{(-1)^m x^{p+2m}}{m! 2^{2m} \Gamma(p+m+1)}. \tag{1.43}$$

Since, equation (1.28) is homogeneous, the solutions can be determined up to a multiplicative constant, which is chosen to be

$$C_0 = \frac{1}{2^n \Gamma(p+1)}. \tag{1.44}$$

We then obtain

$$y(x) = \sum_{m \geq 0} \frac{(-1)^m}{m! \Gamma(p+m+1)} \left(\frac{x}{2}\right)^{p+2m} := J_p(x), \tag{1.45}$$

which is the *Bessel function* of order p and type 1. In a similar way, the following series

$$J_{-p} := \sum_{m \geq 0} \frac{(-1)^m}{m! \Gamma(-p+m+1)} \left(\frac{x}{2}\right)^{-p+2m}, \tag{1.46}$$

corresponding to $r = -p$ represents a second solution of (1.24), linearly independent of J_p. The series (1.45) and (1.46) are obviously convergent for all values of x. In fact, if we write

$$y(x) = \left(\frac{x}{2}\right)^p \sum_{m\geq 0} \frac{(-1)^m}{m!\Gamma(p+m+1)}\zeta^m, \text{ for } \zeta := \left(\frac{x}{2}\right)^2. \tag{1.47}$$

Then the radius of convergence of the power series given above is

$$\begin{aligned} \rho &= \lim_{m\to\infty}\left|\frac{(-1)^m}{m!\Gamma(p+m+1)}\Big/\frac{(-1)^{m+1}}{(m+1)!\Gamma(p+m+2)}\right| \\ &= \lim_{m\to\infty}[(m+1)(p+m+1)] = \infty. \quad \square \end{aligned} \tag{1.48}$$

Remark: For the membrane problem (1.6) in $\mathbb{R}^n$, $n \geq 3$, we have a similar situation because

$$\Delta_{\mathbb{R}^n} = \frac{\partial}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}}, \tag{1.49}$$

and by separation of variables

$$u(x) = f(r)g(\Theta), \tag{1.50}$$

where r is the radius and Θ is the so-called Winkelvariable, we obtain the equivalent equations

$$\begin{cases} \Delta_{S^{n-1}}g = \mu g, \\ r^2 f'' + (n-1)rf' + (\mu + \lambda r)f = 0. \end{cases} \tag{1.51}$$

Moreover, one can show that μ_k are given as follows

$$\mu_k = -k(n+k-2). \tag{1.52}$$

Finally, using the same idea, we can obtain similar results for the free membrane problem (see, [5] for more details.)

Proposition 1.3.4. The only non-trivial solutions of the problem given in equation (1.7) for the disk Ω are:

$$\mu_{0,k} = \frac{{j'_{0,k}}^2}{R^2}, \qquad u_{0,k}(r,\theta) = \sqrt{\frac{1}{\pi}}\frac{1}{R|J_0(j'_{0,k})|}J_0(j'_{0,k}\frac{r}{R}), \qquad k \geq 1, \tag{1.53}$$

$$\mu_{n,k} = \frac{{j'_{n,k}}^2}{R^2}, \qquad u_{n,k}(r,\theta) = \begin{matrix} \sqrt{\frac{2}{\pi}}\frac{j'_{n,k}}{R\sqrt{{j'_{n,k}}^2 - n^2}|J_n(j'_{n,k})|}J_n(j'_{n,k}\frac{r}{R})\cos n\theta \\ \sqrt{\frac{2}{\pi}}\frac{j'_{n,k}}{R\sqrt{{j'_{n,k}}^2 - n^2}|J_n(j'_{n,k})|}J_n(j'_{n,k}\frac{r}{R})\sin n\theta \end{matrix}, \qquad n,k \geq 1, \tag{1.54}$$

where $j'_{n,k}$ is the $k-th$ zero of J'_n (the derivative of the Bessel function J_n).

1.4 Properties of the Eigenvalues

In this section we are going to present the main properties of the eigenvalues. To this end we will concentrate on the Dirichlet eigenvalues, similar results being possible for the Neumann eigenvalues.

Proposition 1.4.1. The eigenvalues λ_k are real and positive. Therefore, we can arrange them in an increasing order such that

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \dots . \tag{1.55}$$

Proof: Let f = $\overline{u}_k$ and g = u$_k$, we know that:

$$\nabla(f\nabla g) = \nabla f \nabla g + f\triangle g. \tag{1.56}$$

Thus

$$\nabla(\bar{u_k}\nabla u_k) = (|\nabla u_k|^2 + \bar{u_k}\triangle u_k) = |\nabla u_k|^2 - \lambda_k |u_k|^2. \tag{1.57}$$

Integrating this relation over Ω, and using the divergence theorem, we get

$$0 = \int_\Omega |\nabla u_k|^2 dx - \lambda_k \int_\Omega |u_k|^2 dx, \qquad k = 1, 2, 3, ..., \tag{1.58}$$

from which, the following relation follows

$$\lambda_k = \frac{\int_\Omega |\nabla u_k|^2 dx}{\int_\Omega |u_k|^2 dx} > 0, \qquad k = 1, 2, 3, \dots . \quad \square \tag{1.59}$$

Proposition 1.4.2. The eigenfunctions u_k and u_j associated to the distinct eigenvalues $\lambda_k \neq \lambda_j$ are two by two orthogonal, that is

$$\int_\Omega u_k u_j dx = 0, \qquad \text{for } \lambda_k \neq \lambda_j. \tag{1.60}$$

Proof: Integrating the following relation

$$u_k \triangle u_j - u_j \triangle u_k = (\lambda_k - \lambda_j) u_k u_j, \tag{1.61}$$

over Ω, and using the Green's formula, we obtain

$$0 = \int_{\partial\Omega} \left(u_k \frac{\partial u_j}{\partial n} - u_j \frac{\partial u_k}{\partial n} \right) ds = (\lambda_k - \lambda_j) \int_\Omega u_k u_j dx, \tag{1.62}$$

and the proof is achieved, since $\lambda_k - \lambda_j \neq 0$. $\square$

Proposition 1.4.3. Each eigenfunction u_k is determined up to a multiplicative constant. Therefore it is always possible to choose an orthonormal system of eigenfunctions, in such a way that

$$\int_\Omega u_k u_j dx = \delta_{kj} = \begin{cases} 0, & \text{if } \lambda_k \neq \lambda_j \\ 1, & \text{if } \lambda_k = \lambda_j \end{cases} . \tag{1.63}$$

1.5 Variational Characterizations of the Eigenvalues

We know that the eigenvalues are only computable for special domains such as rectangles and disks. For other domains, when they are not computable, at least we want to find some bounds for them. To this end, the most important tool is given by the variational characterizations of the eigenvalues.

First, let us introduce some terminology. Let $W^{k,p}(\Omega)$ be the Sobolev space given by

$$W^{k,p}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable} : \exists D^{\alpha}u \in L^{p}, \forall |\alpha| \leq k\}.$$

For $p = 2$ we obtain the space $H^{k}(\Omega) = W^{k,2}(\Omega)$, which is a Hilbert space. In what follows we will often consider the following space

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u_{|\partial\Omega} = 0\}.$$

The first eigenvalue λ_1 of Ω can be characterized as a variational minimum in $H_0^1(\Omega)$ of the following **Rayleigh quotient**

$$R[v] = \frac{\int_{\Omega} |\nabla v|^2 dx}{\int_{\Omega} v^2 dx}. \tag{1.64}$$

This statement is contained in the following variational principle.

Theorem 1.5.1. ***(Rayleigh's Variational Principle)***

$$\lambda_1(\Omega) = \underset{v \in H_0^1(\Omega), v \neq 0}{Min} R(v). \tag{1.65}$$

Note that the variational minimum is obtained if and only if $v = u_1(x)$.

Proof: Let $u(x) \in H_0^1(\Omega)$ be the function that achieves the minimum m of $R(v)$. Let us define $v(x) := u + \epsilon w(x)$, where ϵ is an arbitrary real parameter and $w(x)$ is an arbitrary function in $H_0^1(\Omega)$. By the definition of $u(x)$, we have the following relation

$$\frac{d}{d\epsilon} R(v)\big|_{\epsilon=0} = \frac{d}{d\epsilon} \frac{\int_{\Omega}(|\nabla u|^2 + 2\epsilon \nabla u \nabla w + \epsilon^2 |\nabla w|^2) dx}{\int_{\Omega}(u^2 + 2\epsilon uw + \epsilon^2 w^2) dx}\Bigg|_{\epsilon=0} = 0, \tag{1.66}$$

which leads to

$$\int_{\Omega} u^2 dx \int_{\Omega} \nabla u \nabla w dx - \int_{\Omega} |\nabla u|^2 dx \int_{\Omega} uw dx = 0. \tag{1.67}$$

With $R[u] = m$, (1.67) becomes

$$\int_{\Omega} \nabla u \nabla w dx = m \int_{\Omega} uw dx. \tag{1.68}$$

On the other hand, we have

$$\int_\Omega \nabla u \nabla w dx = -\int_\Omega w\Delta u dx + \int_{\partial\Omega} w\frac{\partial u}{\partial n ds} = -\int_\Omega w\Delta u dx. \qquad (1.69)$$

Combining now the previous two relations, we obtain

$$\int_\Omega (\Delta u + mu) w dx = 0, \qquad \forall w(x) \in H_0^1(\Omega), \qquad (1.70)$$

which implies that

$$\Delta u(x) + mu(x) = 0, \qquad \forall x \in \Omega. \qquad (1.71)$$

We can thus conclude, from (1.71), that

$$u = u_k, \qquad m = \lambda_k,$$

for a certain index k. In fact, we must choose k = 1, since we have

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le ...,$$

and this achieves the proof. □

The eigenvalues of higher order can be also characterized variationally in the same way.

Theorem 1.5.2.

$$\lambda_n(\Omega) = \min_{\substack{v\in H_0^1(\Omega), v\neq 0 \\ <v,u_k>=0, k=1,\dots,n-1}} R(v). \qquad (1.72)$$

We note again that the variational minimum is obtained if and only if $v = u_n(x)$.

Proof: The previous calculations remain valid, provided that additional constraints are imposed, that is

$$< u, u_k >= 0, \qquad k = 1, 2, .., n-1. \qquad (1.73)$$

The eigenfunction $u(x)$ must thus be orthogonal to the first $n-1$ eigenfunctions, so we have

$$u = u_n, \qquad m = \lambda_n,$$

and the proof is achieved. □

The variational characterization indicated by theorem (1.5.2) is of considerable theoretical interest, but is difficult to apply in practice, since the eigenfunctions u_k, $k = 1, ..., n-1$ are generally unknown. However, Poincare variational principle have successfully overcome this difficulty by developing two variational principles which we formulate below.

Theorem 1.5.3. ***(Poincaré Variational Principle)***
Let $f_k(x)$, $k = 1, ..., n$, be n linearly independent functions in $H_0^1(\Omega)$. Let E_n be the vector space generated by these n functions. We then have

$$\lambda_n(\Omega) = \underset{E_n}{Min}(\underset{u \in E_n}{max} R[u]). \tag{1.74}$$

The variational minimum should be computed with respect to the choice of E_n, while the maximum is an ordinary maximum in the vector space E_n.

