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Embedding and global stability in periodic 2-dimensional maps of mixed monotonicity

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Abstract

In this paper, we consider nonautonomous second order difference equations of the form $x_{n+1} = F(n, x_n, x_{n-1})$, where F is p-periodic in its first component, non-decreasing in its second component and non-increasing in its third component. The map F is referred to as periodic of mixed monotonicity, which broadens the notion of maps of mixed monotonicity. We introduce the concept of artificial cycles, and we develop the embedding technique to tackle periodicity and globally attracting cycles in periodic 2-dimensional maps of mixed monotonicity. We present a result on globally attracting cycles and demonstrate its application to periodic systems. The first application is a periodic rational difference equation of second order, and the second application is a population model with periodic stocking. In both cases, we prove the existence of a globally attracting cycle.

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1 Introduction

Two dimensional maps $z = F(x, y)$ that are non-increasing in one component and non-decreasing in the other are called maps of mixed monotonicity. Dynamics of maps with mixed monotonicity have been considered in literature [12, 19, 20]. The main approach to dealing with maps of mixed monotonicity is to embed the dynamical system generated by

$$x_{n+1} = F(x_n, x_{n-1}) \tag{1.1}$$

into a higher dimensional symmetric and monotone dynamical system. The orbits of the symmetric higher-dimensional system are used to squeeze the orbits of the original system and aid in the establishment of a global attractor. For more details on the development history of this approach and its applications in differential equations, integral equations and numerical analysis, we refer the reader to Smith [19] and the references therein. To view Eq. (1.1) in vector form, define $T(x, y) = (F(x, y), x)$ and $X = (x, y)$, then we have

$$X_{n+1} = T(X_n), \quad n \in \mathbb{N} = \mathbb{Z}^+ \cup \{0\}, \tag{1.2}$$

An orbit of Eq.(1.2) through an initial condition $X_0 = (x_0, x_{-1})$ is denoted by $\mathcal{O}_F^+(X_0)$. The authors of [2] recently considered maps of mixed monotonicity with invariant compact domains, then extended the invariant domain from compact to rectangular-compact. This method can be used to achieve global stability for certain maps with compact invariant domains in which the positive orthant is not an invariant domain. In this paper, we will concentrate on periodic discrete equations of the form

$$x_{n+1} = F_n(x_n, x_{n-1}),\ n \in \mathbb{N}, \tag{1.3}$$

in which F_n is p-periodic in n, i.e., $F_{n+p} = F_n$ for all n. Again here, a vector form of Eq. (1.3) is

$$X_{n+1} = T_{n \bmod p}(X_n), \quad \text{where} \quad n \in \mathbb{N},\ T_n(x, y) = (F_n(x, y), x) \quad \text{and} \quad X_0 = (x_0, y_0) \in \mathbb{R}^2_+. \tag{1.4}$$

An orbit of Eq. (1.4) through an initial condition $X_0 = (x_0, x_{-1})$ is denoted by $\mathcal{O}^+_{F_n}(X_0)$. We emphasize here that p-periodic is meant to be of minimal period p. In Eq. (1.3), each map F_n is non-decreasing in x_n and non-increasing in x_{n-1}. For short, we write $F_n(\uparrow, \downarrow)$. Furthermore, we assume all maps $F_n : [0, \infty) \times [0, \infty) \to [0, \infty)$ to be continuous. It is worth noting that during the last two decades, periodic systems in one dimension and their applications have been widely studied in the literature [1, 3, 5, 6, 10]. Forcing between cycles, stability analysis, and the effect of fluctuating environments on population dynamics received the most attention.

Throughout this paper, we consider $\mathbb{R}_+$ to denote the set of nonnegative real numbers, and $Fix(F)$ is used to denote the set of fixed points of F. A compact subset $\Omega \subset [0, \infty) \times [0, \infty)$ is called invariant or forms an invariant domain of F if $T(\Omega) \subseteq \Omega$. Unlike autonomous discrete systems, the starting time $n = n_0$ is crucial in nonautonomous discrete systems, and therefore, we consider $n_0 = 0$ throughout our paper. Another fact that is worth stressing here is the structure of cycles in discrete periodic systems. A k-cycle $C_k = \{x_0, x_1, \ldots, x_{k-1}\}$ of Eq. (1.1) means a periodic solution of minimal period k and we can start at any point in time by taking $F(x_j, x_{j-1})$. However, A k-cycle of Eq. (1.3) is an ordered set, which means a synchronization between time and space must occur. Therefore, we write a k-cycle as $C_k = [x_0, x_1, \ldots, x_{k-1}]$ and we start by $F_0(x_1, x_0)$. Examples are given in the literature in which $[x_0, x_1, \ldots, x_{k-1}]$ is a k-cycle of a periodic system, but its phase shift $[x_1, x_2, \ldots, x_{k-1}, x_0]$ is not a k-cycle [3].