Proof: First of all, we have

$$\lambda_n \leq \underset{u \in E_n}{Max} R(u). \tag{1.75}$$

Indeed, there exists some constants $c_1, ..., c_n$ (not all of them equal to zero), such that

$$u_0(x) := \sum_{k=1}^{n} c_k f_k(x), \tag{1.76}$$

is orthogonal on $u_1(x), ..., u_n(x)$, that is

$$\int_\Omega u_0(x) u_k(x) dx = 0, \qquad k = 1, ..., n-1. \tag{1.77}$$

$u_0(x)$ is thus admissible in the variational characterization of λ_n according to the theorem (1.5.2), we therefore have

$$\lambda_n \leq R(u_0) \leq \underset{u \in E_n}{max} R(u). \quad \square \tag{1.78}$$

1.6 Eigenvalues Monotonicity Property

Theorem 1.6.1. *If $\Omega_1 \subseteq \Omega_2$, then we have*

$$\lambda_n(\Omega_1) \geq \lambda_n(\Omega_2), \quad \forall n = 1, 2, 3... . \tag{1.79}$$

Proof: Let us denote by E_n the vector space generated by the first n eigenvalues $u_1(x), ..., u_n(x)$ corresponding to Ω_1. Let us define the following function

$$\widetilde{u_k}(x) := \begin{cases} u_k(x), & x \in \Omega_1, \\ 0, & x \in \Omega_2 \setminus \Omega_1, \end{cases} \tag{1.80}$$

Clearly, from the definition, the functions $\widetilde{u_k}(x)$ belong to $H_0^1(\Omega_2)$. Next, let us consider

$$\widetilde{E_n} := \text{span}\{\tilde{u_1}, ..., \tilde{u_n}\}, \tag{1.81}$$

which is admissible for the Poincaré variational characterization of $\lambda_n(\Omega_2)$. Therefore, we have

$$\lambda_n(\Omega_2) \leq \underset{u \in \tilde{E}_n}{max} R_{\Omega_2}[u] = \underset{u \in E_n}{max} R_{\Omega_1}[u] = \lambda_n(\Omega_1), \tag{1.82}$$

and the proof is achieved. □

Corollary 1.6.1.1.

$$\lambda_n(\Omega) \to \infty, \text{ when } n \to \infty. \tag{1.83}$$

Proof: Choose $a > 0$ such that the n-cube Q_a contains Ω. The eigenvalues of Q_a are in the following form

$$\left(\frac{\pi}{a}\right)^2 \sum_{k=1}^{n} m_k^2, \tag{1.84}$$

where m_k are any non-negative integers (not all of them equal to zero). Since these eigenvalues are not bounded, we have

$$\lambda_m(\Omega) \geq \lambda_m(Q_a) \to \infty, \tag{1.85}$$

and the proof is achieved. □

The goal of this thesis is to show how one may find some similar types of isoperimetric bounds for frequencies of vibrating membranes and plates, when such bounds are possible, or how to find some universal bounds which are not necessarily optimal, but they are the best known in the literature.

Chapter 2. Dirichlet Case

One of the main tools in the study of isoperimetric inequalities, which lead to optimal shapes that are spherical, is the Schwarz symmetrization, also known as Schwarz rearrangement. The goal of this section is to give a quick overview of this type of symmetrization (see, [7] for more details) and present the results we need later to obtain the Faber-Krahn inequality and the Hardy-Littlewood-Pólya inequality.

2.1 Schwarz Rearrangements of Functions and Sets

Throughout this chapter, $\Omega \subset \mathbb{R}^n$ will represent a measurable set, while $u : \Omega \to \mathbb{R}$ will represent a measurable function.

Definition 2.1.1. (Schwarz Rearrangement of a Set)
The Schwarz rearrangement of Ω is the ball of the same volume as Ω, and it is usually denoted by Ω^*(see Figure 2.1).

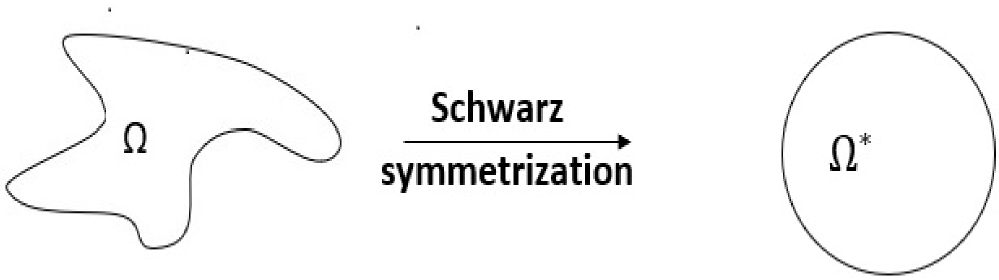


Figure 2.1: Schwarz rearrangement of a set.

Definition 2.1.2. (Schwarz Rearrangement of a Function)
The Schwarz rearrangement of u or the spherical decreasing rearrangement of u, usually denoted by u^*, is the function defined as follows:

$$u^*(x) = sup\{c : x \in \Omega^*(c)\}, \tag{2.1}$$

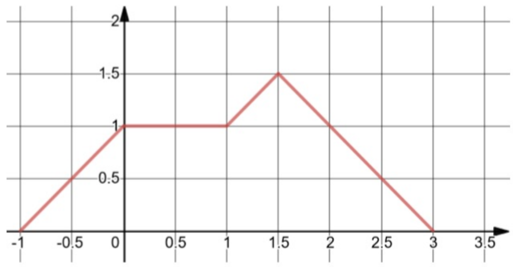
where $\Omega(c)$ is the c-level set of $u(x)$, that is

$$\Omega(c) = \{x \in \Omega : u(x) \geq c\}. \tag{2.2}$$

To better understand and visualize this definition, let us give an example in the one dimension case.

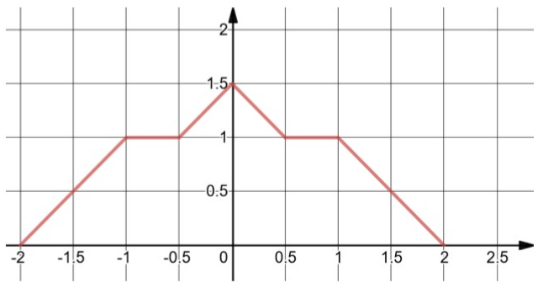
Example 2.1. *Let us consider* $\Omega = [-1, 3]$ *and* $f : \Omega \to \mathbb{R}$ *defined as:*

$$f(x) = \begin{cases} 1 + x & , \ x \in [-1, 0] \\ 1 & , \ x \in [0, 1] \\ x & , \ x \in [1, 3/2] \\ 3 - x & , \ x \in [3/2, 3] \end{cases}$$



Then $\Omega^* = [-2, 2]$ *and*

$$f^*(x) = f^*(-x) = \begin{cases} 3/2 - x & , \ x \in [0, 1/2] \\ 1 & , \ x \in [1/2, 1] \\ 2 - x & , \ x \in [1, 2] \end{cases}$$



In what follows, we consider some basic notations and functions which will be helpful in the next section. First, let us consider the set of all functions that satisfy

$$u(x) \geq 0, \text{ for } x \in \Omega, \tag{2.3}$$

and

$$u = 0, \text{ for } x \in \partial\Omega, \tag{2.4}$$

where $\partial\Omega$ represents the boundary of Ω, and denote this set by $P_0(\Omega)$. Moreover, let us define

$$a(t) := |\Omega(t)|, \tag{2.5}$$

and

$$\Omega(t) := \{x \in \Omega : u(x) > t\}, \tag{2.6}$$

where $a(t)$ is a decreasing function of t. And note that $a(t)$ is continuous if

$$|\{x \in \Omega : u(x) = t\}| = 0, \qquad \forall t \in (0, \bar{u}), \tag{2.7}$$

where $\bar{u} = \sup_{\Omega} u(x)$. On the other hand, if

$$|\{x \in \Omega : u(x) = t\}| \neq 0, \tag{2.8}$$

the function a(t) is discontinuous at $t = t_0$. In this case, its generalized inverse function t(a) is defined by

$$t(a) = t_0, \text{ for } a \in (a(t_0^+), a(t_0^-)), \tag{2.9}$$

where $a(t_0^+) = \lim_{t \to t_0^+} a(t)$ and $a(t_0^-) = \lim_{t \to t_0^-} a(t)$. Now let us establish a formula which expresses the derivative of the function $a(t)$, which is valid if $u(x)$ is analytical and $|\nabla u| \neq 0$ almost everywhere in Ω. First, we define

$$\Gamma(t) := \{x \in \Omega : u(x) = t\}. \tag{2.10}$$

The volume of the domain located between two neighboring level surfaces $\Gamma(t)$ and $\Gamma(t + dt)$ is given by

$$a(t) - a(t + dt) = \int_{\Gamma(t)} dnds + o(dn), \tag{2.11}$$

where ds is the area element on $\Gamma(t)$, dn measures the distance between $\Gamma(t)$ and $\Gamma(t + dt)$ and $dt = |\nabla u|dn + o(dn)$. We then obtain

$$-\frac{da}{dt} = \lim_{dt \to 0} \frac{a(t) - a(t + dt)}{dt} = \int_{\Gamma(t)} \frac{ds}{|\nabla u|}, \quad \underline{u} < t < \bar{u}, \tag{2.12}$$

where $\underline{u} = \inf_{\Omega} u(x)$. Moreover, a generalization of the previous formula is given by the following lemma.

Lemma 2.1.1. If $p(x) \in C(\Omega)$ and $u(x)$ is an analytical function in Ω. We have

$$-\frac{d}{dt} \int_{\Omega(t)} p(x)dx = \int_{\Gamma(t)} p(s) \frac{ds}{|\nabla u|} \quad \underline{u} < t < \bar{u}. \tag{2.13}$$

2.2 Hardy-Littlewood-Pólya Inequality

Theorem 2.2.1. Let us consider

$f(t) : \mathbb{R}_+ \to \mathbb{R}$ a convex non-decreasing function,

$g(t) : \mathbb{R} \to \mathbb{R}_+$ a continuous positive function.

Let $u \in P_0(\Omega)$ be an analytic function such that $|\nabla u| \neq 0$ almost everywhere in Ω. We then have the following inequality

$$\int_{\Omega^*} g(u^*) f(|\nabla u^*|)dx \leq \int_{\Omega} g(u) f(|\nabla u|)dx, \tag{2.14}$$

with equality when either

$$\Omega = \Omega^* \text{ and } u = u^*,$$

or

$$f \equiv const.$$

In particular, we have

$$\int\limits_{\Omega^*} (u^*)^2 \, dx = \int\limits_{\Omega} u^2 dx, \tag{2.15}$$

$$\int\limits_{\Omega^*} |\nabla u^*|^2 \, dx \leq \int\limits_{\Omega} |\nabla u|^2 \, dx. \tag{2.16}$$

For the proof the following two classical inequalities will play an important role. We state them in the following two lemmas.

Lemma 2.2.1. ***(Jensen Inequality)***
Let $f(t)$ be a convex function of a real variable and E be a measurable set. Then, we have the following inequality:

$$f\left(\frac{\int_E c(s)x(s)ds}{\int_E c(s)ds}\right) \leq \frac{\int_E c(s)f(x(s))ds}{\int_E c(s)ds}, \tag{2.17}$$

where $x(s) \in C^1(E), c(s) \geq 0$ is a measurable function. Moreover, if f is strictly convex, then the equality holds if and only if $x(s)$ is identically constant.

Lemma 2.2.2. ***(Geometric Isoperimetric Inequality)***
Among all bounded domains of class C^1 in $\mathbb{R}^n$ of given volume $|\Omega|$, the ball Ω^* has the smallest area. In other words

$$|\partial\Omega|^n \geq |\partial\Omega^*|^n = w_n(n|\Omega|)^{n-1}, \tag{2.18}$$

where $w_n := \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is the area of the unit sphere in $\mathbb{R}^n$. The equality in (2.18) holds if and only if $\Omega = \Omega^*$.

Proof of Theorem 2.2.1

Now we apply the Jensen's inequality (lemma 2.2.1) with

$$E = \Gamma(t), \quad c(s) = \frac{1}{|\nabla u|}, \quad x(s) = |\nabla u|.$$

We thus obtain

$$\int_{\Gamma(t)} \frac{f(\nabla u)}{|\nabla u|} ds \geq \int_{\Gamma(t)} \frac{ds}{|\nabla u|} f\left(\frac{\int_{\Gamma(t)} ds}{\int_{\Gamma(t)} \frac{ds}{|\nabla u|}}\right). \tag{2.19}$$

Since $u \in P_0(\Omega)$, the level surfaces $\Gamma(t)$ are closed and the geometric isoperimetric inequality from Lemma (2.2.2) allows us to write

$$|\Gamma(t)| := \int_{\Gamma(t)} ds \geq (na(t))^{\frac{n-1}{n}} w_n^{\frac{1}{n}}. \tag{2.20}$$

Now combining the previous two inequalities and using

$$\int_{\Gamma(t)} \frac{ds}{|\nabla u|} = -a^{'}(t) > 0 \tag{2.21}$$

we get

$$\int_{\Gamma(t)} \frac{f(|\nabla u|)}{|\nabla u|} ds \geq -a^{'}(t) f\left(\frac{(na(t))^{\frac{n-1}{n}} w_n^{\frac{1}{n}}}{-a^{'}(t)}\right). \tag{2.22}$$

Next, multiplying (2.22) by $g(u(t)) > 0$ and integrating the result from t_0 to $\bar{u}$, we obtain

$$\begin{aligned}\int_{\Omega(t_0)} g(u) f(|\nabla u|) dx &= \int_{t_0}^{\bar{u}} \int_{\Gamma(t)} \frac{g(u) f(|\nabla u|)}{|\nabla u|} ds dt \\ &\geq \int_{t_0}^{\bar{u}} g(t) f\left((na(t))^{\frac{n-1}{n}} w_n^{\frac{1}{n}} (-\frac{dt}{da})\right) da.\end{aligned} \tag{2.23}$$

We thus obtain a lower bound for

$$\int_{\Omega(t_0)} g(u) f(|\nabla u|) dx, \tag{2.24}$$

which only depends on the function a(t) and $\overline{u}$. Therefore, this lower bound also applies to the quantity

$$\int_{\Omega(t_0)^*} g(u^*) f(|\nabla u^*|) dx, \tag{2.25}$$

since we have $a(t) = a^*(t)$ and $\overline{u} = \overline{u^*}$. Moreover, there is equality in the latter case since $\Omega(t)^*$ are balls and $|\nabla u^*| = const.$ on $\Gamma(t)^*$, which completes the proof of the theorem. □

2.3 Faber-Krahn Inequality

Theorem 2.3.1. Among all open bounded sets $\Omega \subseteq \mathbb{R}^n$ of given volume, the ball minimizes the first eigenvalue of the Dirichlet Laplacian. In other words, if c is a positive number and B is the ball of volume c, then:

$$\lambda_1(B) = min\{\lambda_1(\Omega) : \Omega \text{ open bounded set of } \mathbb{R}^n, |\Omega| = c\}, \tag{2.26}$$

or, equivalently,

$$\lambda_1(\Omega^*) \leq \lambda_1(\Omega). \tag{2.27}$$

Proof: To establish the previous result, we apply the Rayleigh's principle to estimate $\lambda_1(\Omega^*)$. More precisely, we choose u_1^*, the Schwarz rearrangement of the first eigenfunction u_1, corresponding to Ω, as the test function in the variational characterization of $\lambda_1(\Omega^*)$. We then have

$$\lambda_1(\Omega^*) \leq \frac{\int_{\Omega^*} |\nabla u_1^*|^2 dx}{\int_{\Omega^*} (u_1^*)^2 dx}. \tag{2.28}$$

On the other hand, from Hardy-Littlewood-Pólya inequality we know that

$$\frac{\int_{\Omega^*} |\nabla u_1^*|^2 dx}{\int_{\Omega^*} (u_1^*)^2 dx} \leq \frac{\int_{\Omega} |\nabla u_1|^2 dx}{\int_{\Omega} u_1^2 dx} = \lambda_1(\Omega). \tag{2.29}$$

In conclusion, combining (2.28) and (2.29), we obtain

$$\lambda_1(\Omega^*) \leq \lambda_1(\Omega). \quad \square$$

3.1 Conformal Transplantation: Szegö's Inequality

A conformal mapping, also called a conformal transformation, is a transformation that preserves local angles. To prove isoperimetric inequality for the first frequency of the free membrane, G.Szegö in [11] considered as test functions in the variational characterization of μ_1 and μ_2 some conformal transplantations of the first two eigenfunctions on the unit disk.

Theorem 3.1.1. ***(Szegö's Isoperimetric Inequality)***

$$\frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)} \geq \frac{1}{\mu_1(\Omega^*)} + \frac{1}{\mu_2(\Omega^*)}. \tag{3.1}$$

In particular, we have

$$|\Omega|\mu_1(\Omega) \leq |\mathbb{D}|\mu_1(\mathbb{D}), \tag{3.2}$$

or, equivalently,

$$\mu_1(\Omega) \leq \mu_1(\Omega^*). \tag{3.3}$$

Proof: Let $f : \mathbb{D} \to \Omega$ be a conformal mapping of the unit disk $\mathbb{D}$ into Ω.
We consider two cases:
First Case: Ω is symmetric of order 2, which means

$$z \in \Omega \Leftrightarrow -z \in \Omega, \tag{3.4}$$

that is, by doing a rotation of angle π we get the same domain (see Figure 3.1). In such a case, we clearly have $f(0) = 0$ and $f(-\mathfrak{z}) = -f(\mathfrak{z})$.

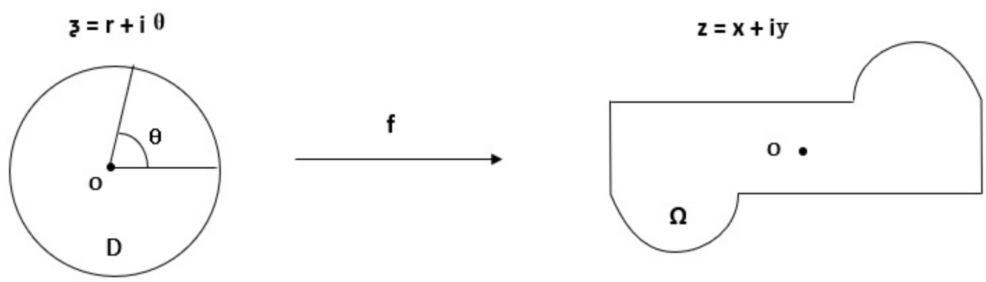


Figure 3.1: Conformal mapping.

Next, let us define

$$U_k(z,\gamma) := u_k(f^{-1}(z),\gamma), \text{ for } k=1,2, \tag{3.5}$$

where

$$u_1(\mathfrak{z},\gamma) = J_1(j'_{1,1}r)\cos(\theta+\gamma),$$

and

$$u_2(\mathfrak{z},\gamma) = J_1(j'_{1,1}r)\sin(\theta+\gamma),$$

are the known first two eigenfunctions on the unit disk, γ is an arbitrary constant to be chosen and $j'_{1,1} \approx 1.84$ is the first positive zero of the derivative of $J_1(t)$. Clearly,

$$\int_\Omega U_1(z,\frac{\pi}{2})U_2(z,\frac{\pi}{2})dAz = -\int_\Omega U_1(z,0)U_2(z,0)dAz. \tag{3.6}$$

Therefore, by the intermediate value theorem, there exists $\gamma_0 \in [0,\frac{\pi}{2}]$ such that

$$\int_\Omega U_1(z,\gamma_0)U_2(z,\gamma_0)dAz = 0. \tag{3.7}$$

In what follows, we will fix $\gamma=\gamma_0$, omit γ as an argument of u_k and U_k, $k=1,2$. And make use of the following immediate properties:

1. $\int_\Omega U_1U_2dAz = 0$ (by the appropriate choice of γ).

2. $\int_\Omega U_kdAz = 0$ for k = 1,2 (since $U_k(-z) = -U_k(z)$).

3. $\int_\Omega |\nabla U_1|^2dAz = \int_{\mathbb{D}} |\nabla u_1|^2dA\mathfrak{z} = \int_{\mathbb{D}} |\nabla u_2|^2dA\mathfrak{z} = \int_\Omega |\nabla U_2|^2dAz.$

4. $\int_\Omega \nabla U_1\nabla U_2dAz = \int_{\mathbb{D}} \nabla u_1\nabla u_2dA\mathfrak{z} = 0$ (by conformal invariance).

Using U_1 and U_2 as test functions in the variational characterization of μ_1, we have

$$\mu_1(\Omega) \le R[U_k], \tag{3.8}$$

where

$$R[U_k] := \frac{\int_\Omega |\nabla U_k|^2dAz}{\int_\Omega U_k^2dAz}, \qquad k=1,2. \tag{3.9}$$

This implies that

$$\mu_1(\Omega) \le min\{R[U_1],R[U_2]\}. \tag{3.10}$$

Now, let us consider the following linear combination

$$U(z) = c_1 U_1(z) + c_2 U_2(z). \tag{3.11}$$

Then,

$$R[U] = \frac{\int_\Omega |\nabla U|^2 dAz}{\int_\Omega U^2 dAz} = \epsilon_1 R[U_1] + \epsilon_2 R[U_2] \leq max\{R[U_1], R[U_2]\}, \tag{3.12}$$

with

$$\epsilon_k := \frac{c_k \int_\Omega U_k^2 dAz}{c_1^2 \int_\Omega U_1^2 dAz + c_2^2 \int_\Omega U_2^2 dAz} \quad \text{and} \quad \epsilon_1 + \epsilon_2 = 1. \tag{3.13}$$

Clearly there exists $(c_1, c_2) \neq (0,0)$ such that

$$\int_\Omega U_1(z) U_2(z) dAz = 0. \tag{3.14}$$

Using now the variational characterization of μ_2, we have

$$\mu_2 \leq R[U] \leq max\{R[U_1], R[U_2]\}. \tag{3.15}$$

Next, we will find some lower bounds for $R[U_k]$ where $k = 1, 2$. With

$$z = f(\mathfrak{z}) = \sum_{n=1}^{\infty} C_n \mathfrak{z}^n, \text{ where } \mathfrak{z} = re^{i\theta}, \tag{3.16}$$

we compute

$$z' = \sum_{n=1}^{\infty} n C_n \mathfrak{z}^{n-1}, \tag{3.17}$$

$$|z'|^2 = z' \bar{z'} = \sum_{n,k=1}^{\infty} nk C_n \bar{C}_k r^{n+k-2} e^{i(n-k)\theta}, \tag{3.18}$$

and

$$|\Omega| = \int_\Omega dAz = \int_{\mathbb{D}} |z'|^2 r dr d\theta = 2\pi \sum_{n=1}^{\infty} n^2 |C_n|^2 \int_0^1 r^{2n-1} dr = \pi \sum_{n=1}^{\infty} n |C_n|^2. \tag{3.19}$$

Moreover, with

$$u_1(\mathfrak{z}) = J_1(j'_{1,1} r) \cos\theta \quad \text{and} \quad u_2(\mathfrak{z}) = J_1(j'_{1,1} r) \sin\theta, \tag{3.20}$$

we have

$$\int_\Omega U_1^2 dAz = \int_{\mathbb{D}} u_1^2 |z'|^2 r dr d\theta = \pi \sum_{n=1}^{\infty} n |C_n|^2 M_n + \alpha, \tag{3.21}$$

and

$$\int_\Omega U_2^2 dAz = \int_{\mathbb{D}} u_2^2 |z'|^2 r dr d\theta = \pi \sum_{n=1}^{\infty} n|C_n|^2 M_n - \alpha, \tag{3.22}$$

where

$$\alpha := \pi \sum_{n=1}^{\infty} n(n+2) Re(C_n \overline{C_{n+2}}) \int_0^1 J_1^2(j_{1,1}' r) r^{2n+1} dr, \tag{3.23}$$

and

$$M_n := n \int_0^1 J_1^2(j_{1,1}' r) r^{2n-1} dr. \tag{3.24}$$

Now, we can easily notice that

$$M_1 < M_2 < ... <, \tag{3.25}$$

which implies

$$M_n \geq M_1 = \int_0^1 J_1^2(j_{1,1}' r) r dr. \tag{3.26}$$

In conclusion

$$\int_\Omega (U_1^2 + U_2^2) dAz = 2\pi \sum_{n=1}^{\infty} n|C_n|^2 M_n \geq 2M_1 |\Omega|,$$

and

$$\begin{aligned} \int_\Omega |\nabla U_k| dAz &= \int_{\mathbb{D}} |\nabla u_k|^2 r dr d\theta = \mu_k(\mathbb{D}) \int_{\mathbb{D}} u_k^2 r dr d\theta \\ &= \pi \mu_k(\mathbb{D}) \int_0^1 J_1^2(j_{1,1}' r) r dr = \pi M_1 \mu_k(\mathbb{D}), \qquad k = 1, 2. \end{aligned} \tag{3.27}$$

It then follows that

$$\begin{aligned} \frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)} &\geq R^{-1}[U_1] + R^{-1}[U_2] = \frac{\int_\Omega U_1^2 + U_2^2 dAz}{\int_\Omega |\nabla u_1|^2 dAz} \\ &\geq \frac{2|\Omega|}{\pi} . \frac{1}{\mu_1(\mathbb{D})} = \frac{|\Omega|}{|\mathbb{D}|} \left(\frac{1}{\mu_1(\mathbb{D})} + \frac{1}{\mu_2(\mathbb{D})} \right), \end{aligned} \tag{3.28}$$

and inequality (3.1) is achieved in this case. Obviously, in particular we have

$$|\Omega| \mu_1(\Omega) \leq |\mathbb{D}| \mu_1(\mathbb{D}).$$

Second Case: (with no symmetry assumption on Ω)

The orthogonality conditions $\int_\Omega U_k dAz = 0$, for $k = 1, 2$, are not satisfied in general. However, Szegö adjusts the situation replacing $z = f(\mathfrak{z})$ by

$$z = f\left(\frac{\mathfrak{z} - a}{1 - \bar{a}\mathfrak{z}} \right), \quad \text{with } |a| < 1,$$

and using a topological argument which guarantee that there exists $\mathfrak{z}$ and a such that the orthogonality conditions mentioned above are satisfied (see, [11] for more details). □

3.2 Radial Extension: Weinberger's Inequality

Szegö's inequality (3.3) was extended to higher dimension by Weinberger in [13].

Theorem 3.2.1. The ball maximizes the second Neumann eigenvalue among Lipschitz open sets of given volume. In other words,

$$\mu_1(\Omega) \leq \mu_1(\Omega^*), \tag{3.29}$$

where Ω is a Lipschitz bounded domain in $\mathbb{R}^N$ and Ω^* is the ball of the same volume.

Proof: Let R be the radius of Ω^* and μ_2^* its second eigenvalue, which has multiplicity N, and is associated to the following N eigenfunctions:

$$\frac{g(r)}{r}x_i, \text{ for } i = 1, 2, ..., N, \tag{3.30}$$

where g is given by the Bessel function $J_{N/2}$, that is

$$g(r) = J_{N/2}(j'_{N/2,1}\frac{r}{R}), \tag{3.31}$$

and

$$\mu_2^* = \left(\frac{j'_{N/2,1}}{R}\right)^2. \tag{3.32}$$

Now, let us notice that R is the first zero of g', while g satisfies the following ordinary differential equation

$$g''(r) + \frac{N-1}{r}g'(r) + (\mu_2^* - \frac{N-1}{r^2})g(r) = 0. \tag{3.33}$$

Next, we define the continuous extension of g:

$$G(r) = \begin{cases} g(r), & r \leq R, \\ g(R), & r > R. \end{cases} \tag{3.34}$$

Next, we are going to use the following variational characterization of μ_2

$$\mu_2(\Omega) = \min_{\substack{v \in H_0^1(\Omega), v \neq 0, \\ \int_\Omega v = 0}} \frac{\int_\Omega |\nabla v(x)|^2 dx}{\int_\Omega v^2(x) dx}. \tag{3.35}$$

To this end, we are going to introduce the following functions

$$f_i(x) := G(r)\frac{x_i}{r}, \tag{3.36}$$

and try to use them as test functions in the variational characterization given in (3.35). Let us now compute

$$\frac{\partial f_i}{\partial x_j} = \frac{G^{'}(r)x_i x_j}{r^2} - \frac{G(r)x_j x_i}{r^3} + \delta_{ij}\frac{G(r)}{r}, \tag{3.37}$$

where δ_{ij} is the kronecker symbol. We then get, for $i = 1, ..., N$, the following inequality

$$\mu_2(\Omega) \leq \frac{\int_\Omega [G^{'2}(r)\frac{x_i^2}{r^2} + G^2(r)(1 - \frac{x_i^2}{r^2})]/r^2 dx}{\int_\Omega [G^2(r)\frac{x_i^2}{r^2}]dx}. \tag{3.38}$$

By multiplying each of these inequalities by the denominator on the right and adding the resulting inequalities, we obtain

$$\mu_2(\Omega) \leq \frac{\int_\Omega [G^{'2}(r) + (N-1)\frac{G^2(r)}{r^2}]dx}{\int_\Omega G^2(r)dx}. \tag{3.39}$$

Now, let us denote by Ω_1 the intersection of Ω with the ball Ω^*. Since R is the first zero of $g^{'}$, we have that G(r) is nondecreasing for $r > 0$. Therefore,

$$\begin{aligned} \int_\Omega G^2(r)dx &= \int_{\Omega_1} G^2(r)dx + \int_{\Omega \diagup \Omega_1} G^2(r)dx \\ &\geq \int_{\Omega_1} G^2(r)dx + G^2(R)\int_{\Omega \diagup \Omega_1} dx, \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} \int_{\Omega^*} G^2(r)dx &= \int_{\Omega_1} G^2(r)dx + \int_{\Omega^* \diagup \Omega_1} G^2(r)dx \\ &\leq \int_{\Omega_1} G^2(r)dx + G^2(R)\int_{\Omega^* \diagup \Omega_1} dx. \end{aligned} \tag{3.41}$$

Since Ω and Ω^* have the same volume, from (3.40) and (3.41) we obtain

$$\int_\Omega G^2(r)dx \geq \int_{\Omega^*} G^2(r)dx = \int_{\Omega^*} g^2(r)dx. \tag{3.42}$$

Differentiating now the integrand in the numerator of (3.39), we obtain

$$\frac{d}{dr}\left[G^{'2}(r) + (N-1)\frac{G^2(r)}{r^2}\right] = 2G^{'}G^{''} + \frac{2(N-1)(rGG^{'} - G^2)}{r^3}, \tag{3.43}$$

which leads us to

$$\frac{d}{dr}\left[G^{'2}(r) + (N-1)\frac{G^2(r)}{r^2}\right] = -2\mu^* GG^{'} - \frac{(N-1)(rG - G)^2}{r^3} < 0. \tag{3.44}$$

Therefore, the integrand in the numerator of (3.39) is decreasing, for $r > 0$, and we can prove that

$$\int_\Omega \left[G^{'2}(r) + (N-1)\frac{G^2(r)}{r^2}\right]dx \leq \int_{\Omega^*} [g^{'2}(r) + (N-1)\frac{g^{'}(r)}{r^2}]dx, \tag{3.45}$$

where equality holds for the ball. On the other hand, using integration by parts, we get

$$\int_{\Omega^*}\left[g^{'2}(r)+(N-1)\frac{g^2(r)}{r^2}\right]dx=\mu_2^*\int_{\Omega^*}g^2dx. \tag{3.46}$$

Finally, the combination of (3.45), (3.44), (3.41) and (3.38) yields to the desired result, that is

$$\mu_2(\Omega)\leq\mu_2(\Omega^*),$$

and the proof is achieved. □

Chapter 4. Universal Bounds

In this chapter, we are going to present some universal bounds for frequencies of membranes and plates. In the first section, we will focus on some seminal paper of L.E. Payne, G. Pólya and H.F. Weinberger in [9]. More precisely, we present the inequalities obtained in their paper for ratios of low frequencies of membranes and plates. Thereafter, in the next section, we present an extension to higher dimension, obtained by C.J. Thompson in [12]. Finally we will improve and extend PPW's universal inequalities for the clamped and buckled plates.

4.1 Payne-Pólya-Weinberger Inequalities

Let Ω be a bounded domain in the xy-plane with a smooth boundary, $\partial\Omega$. Let us consider the following three eigenvalue problems, related with the fixed membrane, the clamped plate and the buckled plate, respectively.

$$\begin{cases} \Delta u + \lambda u = 0, & \Omega \subseteq R^2, \\ u = 0, & \partial\Omega, \end{cases} \tag{4.1}$$

$$\begin{cases} \Delta\Delta u - \mu u = 0, & \Omega \subseteq R^2, \\ u = \frac{\partial u}{\partial n} = 0, & \partial\Omega, \end{cases} \tag{4.2}$$

$$\begin{cases} \Delta\Delta u + \nu\Delta u = 0, & \Omega \subseteq R^2, \\ u = \frac{\partial u}{\partial n} = 0, & \partial\Omega, \end{cases} \tag{4.3}$$

where Δ denotes the Laplace operator and n the outer unit normal to $\partial\Omega$.
The eigenvalues of the above problems are denoted as follows

$$\lambda_1 < \lambda_2 \le \lambda_3 \le \dots\,,$$

$$\mu_1 \le \mu_2 \le \mu_3 \le \dots\,,$$

$$\nu_1 \le \nu_2 \le \nu_3 \le \dots\,.$$

This section will be divided into five subsections. In subsection 1 we present some general results which apply equally to the case of the fixed membrane problem and the clamped plate problem. Subsections 2, 3 and 4 will reveal some universal bounds corresponding respectively to the eigenvalue problems (4.1), (4.2) and (4.3) and the proofs for finding them. Finally, subsection 5 will contain some final remarks.

4.1.1 Membranes and plates. Let us denote by $u_1, u_2, ..., u_n$, the first n eigenfunctions corresponding equally to problems (4.1), (4.2) and (4.3). Clearly, these eigenfunctions satisfy the following properties:

$$u_i = 0 \text{ on } \partial\Omega, \qquad i = 1, ..., n, \tag{4.4}$$

$$\int_\Omega u_i u_j = \begin{cases} 1 \text{ when } i = j, \\ 0 \text{ when } i \neq j, \end{cases} \qquad i, j = 1, ..., n. \tag{4.5}$$

Let us choose the direction of the coordinate axes such that:

$$\sum_{i=1}^{n} \int_\Omega u_{ix}^2 dA = \sum_{i=1}^{n} \int_\Omega u_{iy}^2 dA. \tag{4.6}$$

where u_{ix} denotes $\frac{\partial u_i}{\partial x}$ and u_{iy} denotes $\frac{\partial u_i}{\partial y}$. This choice is possible since if we admit that we have an inequality (4.6), then a rotation of $90°$ with respect to the origin will interchange the axes and reverse the inequality. Therefore, there is an intermediate position in which (4.6) holds. Now, let us build n functions $\varphi_1, \varphi_2, ..., \varphi_n$ which satisfy:

$$\varphi_i = 0 \text{ on } \partial\Omega, \tag{4.7}$$

$$\int_\Omega \varphi_i u_j dA = 0, \qquad i, j = 1, ..., n. \tag{4.8}$$

To this end, we choose

$$\varphi_i = x u_i - \sum_{k=1}^{n} a_{ik} u_k, \tag{4.9}$$

where a_{ik} are some appropriate constant to be chosen. Clearly (4.7) is satisfied by the virtue of (4.4). We will now choose a_{ik} such that (4.8) is satisfied too. We then have

$$a_{ij} = \int_\Omega x u_i u_j dA = a_{ji}. \tag{4.10}$$

Moreover from (4.8) and (4.9) we have:

$$\int_\Omega \varphi_i^2 dA = \int_\Omega x u_i \varphi_i dA. \tag{4.11}$$

Next, let us compute the following sum, which will be useful in later computations.

$$-2\sum_{i=1}^{n}\int_{\Omega}u_{ix}\varphi_i dA = -\sum_{i=1}^{n}\int_{\Omega}2xu_iu_{ix}dA + 2\sum_{i=1}^{n}a_{ik}\int_{\Omega}u_{ik}u_k dA, \tag{4.12}$$

Now, let us note that

$$(u_iu_k)_x = u_{ix}u_k + u_iu_{kx}, \tag{4.13}$$

so

$$\int_{\Omega}u_{ix}u_k dA = -\int_{\Omega}u_iu_{kx}dA. \tag{4.14}$$

Therefore, we can conclude that $\int_{\Omega}u_{ix}u_k dA$ is antisymmetric. In addition, using the fact that $a_{ik} = a_{ki}$, we get:

$$\sum_{i,k=1}^{n}a_{ik}\int_{\Omega}u_{ik}u_k dA = 0. \tag{4.15}$$

On the other hand,

$$(2xu_iu_i)_x = (2xu_i)_x u_i + 2xu_iu_{ix} = 4xu_iu_{ix} + 2u_i^2, \tag{4.16}$$

so

$$-2\int_{\Omega}xu_iu_{ix}dA = \int_{\Omega}u_i^2 dA = 1. \tag{4.17}$$

Now, using (4.17) and (4.15), we obtain

$$-2\sum_{i=1}^{n}\int_{\Omega}u_{ix}\varphi_i dA = \sum_{i=1}^{n}1 = n. \tag{4.18}$$

Using now, Schwarz's inequality, we can derive the following useful inequality

$$n^2 \le 4\sum_{i=1}^{n}\int_{\Omega}\varphi_i^2 dA\sum_{i=1}^{n}\int_{\Omega}u_{ix}^2 dA = 2\sum_{i=1}^{n}\int_{\Omega}\varphi_i^2 dA. \tag{4.19}$$

4.1.2 Universal bounds for the eigenvalues of the fixed membrane.

Theorem 4.1.1. If $\lambda_1 < \lambda_2 \le \lambda_3 \le ...$, are the eigenvalues for problem (4.1), then

$$\lambda_{n+1} \le \lambda_n + \frac{2(\lambda_1 + ... + \lambda_n)}{n}, \qquad n \ge 1. \tag{4.20}$$

In particular, we have

$$\lambda_{n+1} \le 3\lambda_n, \qquad n \ge 1. \tag{4.21}$$

Proof: We regard as known the first n eigenfunctions $u_1, u_2, ..., u_n$ and the corresponding eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of problem (4.1). Obviously, these eigenfunctions also satisfy the additional property:

$$\Delta u_i + \lambda_i u_i = 0 \text{ on } \Omega. \tag{4.22}$$

Now, by (4.22) and (4.5), we obtain

$$\int_\Omega \Delta u_i . u_i dA = -\lambda_i \int_\Omega u_i^2 dA = -\lambda_i. \tag{4.23}$$

On the other hand, using Green's theorem, we have

$$\int_\Omega u_i \Delta u_i dA + \int_\Omega |\nabla u|^2 dA = 0, \tag{4.24}$$

since $u_i = 0$ on $\partial\Omega$. Combining the previous two equations, we obtain

$$\int_\Omega |\nabla u_i|^2 dA = \int_\Omega (u_{ix}^2 + u_{iy}^2) dA = \lambda_i. \tag{4.25}$$

Next, from the variational characterization of λ_{n+1}, we have

$$\lambda_{n+1} \leq \frac{-\int_\Omega \varphi_i \Delta \varphi_i dA}{\int_\Omega \varphi_i^2 dA}, \tag{4.26}$$

which is valid for any function φ_i, $i = 1, ..., n$, since it satisfies (4.7) and (4.8).

Next, let us compute

$$\begin{aligned} \Delta \varphi_i &= \Delta x u_i + 2\nabla x \nabla u_i + x \Delta u_i - \sum_{k=1}^{n} a_{ik} \Delta u_k \\ &= 2u_{ix} - \lambda_i x u_i + \lambda_k \sum_{k=1}^{n} a_{ik} u_k. \end{aligned} \tag{4.27}$$

Then, from (4.8) and (4.11), it follows that

$$-\int_\Omega \varphi_i \Delta \varphi_i dA = \lambda_i \int_\Omega x u_i \varphi_i dA - 2\int_\Omega u_{ix} \varphi_i dA - \lambda_k \sum_{k=1}^{n} a_{ik} \int_\Omega u_k \varphi_i dA. \tag{4.28}$$

So, by (4.15), we have

$$-\int_\Omega \varphi_i \Delta \varphi_i dA = \lambda_i \int_\Omega \varphi_i^2 dA - 2\int_\Omega u_{ix} \varphi_i dA. \tag{4.29}$$

Substituting (4.29) into (4.26), we get

$$\lambda_{n+1} \leq \lambda_i - \frac{2\int_\Omega u_{ix}\varphi_i dA}{\int_\Omega \varphi_i^2 dA} \leq \lambda_n - \frac{2\int_\Omega u_{ix}\varphi_i dA}{\int_\Omega \varphi_i^2 dA}. \tag{4.30}$$

Finally, using (4.18), (4.19) and (4.25), we obtain successively

$$\begin{aligned} \lambda_{n+1} - \lambda_n &\leq -\frac{2\int_\Omega u_{ix}\varphi_i dA}{\int_\Omega \varphi_i^2 dA} = \frac{n}{\int \varphi_i^2 dA} \\ &\leq \frac{2\sum_{i=1}^n \int_\Omega |\nabla u_i|^2 dA}{n} = \frac{2\sum_{i=1}^n \lambda_i}{n}, \end{aligned} \tag{4.31}$$

which implies

$$\lambda_{n+1} \leq \lambda_n + \frac{2(\lambda_1 + ... + \lambda_n)}{n}, \tag{4.32}$$

and the proof is achieved. □

4.1.3 Universal bounds for the eigenvalues of the clamped plate.

Theorem 4.1.2. If $\mu_1 \leq \mu_2 \leq ...$, are the eigenvalues for problem (4.2), then

$$\mu_{n+1} \leq \mu_n + \frac{8(\mu_1 + ... + \mu_n)}{n}, \qquad n \geq 1. \tag{4.33}$$

In particular, we have

$$\mu_{n+1} \leq 9\mu_n, \qquad n \geq 1. \tag{4.34}$$

Proof: We regard now as known the first n eigenfunctions $u_1, u_2, ..., u_n$ and the corresponding eigenvalues $\mu_1, \mu_2, ..., \mu_n$ of problem (4.2). Obviously, these eigenfunctions also satisfy the following two additional properties specific to this problem:

$$\Delta\Delta u_i - \mu_i u_i = 0 \qquad \text{on} \qquad \Omega, \tag{4.35}$$

and

$$\frac{\partial u_i}{\partial n} = 0 \qquad \text{on} \qquad \partial\Omega. \tag{4.36}$$

Clearly, by Green's theorem and the boundary conditions, we have

$$\int_\Omega \Delta u_i \Delta u_i dA - \int_\Omega u_i \Delta\Delta u_i dA = 0, \tag{4.37}$$

which leads us to

$$\mu_i = \mu_i \int_\Omega u_i^2 dA = \int_\Omega (\Delta u_i)^2 dA. \tag{4.38}$$

Moreover, from the variational characterization of μ_{n+1}, we have

$$\mu_{n+1} \leq \frac{\int_\Omega (\Delta \varphi_i)^2 dA}{\int_\Omega \varphi_i^2 dA} = \frac{\int_\Omega \varphi_i \Delta\Delta\varphi_i dA}{\int_\Omega \varphi_i^2 dA}, \tag{4.39}$$

which is valid for any function φ_i, satisfying (4.4) and (4.5).

Next, let us compute successively,

$$\varphi_i = x u_i - \sum_{k=1}^n a_{ik} u_k, \tag{4.40}$$

$$\Delta\varphi_i = 2u_{ix} + x\Delta u_i - \sum_{k=1}^n a_{ik}\Delta u_k, \tag{4.41}$$

and

$$\Delta\Delta\varphi_i = 4\Delta u_{ix} + \mu_i x u_i - \sum_{k=1}^n \mu_k a_{ik} u_k. \tag{4.42}$$

Now, multiplying (4.42) by φ_i and integrating the result, we get

$$\int_\Omega \varphi_i \Delta\Delta\varphi_i dA = 4\int_\Omega \varphi_i \Delta u_{ix} dA + \mu_i \int_\Omega x u_i \varphi_i dA - \sum_{k=1}^n \mu_k a_{ik} \int_\Omega u_k \varphi_i dA, \tag{4.43}$$

which yields to

$$\int_\Omega \varphi_i \Delta\Delta\varphi_i dA = 4\int_\Omega \varphi_i \Delta u_{ix} dA + \mu_i \int_\Omega \varphi_i^2 dA. \tag{4.44}$$

Substituting now (4.44) into (4.39), we obtain

$$\mu_{n+1} \leq \mu_i + \frac{4\int_\Omega \varphi_i \Delta u_{ix} dA}{\int_\Omega \varphi_i^2 dA}, \tag{4.45}$$

so

$$\mu_{n+1} \leq \mu_n + \frac{4\sum_{i=1}^n \int_\Omega \varphi_i \Delta u_{ix} dA}{\sum_{i=1}^n \int_\Omega \varphi_i^2 dA}. \tag{4.46}$$

Next, let us compute the numerator of the right hand side of (4.46). Using (4.40), we get

$$2\sum_{i=1}^n \int_\Omega \varphi_i \Delta u_{ix} dA = 2\sum_{i=1}^n \int_\Omega x u_i \Delta u_{ix} dA - 2\sum_{i=1}^n a_{ik} \int_\Omega u_k \Delta u_{ix} dA. \tag{4.47}$$

On the other hand, since

$$(u_k \Delta u_i)_x = u_k \Delta u_{ix} + u_{kx} \Delta u_i, \tag{4.48}$$

we have

$$\int_\Omega u_k \Delta u_{ix} dA = -\int_\Omega u_{kx} \Delta u_i dA = -\int_\Omega u_i \Delta u_{kx} dA. \tag{4.49}$$

Moreover, since the matrix a_{ik} is symmetric, we have

$$\sum_{i,k=1}^{n} a_{ik} \int_\Omega u_k \Delta u_{ix} dA = 0. \tag{4.50}$$

Next, let us compute

$$J_i = 2 \int_\Omega x u_i \Delta u_{ix} dA. \tag{4.51}$$

To this end, we first note that

$$(x u_i \Delta u_i)_x = x u_i \Delta u_{ix} + u_i \Delta u_i + x u_{ix} \Delta u_i, \tag{4.52}$$

so

$$\begin{aligned} 2 \int_\Omega x u_i \Delta u_{ix} dA &= -2 \int_\Omega u_i \Delta u_i dA - 2 \int_\Omega x u_{ix} \Delta u_i dA \\ &= -2 \int_\Omega (u_i + x u_{ix}) \Delta u_i dA \\ &= -2 \int_\Omega (u_i + x u_{ix}) u_{ixx} dA - 2 \int_\Omega (u_i + x u_{ix}) u_{iyy} dA. \end{aligned} \tag{4.53}$$

Moreover, we also note that

$$[(x u_i)_x u_{iy}]_y = (x u_i)_{xy} u_{iy} + (x u_i)_x u_{iyy}, \tag{4.54}$$

yields to

$$\int_\Omega (u_i + x u_{ix}) u_{iyy} dA = -\int_\Omega (u_{iy} + u_{iyx}) u_{iy} dA. \tag{4.55}$$

In conclusion, using (4.55) and (4.53), we have

$$J_i = -2 \int_\Omega u_i u_{ixx} dA - 2 \int_\Omega x u_{ix} u_{ixx} dA + 2 \int_\Omega u_{iy}^2 dA + 2 \int_\Omega x u_{ixy} u_{iy} dA. \tag{4.56}$$

Next, the identity

$$(u_i u_{ix})_x = u_i u_{ixx} + u_{ix}^2,$$

implies

$$\int_\Omega u_i u_{ixx} dA = -\int_\Omega u_{ix}^2 dA, \tag{4.57}$$

while the identity

$$(xu_{ix}^2)_x = 2xu_{ix}u_{ixx} + u_{ix}^2,$$

implies

$$2\int_\Omega xu_{ix}u_{ixx} dA = -\int_\Omega u_{ix}^2 dA. \tag{4.58}$$

Also, in a similar way, the identity

$$(xu_{iy}^2)_y = 2xu_{iy}u_{iyy} + u_{iy}^2,$$

leads to

$$2\int_\Omega xu_{ixy}u_{iy} dA = -\int_\Omega u_{iy}^2 dA. \tag{4.59}$$

Therefore substituting (4.57), (4.58) and (4.59) into (4.56), we obtain

$$J_i = 3\int_\Omega u_{ix}^2 dA + \int_\Omega u_{iy}^2 dA, \tag{4.60}$$

Combining now (4.60), (4.50) and (4.47), we get

$$2\sum_{i=1}^n \int_\Omega \varphi_i \Delta u_{ix} dA = 2\sum_{i=1}^n J_i = 2\sum_{i=1}^n (u_{ix}^2 + u_{iy}^2) dA. \tag{4.61}$$

Finally, using (4.46), (4.61) and Schwarz's inequality, we have

$$\mu_{n+1} - \mu_n \le \frac{4\sum_{i=1}^n \int_\Omega |\nabla u_i|^2 dA}{\sum_{i=1}^n \int_\Omega \varphi_i^2 dA} = \frac{-4\sum_{i=1}^n \int_\Omega u_i \Delta u_i dA}{\sum_{i=1}^n \int \varphi_i^2 dA}$$

$$= \frac{8[\sum_{i=1}^n \int_\Omega u_i \Delta u_i dA]^2}{2\sum_{i=1}^n \int_\Omega \varphi_i^2 dA . \sum_{i=1}^n \int_\Omega |\nabla u_i|^2 dA}$$

$$\le \frac{8\sum_{i=1}^n \int_\Omega u_i^2 dA . \sum_{i=1}^n \int_\Omega (\Delta u_i)^2 dA}{n^2} = \frac{8\sum_{i=1}^n \mu_i}{n},$$

and the proof is achieved. □

4.1.4 Universal bounds for the eigenvalues of the buckled plate.

Theorem 4.1.3. If $\nu_1 \le \nu_2 \le ...$, are the eigenvalues for problem (4.3), then

$$\nu_2 \le 3\nu_1. \tag{4.62}$$

Proof: We regard as known the first eigenfunction u and its corresponding eigenvalue ν_1 of problem (4.3). Obviusly, this eigenfunction satisfies

$$\begin{cases} \Delta\Delta u + \nu_1 \Delta u = 0, & \Omega \subseteq R^2, \\ u = \dfrac{\partial u}{\partial n} = 0, & \partial\Omega. \end{cases} \tag{4.63}$$

Next, let us introduce the following notations:

$$D(f,g) := \int_\Omega \nabla f \nabla g dA, \qquad D(f) := D(f,f). \tag{4.64}$$

It then follows, from (4.64) and Green's theorem, that

$$\int_\Omega \Delta\Delta u dA = \int_\Omega (\Delta u)^2 dA = D(u_x) + D(u_y) = \nu_1 D(u). \tag{4.65}$$

On the other hand, from the variational characterization of ν_2, we have

$$\nu_2 \le \frac{\int_\Omega (\Delta\varphi)^2 dA}{D(\varphi)} = \frac{\int_\Omega \varphi \Delta\Delta\varphi dA}{D(\varphi)}, \tag{4.66}$$

provided that a trial function φ satisfies

$$\varphi = \frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial\Omega, \tag{4.67}$$

and

$$D(u,\varphi) = 0. \tag{4.68}$$

Next, we know that there exists a system of coordinates such that $\varphi = xu$ is admissible for the variational characterization of ν_2. Indeed, let us cover Ω with matter of surface density $u_x^2 + u_y^2$. So we can assume that the center of gravity of this mass is at the origin. Hence we can find a system of coordinates such that

$$\int_\Omega x(u_x^2 + u_y^2) dA = \int_\Omega y(u_x^2 + u_y^2) dA = 0. \tag{4.69}$$

Therefore,

$$D(u,\varphi) = \int_\Omega [u_x(xu_x+u)+u_y xu_y]dA = \int_\Omega uu_x dA + \int_\Omega x(u_x^2+u_y^2)dA = 0. \quad (4.70)$$

In conclusion $\varphi = xu$ is indeed admissible for the variational characterization of ν_2.

Now, let us compute

$$D(\varphi) = \int_\Omega [(xu_x+u)^2 + (xu_y)^2]dA = \int_\Omega x^2(u_x^2+u_y^2)dA = \int_\Omega x^2|\nabla u|^2 dA. \quad (4.71)$$

Next, we also compute

$$\Delta\varphi = \Delta(xu) = x\Delta u + 2u_x, \quad (4.72)$$

and

$$\Delta\Delta\varphi = x\Delta\Delta u + 4\Delta u_x = -x\nu_1\Delta u + 4\Delta u_x. \quad (4.73)$$

Multiplying now (4.73) by φ and integrating the result, we get

$$\int_\Omega \varphi\Delta\Delta\varphi dA = -\nu_1\int_\Omega x^2 u\Delta u dA + 4\int_\Omega \varphi\Delta u_x dA. \quad (4.74)$$

In addition, since

$$\nabla(x^2u\nabla u) = x^2u\Delta u + 2xuu_x + x^2|\nabla u|^2, \quad (4.75)$$

we have

$$\begin{aligned} -\int_\Omega x^2u\Delta u dA &= 2\int_\Omega xuu_x dA + \int_\Omega x^2|\nabla u|^2 dA \\ &= D(\varphi) - \int_\Omega u^2 dA < D(\varphi). \end{aligned} \quad (4.76)$$

Similarly, since

$$\nabla(\varphi\nabla u_x) = \varphi\Delta u_x + \nabla\varphi\nabla u_x, \quad (4.77)$$

we also have

$$\begin{aligned} 2\int_\Omega \varphi\Delta u_x dA &= -2D(\varphi,u_x) = -2\int_\Omega [(xu_x+u)u_{xx} + xu_y u_{xy}]dA \\ &= -\int_\Omega x(2u_xu_{xx} + 2u_yu_{yx})dA - 2\int_\Omega uu_{xx}dA \\ &= 3\int_\Omega u_x^2 dA + \int_\Omega u_y^2 dA. \end{aligned} \quad (4.78)$$

Now, substituting (4.78) into (4.74) and using (4.66), we have

$$\nu_2 \leq \frac{\int_\Omega \varphi\Delta\Delta\varphi dA}{D(\varphi)} = -\nu_1 \frac{\int_\Omega x^2 u\Delta u dA}{D(\varphi)} + \frac{4\int_\Omega \varphi\Delta u_x dA}{D(\varphi)}. \tag{4.79}$$

We hence derive, by using (4.74), (4.76)and (4.79), that

$$\begin{aligned}\nu_2 &< -\nu_1 \frac{\int_\Omega x^2 u\Delta u dA}{-\int_\Omega x^2 u\Delta u dA} + \frac{4\int_\Omega \varphi\Delta u_x dA}{D(\varphi)} \\ &= -\nu_1 + \frac{8(\int_\Omega \varphi\Delta u_x dA)^2}{D(\varphi)[3\int_\Omega u_x^2 dA + \int_\Omega u_y^2 dA]}.\end{aligned} \tag{4.80}$$

Next, using Schwarz's inequality in the numerator of (4.80), we get

$$\nu_2 < \nu_1 + \frac{8D(u_x)}{3\int_\Omega u_x^2 dA + \int_\Omega u_y^2 dA}. \tag{4.81}$$

Finally, we note that all that we have computed so far, with the help of the trial function $\varphi = xu$, remain valid for the trial function $\psi = yu$, so that we also have

$$\nu_2 < \nu_1 + \frac{8D(u_y)}{\int_\Omega u_x^2 dA + 3\int_\Omega u_y^2 dA}. \tag{4.82}$$

Combining now (4.81) and (4.82), we obtain

$$\begin{aligned}\nu_2 &< \nu_1 + \frac{8[D(u_x) + D(u_y)]}{4\int_\Omega u_x^2 dA + 4\int_\Omega u_y^2 dA} \\ &= \nu_1 + \frac{2[D(u_x) + D(u_y)]}{D(u)} = 3\nu_1,\end{aligned} \tag{4.83}$$

and the proof is achieved. □

4.1.5 Remarks. Let us consider as known the first eigenvalue λ_1 and the corresponding eigenfunction $u := u_1$ of problem (4.1), normalized so that $\int_\Omega u^2 dA = 1$. We then have

$$\lambda_1 = \int_\Omega |\nabla u|^2 dA. \tag{4.84}$$

Now, let us cover Ω with a matter of surface density u^2, and choose the coordinate axes such that the center of gravity of this mass coincides with the origin and the principal axes of interia coincide with the coordinate axes, that is

$$\int_{\Omega} xu^2 dA = \int_{\Omega} yu^2 dA = \int_{\Omega} xyu^2 dA = 0. \tag{4.85}$$

Next, we consider a trial function that depend linearly on 3 parameters α, β and γ,

$$\varphi = \alpha aux + \beta buy + \gamma u. \tag{4.86}$$

Clearly, from this definition, we have $\varphi = 0$ on $\partial\Omega$. Moreover, we choose a and b such that:

$$a^2 \int_{\Omega} u^2 x^2 dA = b^2 \int_{\Omega} u^2 y^2 dA = 1. \tag{4.87}$$

Then we compute the following

$$\begin{aligned} \int_{\Omega} \varphi^2 dA = \alpha^2 a^2 \int_{\Omega} u^2 x^2 dA + \beta^2 b^2 \int_{\Omega} u^2 y^2 dA + \gamma^2 \int_{\Omega} u^2 dA \\ + 2\alpha\beta ab \int_{\Omega} u^2 xy dA + 2\alpha\gamma a \int_{\Omega} u^2 x dA + 2\beta\gamma \int_{\Omega} u^2 y dA. \end{aligned} \tag{4.88}$$

Substituting (4.85) and (4.87) into (4.88), we get

$$\int_{\Omega} \varphi^2 dA = \alpha^2 + \beta^2 + \gamma^2. \tag{4.89}$$

In addition, let us note that

$$\Delta(ux) = x\Delta u + 2u_x \tag{4.90}$$

and

$$\Delta(uy) = y\Delta u + 2u_y. \tag{4.91}$$

Therefore,

$$\Delta\varphi = -\lambda_1 u(\alpha ax + \beta by + \gamma) + 2\alpha au_x + 2\beta bu_y, \tag{4.92}$$

implies, by using (4.86), that

$$\begin{aligned} \varphi\Delta\varphi = &-\lambda_1 \alpha au^2(\alpha ax + \beta by + \gamma) + 2\alpha^2 a^2 xuu_x + 2\alpha\beta abxuu_y \\ &- \lambda_1 \beta bu^2(\alpha ax + \beta by + \gamma) + 2\alpha\beta abuu_x y + 2\beta^2 b^2 yuu_y \\ &- \gamma\lambda_1 u^2(\alpha ax + \beta by + \gamma) + 2\alpha\gamma auu_x + \alpha\beta\gamma buu_y. \end{aligned} \tag{4.93}$$

Integrating now (4.93), we get

$$-\int_{\Omega}\varphi\Delta\varphi dA=\alpha^2(a^2+\lambda_1)+\beta^2(b^2+\lambda_1)+\lambda_1\gamma^2. \tag{4.94}$$

Next, according to Poincaré's variational characterization of λ_3, we have

$$\lambda_3=\underset{L_3}{Min}\quad\underset{\varphi\in L_3}{max}\frac{D(\varphi)}{\int_{\Omega}\varphi^2dA}. \tag{4.95}$$

As $\varphi\in L_3=span\{u,u_x,u_y\}$, we obtain

$$\lambda_3\leq\underset{\varphi\in L_3}{max}\frac{\Delta(\varphi)}{\int_{\Omega}\varphi^2dA}\leq max\frac{\alpha^2(a^2+\lambda_1)+\beta^2(b^2+\lambda_1)+\lambda_1\gamma^2}{\alpha^2+\beta^2+\gamma^2}. \tag{4.96}$$

Therefore, we can conclude that

$$\lambda_1+\lambda_2+\lambda_3\leq a^2+\lambda_1+b^2+\lambda_1+\lambda_1=3\lambda_1+a^2+b^2, \tag{4.97}$$

and taking into account that

$$1=(\int_{\Omega}u^2dA)^2=(-2\int_{\Omega}xuu_xdA)^2\leq 4\int_{\Omega}u^2x^2dA\int_{\Omega}u_x^2dA, \tag{4.98}$$

we obtain

$$a^2\leq 4\int_{\Omega}u_x^2dA, \tag{4.99}$$

and

$$b^2\leq 4\int_{\Omega}u_y^2dA. \tag{4.100}$$

Adding these two last inequalities, we obtain

$$a^2+b^2\leq 4D(u)=4\lambda_1. \tag{4.101}$$

Finally, substituting (4.101) into (4.97), we obtain

$$\lambda_1+\lambda_2+\lambda_3\leq 3\lambda_1+4\lambda_1=7\lambda_1, \tag{4.102}$$

which implies

$$\lambda_2+\lambda_3\leq 6\lambda_1. \tag{4.103}$$

Clearly, as immediate consequences of (4.103), we have

$$\lambda_2\leq 3\lambda_1, \tag{4.104}$$

and

$$\lambda_3\leq 5\lambda_1. \tag{4.105}$$

Finally, we also note that one can prove in the same analog of (4.103), for the clamped and buckled plates. However, we should replace 6 by a less sharp constant.

4.2 Thompson's Inequality

4.2.1 On the ratio of consecutive eigenvalues in N-dimensions. Let us consider the N-dimensional eigenvalue problem of the fixed vibrating membrane, that is

$$\begin{cases} \Delta u + \lambda u = 0, & \Omega \subseteq R^N, \\ u = 0, & \partial\Omega, \end{cases} \tag{4.106}$$

where Ω is a bounded domain with smooth boundary. From the previous section the following Payne-Pólya-Weinberger inequality is known in the case $N = 2$:

$$\lambda_2 \leq 3\lambda_1,$$

where λ_1 and λ_2 are the first two eigenvalues of problem (4.106). In the following theorem C.J. Thompson [12] obtained an extension of this inequality to higher dimensions.

Theorem 4.2.1. If $\lambda_1 < \lambda_2 \leq ...$, are the eigenvalues of problem (4.106), then

$$\lambda_2 \leq (1 + \frac{4}{N})\lambda_1. \tag{4.107}$$

Proof: First we choose the center of coordinates axes to be the center of gravity of Ω, with mass distribution u^2, where $u := u_1$. We then have

$$\int_\Omega x_l u^2 dA = 0 \qquad \text{for} \qquad l = 1, 2, ..., N. \tag{4.108}$$

Therefore, $\varphi^l = x_l u$, $l = 1, ..., N$, become legitimate test functions for the variational characterization of λ_2, as

$$\varphi^l = 0 \text{ on } \partial\Omega, \tag{4.109}$$

and

$$\int_\Omega \varphi^l u dA = \int_\Omega x_l u^2 dA = 0. \tag{4.110}$$

Therefore,

$$\lambda_2 \leq \frac{\int_\Omega |\nabla\varphi^l|^2 dA}{\int_\Omega (\varphi^l)^2 dA} = \frac{-\int_\Omega \varphi^l \Delta\varphi^l dA}{\int_\Omega (\varphi^l)^2 dA}. \tag{4.111}$$

Clearly,

$$\Delta\varphi^l = \Delta(x_l u) = -\lambda_1\varphi^l + 2u_{x_l}. \tag{4.112}$$

Now, multiplying (4.112) by $-\varphi^l$ and integrating the result, we get

$$\int_\Omega -\varphi^l \Delta\varphi^l dA = \lambda_1 \int_\Omega (\varphi^l)^2 dA - 2\int_\Omega \varphi^l u_{x_l} dA. \tag{4.113}$$

On the other hand, we also have

$$-2\int_{\Omega}\varphi^l u_{x_l}dA = 2\int_{\Omega}x_l u u_{x_l}dA = \int_{\Omega}x_l(u^2)_{x_l}dA = \int_{\Omega}u^2 dA = 1, \tag{4.114}$$

where the last equality follows from the normalization of u. Using now (4.114) and (4.113) in (4.111), we get

$$\lambda_2 \leq \lambda_1 + \frac{1}{\int_{\Omega}(\varphi^l)^2 dA}. \tag{4.115}$$

Next, using Schwarz's inequality, we have

$$4\int_{\Omega}(\varphi^l)^2 dA\int_{\Omega}u_{x_l}^2 dA \geq (-2\int_{\Omega}\varphi^l u_{x_l}dA)^2, \tag{4.116}$$

which implies

$$\frac{1}{\int_{\Omega}(\varphi^l)^2 dA} \leq 4\int_{\Omega}u_{x_l}^2 dA. \tag{4.117}$$

Finally, substituting (4.117) into (4.115), we obtain

$$\sum_{l=1}^{N}(\lambda_2-\lambda_1) \leq \sum_{l=1}^{N}\frac{1}{\int_{\Omega}(\varphi^l)^2 dA} \leq 4\sum_{l=1}^{N}\int_{\Omega}u_{x_l}^2 dA = 4\int_{\Omega}|\nabla u|^2 dA, \tag{4.118}$$

which implies the desired inequality

$$\lambda_2 \leq (1+\frac{4}{N})\lambda_1,$$

and the proof is thus achieved. □

4.3 New Results

4.3.1 The clamped plate problem. Let us consider the N-dimensional eigenvalue problem of the clamped plate

$$\begin{cases} \Delta\Delta u - \mu u = 0, & \Omega \subseteq R^N, \\ u = \dfrac{\partial u}{\partial n} = 0, & \partial\Omega, \end{cases} \tag{4.119}$$

where Ω is a bounded domain with a smooth boundary. We order and denote the eigenvalues of this problem as

$$0 < \mu_1 \leq \mu_2 \leq ...,$$

and its corresponding eigenfunctions

$$u := u_1, u_2, u_3, ...$$

From section (4.1) we know that in the case $N = 2$ the following Payne-Pólya-Weinberger holds

$$\mu_2 \leq 9\mu_1.$$

Our goal here is to improve and extend this result to higher dimensions. We thus have:

Theorem 4.3.1. If $\mu_1 \leq \mu_2 \leq ...$, are the eigenvalues of the problem given in equation (4.119), then

$$\mu_2 \leq (1 + \frac{8}{N})\mu_1. \tag{4.120}$$

In particular, when $N = 2$ we have

$$\mu_2 \leq 5\mu_1. \tag{4.121}$$

Proof: We first choose the center of coordinates axes to be the center of gravity of Ω with mass distribution u^2, that is

$$\int_\Omega x_l u^2 dA = 0 \qquad \text{for} \qquad l = 1, 2, ..., N. \tag{4.122}$$

Thus, $\varphi^l = x_l u$, $l = 1, ..., N$, become legitimate test functions for the variational characterization of μ_2, since

$$\varphi^l = 0 \text{ on } \partial\Omega, \tag{4.123}$$

and

$$\int_\Omega \varphi^l u dA = \int_\Omega x_l u^2 dA = 0. \tag{4.124}$$

We hence derive, using the variational characterization of μ_2, the following inequality

$$\mu_2 \leq \frac{\int_\Omega (\Delta\varphi^l)^2 dA}{\int_\Omega (\varphi^l)^2 dA} = \frac{\int_\Omega \varphi^l \Delta\Delta\varphi^l dA}{\int_\Omega (\varphi^l)^2 dA}. \tag{4.125}$$

Next, we compute

$$\Delta(\varphi^l) = \Delta(x_l u) = x_l \Delta u + 2u_{,l}, \tag{4.126}$$

and

$$\Delta\Delta(\varphi^l) = \Delta(x_l \Delta u) + \Delta(2u_{,l}) = 4\Delta u_{,l} + \mu_1 x_l u, \tag{4.127}$$

where $u_{,l} = \dfrac{\partial u}{\partial x_l}$. Then multiplying (4.127) by φ^l and integrating the result, we obtain

$$\int_\Omega \varphi^l \Delta\Delta\varphi^l dA = 4\int_\Omega \varphi^l \Delta u_{,l} dA + \mu_1 \int_\Omega \varphi^l x_l u dA. \tag{4.128}$$

On the other hand, since

$$(x_l u\Delta u)_{,l} = u\Delta u + x_l u_{,l}\Delta u + x_l u\Delta u_{,l}, \tag{4.129}$$

we have

$$\begin{aligned}\int_\Omega \varphi^l \Delta u_{,l} dA &= -\int_\Omega u\Delta u dA - \int_\Omega x_l u_{,l}\Delta u dA = -\int_\Omega (u + x_l u_{,l})\Delta u dA \\ &= -\int_\Omega (x_l u)_{,l}\Delta u dA = -\sum_{k=1}^{N}\int_\Omega (x_l u)_{,l} u_{,kk} dA.\end{aligned} \tag{4.130}$$

Similarly, since

$$[(x_l u)_{,l} u_{,k}]_{,k} = (x_l u)_{,lk} u_{,k} + (x_l u)_{,l} u_{,kk}, \tag{4.131}$$

we also have

$$\begin{aligned}\int_\Omega (x_l u)_{,l} u_{,kk} dA &= -\int_\Omega (x_l u)_{,lk} u_{,k} dA = -\int_\Omega u_{,k}^2 dA - \int_\Omega x_l u_{,kl} u_{,k} dA \\ &= -\int_\Omega u_{,k}^2 dA + \frac{1}{2}\int_\Omega u_{,k}^2 dA = -\frac{1}{2}\int_\Omega u_{,k}^2 dA.\end{aligned} \tag{4.132}$$

Now, using (4.132) and (4.130) into (4.128), we get

$$\int_\Omega \varphi^l \Delta\Delta\varphi^l dA = 2\int_\Omega |\nabla u|^2 dA + \mu_1 \int_\Omega (\varphi^l)^2 dA. \tag{4.133}$$

Next, substituting (4.133) into (4.125), we obtain

$$\mu_2 \leq \mu_1 + \frac{2\int_\Omega |\nabla u|^2 dA}{\int_\Omega (\varphi^l)^2 dA}, \tag{4.134}$$

so

$$\mu_2 - \mu_1 \leq \frac{2N\int_\Omega |\nabla u|^2 dA}{\sum_{l=1}^{N}\int_\Omega (\varphi^l)^2 dA}. \tag{4.135}$$

Next, let us find an upper bound for the denominator of the right hand side of (4.135). To this end, we first note that we have

$$(x_l u^2)_{,l} = u^2 + 2x_l u u_{,l}, \tag{4.136}$$

so, by integration, we get

$$-2\int_\Omega x_l u u_{,l} dA = \int_\Omega u^2 dA = 1, \tag{4.137}$$

where the last equality follows from the normalization of u. This implies

$$4\Big[\sum_{l=1}^{N}\int_\Omega x_l u u_{,l} dA\Big]^2 = N^2. \tag{4.138}$$

Next, using Schwarz's inequality, we have

$$N^2 \le 4\sum_{l=1}^{N}\int_\Omega (\varphi^l)^2 dA \sum_{l=1}^{N}\int_\Omega u_{,l}^2 dA$$

$$(4.139)$$

$$= 4\sum_{l=1}^{N}\int_\Omega (\varphi^l)^2 dA \int_\Omega |\nabla u|^2 dA,$$

which yields to

$$\sum_{l=1}^{N}\int_\Omega (\varphi^l)^2 dA \ge \frac{N^2}{4\int_\Omega |\nabla u|^2 dA}. \tag{4.140}$$

Substituting (4.140) into (4.135), we obtain

$$\mu_2 - \mu_1 \le \frac{8}{N}\Big[\int_\Omega |\nabla u|^2 dA\Big]^2. \tag{4.141}$$

On the other hand, by Schwarz inequality, we have

$$\Big[\int_\Omega |\nabla u|^2 dA\Big]^2 = \Big[\int_\Omega -u\Delta u dA\Big]^2 \le \int_\Omega (-u)^2 dA \int_\Omega (\Delta u)^2 dA. \tag{4.142}$$

Finally, using (4.142) in (4.135), we are lead to

$$\mu_2 - \mu_1 \le \frac{8}{N}\mu_1,$$

and the proof is thus achieved. □

Remarks: We note that other results on the ratio of the first two frequencies of a clamped plate are already known in the literature. For comparison with our result and completeness, we list them below. First, let us recall that L.E. Payne, G. Pólya and H.F. Weinberger in [9] proved in two dimensions inequality (4.34), that is

$$\mu_2 \le 9\mu_1.$$

However, if we take $N = 2$ in our inequality (4.120), we obtain

$$\mu_2 \leq 5\mu_1,$$

which is clearly a better inequality than that of Payne, Polya and Weinberger.
Next, G.N. Hile and R.Z. Yeh in [6] proved that

$$\mu_2 \leq \left(1 + \frac{4}{N}\right)^2 \mu_1, \tag{4.143}$$

which is clearly weaker than our inequality (4.121). The same preceding inequality has been obtained later by Q.M. Cheng and H. Yang in [2]. In fact they proved something more general, that is

$$\mu_{k+1} - \frac{1}{k}\sum_{i=1}^{k} \mu_i \leq \left[\frac{8(N+2)}{N^2}\right]^{\frac{1}{2}} \frac{1}{k}\sum_{i=1}^{k} \left[\mu_i(\mu_{k+1} - \mu_i)\right]^{\frac{1}{2}}, \tag{4.144}$$

which in the case $k = 1$ becomes (4.143). Finally, in 2013 Q.M. Cheng and G. Wei conjectured in [4] that the following inequality should hold:

$$\sum_{i=1}^{k} \left(\mu_{k+1} - \mu_i\right)^2 \leq \frac{8}{N}\sum_{i=1}^{k}(\mu_{k+1} - \mu_i)\mu_i. \tag{4.145}$$

If we take $k = 1$ in (4.145), we get exactly our inequality (4.121). In conclusion we have proved that the case $k = 1$ of this conjecture is indeed true.

4.3.2 The buckled plate problem. Let us now consider the N-dimensional eigenvalue problem of the buckled plate

$$\begin{cases} \Delta\Delta u + \nu\Delta u = 0, & \Omega \subseteq R^2, \\ u = \dfrac{\partial u}{\partial n} = 0, & \partial\Omega, \end{cases} \tag{4.146}$$

where Ω is a bounded domain with a smooth boundary. We order and denote the eigenvalues of this problem as

$$0 < \nu_1 \leq \nu_2 \leq \ldots,$$

and its corresponding eigenfunctions

$$u = u_1, u_2, u_3, \ldots .$$

From section (4.1), we know that in the case $N = 2$ the following Payne-Pólya-Weinberger inequality holds:

$$\nu_2 \leq 3\nu_1.$$

Our goal here is to extend this inequality to higher dimensions.

Theorem 4.3.2. If $\nu_1 \leq \nu_2 \leq ...$, are the eigenvalues of problem (4.146), then

$$\nu_2 \leq (1 + \frac{8}{N+2})\nu_1. \tag{4.147}$$

In particular, when $N = 2$ we have

$$\nu_2 \leq 3\nu_1. \tag{4.148}$$

Proof: First of all, let us cover Ω with matter of density $|\nabla u|^2$ and choose the coordinate axes such that

$$\int_\Omega x_l |\nabla u|^2 dA = 0 \qquad \text{for} \qquad l = 1, 2, .., N. \tag{4.149}$$

Next, we consider $\varphi^l := x_l u$, $l = 1, ..., N$, which become legitimate test functions in the variational characterization of ν_2, since

$$\varphi^l = \frac{\partial \varphi^l}{\partial n} = 0 \text{ on } \partial\Omega. \tag{4.150}$$

Next, let us compute

$$\int_\Omega |\nabla \varphi^l|^2 dA = \int_\Omega \left[\sum_{k=1}^{N} x_l u_{,k}^2 + 2x_l u u_{,l} + u^2 \right] dA = \int_\Omega x_l^2 |\nabla u|^2 dA, \tag{4.151}$$

and note that by Green's theorem we have

$$\int_\Omega \varphi^l \Delta\Delta\varphi^l dA = \int_\Omega (\Delta\varphi^l)^2 dA. \tag{4.152}$$

Now using the fact that

$$\Delta\varphi^l = \Delta(x_l u) = x_l \Delta u + 2u_{,l}, \tag{4.153}$$

and

$$\Delta\Delta\varphi^l = \Delta(x_l \Delta u) + \Delta(2u_{,l}) = 4\Delta u_{,l} - \nu_1 x_l \Delta u, \tag{4.154}$$

we get

$$\int_\Omega \varphi^l \Delta\Delta\varphi^l dA = 4\int_\Omega \varphi^l \Delta u_{,l} dA - \nu_1 \int_\Omega x_l^2 u \Delta u dA. \tag{4.155}$$

On the other hand, since

$$\nabla(x_l^2 u\nabla u) = x_l^2 u\Delta u + 2x_l u u_{,l} + x_{,l}^2|\nabla u|^2, \tag{4.156}$$

we also have

$$-\int_\Omega x_l^2 u\Delta u dA = 2\int_\Omega x_l u u_{,l} dA + \int_\Omega x_l^2|\nabla u|^2 dA. \tag{4.157}$$

Moreover, using φ^l in the variational characterization of ν_2, we have

$$\nu_2 \leq \frac{\int_\Omega \varphi^l \Delta\Delta\varphi^l dA}{\int_\Omega |\nabla\varphi^l|^2 dA} = \frac{4\int_\Omega \varphi^l \Delta u_{,l} dA}{\int_\Omega |\nabla\varphi^l|^2 dA} - \nu_1 \frac{\int_\Omega x_l^2 u\Delta u dA}{\int_\Omega |\nabla\varphi^l|^2 dA}. \tag{4.158}$$

Now, let us compute $\int_\Omega \varphi^l \Delta u_{,l} dA$. First, we note that

$$\nabla(\varphi^l \nabla u_{,l}) = \varphi^l \Delta u_{,l} + \nabla\varphi^l \nabla u_{,l}, \tag{4.159}$$

so

$$2\int_\Omega \varphi^l \Delta u_{,l} dA = -2\int_\Omega \nabla\varphi^l \nabla u_{,l} dA = 2\int_\Omega u_{,l}^2 dA + \int_\Omega |\nabla u|^2 dA. \tag{4.160}$$

We hence derive, by using (4.158), (4.157), (4.160), Schwarz's inequality and Green's theorem, that

$$\begin{aligned} \nu_2 &\leq \nu_1 + \frac{8\big[\int_\Omega \varphi^l \Delta u_{,l} dA\big]^2}{\int_\Omega |\nabla\varphi^l|^2 dA\big[2\int_\Omega u_{,l}^2 dA + \int_\Omega |\nabla u|^2 dA\big]} \\ &\leq \nu_1 + \frac{8\int_\Omega |\nabla u_{,l}|^2 dA}{2\int_\Omega u_{,l}^2 dA + \int_\Omega |\nabla u|^2 dA}. \end{aligned} \tag{4.161}$$

Next, we compute

$$8\int_\Omega |\nabla u_{,l}|^2 dA = 8\int_\Omega u_{,l}\Delta u_{,l} dA = 8\int_\Omega u\Delta u_{,ll} dA. \tag{4.162}$$

Substituting it into (4.161), we get

$$(\nu_2 - \nu_1)\big[2\int_\Omega u_{,l}^2 dA + \int_\Omega |\nabla u|^2 dA\big] \leq 8\int_\Omega u\Delta u_{,ll} dA, \tag{4.163}$$

which implies

$$(\nu_2 - \nu_1)\big[2\int_\Omega |\nabla u|^2 dA + N\int_\Omega |\nabla u|^2 dA\big] \leq 8\int_\Omega u\Delta\Delta u dA = 8\nu_1\int_\Omega |\nabla u|^2 dA,$$

or, equivalently,

$$\nu_2 - \nu_1 \leq \frac{8}{N+2}\nu_1,$$

and the proof is thus achieved. □

Remarks: Some results related to our work, about bounds for the ratio of the first two frequencies of the buckled plate, have been already obtained before. For comparison with our result and completeness we mention them below. First let us recall that L.E. Payne, G. Pólya and H.F. Weinberger in [9] proved in two dimensions inequality (4.62), that is

$$\nu_2 \leq 3\nu_1.$$

If we take $N = 2$ in our inequality we obtain (4.147) the same inequality, so our result represents a natural extension to any higher dimension of (4.62).

Next, M.S. Ashbaugh proved in [1] the following general inequality

$$\sum_{k=1}^{N} \nu_{k+1} \leq (N+4)\nu_1, \tag{4.164}$$

which implies in particular that

$$\nu_2 \leq (1 + \frac{4}{N})\nu_1. \tag{4.165}$$

This result gives a better bound than (1.147). Moreover, it proves for $k = 1$ the following recent conjecture, presented by Q.M. Cheng and H. Yang in [3],

$$\sum_{i=1}^{k}(\nu_{k+1} - \nu_i)^2 \leq \frac{4}{N}\sum_{i=1}^{k}(\nu_{k+1} - \nu_i)\nu_i. \tag{4.166}$$

Therefore, some more work should be done by using other test functions to improve not only our inequality, but also (4.164).

Conclusion and Future Work

In this thesis, we tried to find some universal bounds for the eigenvalues of the clamped and buckled plate problems. On one hand, I hope that what we presented was as straightforward and clear as possible. On the other hand, it required background on the eigenvalues of different differential operators, mathematical tools and some pre-proven results.

That being said, this thesis was divided into four chapters. In chapter 1, we have provided an overview of isoperimetric inequalities and modal problems for frequencies of free, fixed and elastically supported membranes. Moreover, we interpreted an important tool, which was very useful in our work, that is the Variational Characterizations.

In both chapters 2 and 3, we introduced two additional main tools in the study of isoperimetric inequalities, Schwarz rearrangements and conformal mappings respectively. These tools were mainly used in the proof of many important inequalities such as Faber-Krahn's inequality, Hardy-Littlewood-Pólya inequality, Szegö's inequality and Weinberger's inequality.

Finally, in chapter 4, we presented some of the pre-proven results about universal bounds for the eigenvalues of the fixed membrane, clamped plate and buckled plate, obtained by L.E. Payne, G. Pólya and H.F. Weinberger, respectively C.J.Thompson. In addition to that, we have included our new related results.

In our future work, we will try to extend our results to higher order eigenvalues or improve some known results.

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Vita

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