Consider the Euclidean metric on $\mathbb{R}^2_+$. A ball $B_r(x, y)$ denotes the disk centered at (x, y) with radius r. An equilibrium solution $\bar{x}$ of Eq. (1.1) is called stable if for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $(x_0, x_{-1}) \in B_\delta(\bar{x}, \bar{x})$, we obtain $(x_n, x_{n-1}) \in B_\epsilon(\bar{x}, \bar{x})$ for all $n \in \mathbb{N}$. $\bar{x}$ is called attracting if there exists $c > 0$ such that for all $(x_0, x_{-1}) \in B_c(\bar{x}, \bar{x})$, we obtain $\lim x_n = \bar{x}$. $\bar{x}$ is called asymptotically stable if it is stable and attracting. $\bar{x}$ is globally attracting with respect to an invariant region Ω if for all $(x_0, x_{-1}) \in \Omega$, we obtain $\lim x_n = \bar{x}$. $\bar{x}$ is globally stable (or a global attractor) if it is stable and globally attracting with respect to a specific invariant domain. A k-cycle $[\bar{x}_0, \ldots, \bar{x}_{k-1}]$ of Eq. (1.3) is called stable, attracting, asymptotically stable, globally attracting or globally stable if the equilibrium solution $(\bar{x}_0, \bar{x}_0)$ of $X_{n+1} = \widehat{T}_0(X_n)$, where $\widehat{T}_0 = T_{p-1} \circ \cdots \circ T_0$, is stable, attracting, asymptotically stable, globally attracting or globally stable, respectively. Note that the continuity of the maps F_j extends the dynamics of $X_{n+1} = \widehat{T}_0(X_n)$ at $(\bar{x}_0, \bar{x}_0)$ to the dynamics of $X_{n+1} = \widehat{T}_j(X_n)$ at $(\bar{x}_j, \bar{x}_j)$, where $\widehat{T}_j = T_{j-1} \circ \cdots \circ T_0 \circ T_{p-1} \circ \cdots \circ T_j$.

The structure of this paper is as follows: In section two, we go through the embedding strategy for maps $F(\uparrow, \downarrow)$. We introduce the concept of artificial cycles and present some preliminary results.

In addition, we discuss some recent results regarding extensions of such maps. In section three, we investigate the dynamics of the p-periodic difference equation (1.3), and embed the periodic dynamical system into a 4-dimensional periodic symmetric system. We demonstrate the connection between the orbits, generalize the concept of artificial cycles to the periodic case, then give a classification result for cycles when $p = 2$. In section four, we develop the needed machinery for establishing globally attractive cycles and give some results. In section five, we consider two illustrative examples of p-periodic equations, and investigate their dynamics using the methodologies and results presented in the preceding sections. Finally, we give a conclusion section in which we summarize our findings, then discuss some answered and unanswered questions.

2 Embedding in higher dimension

Consider $\leq_{se}$ to denote the southeast partial order on the positive orthant $\mathbb{R}_+^2 = [0, \infty) \times [0, \infty)$, i.e, $(x_1, y_1) \leq_{se} (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_2 \leq y_1$. When $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ satisfies $F(\uparrow, \downarrow)$, the symmetric map $G^* : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ defined by

$$G^*(u, v) = (F(u, v), F(v, u))$$

is non-decreasing with respect to $\leq_{se}$. However, the orbits of G^* are not related to the orbits of F [12]. To make the notion of symmetric maps more fruitful, we need to embed in a higher dimension [19, 20]. Consider both $\mathbb{R}_+^2$ and $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ with $\leq_{se}$. Then define $g : \mathbb{R}_+^4 \to \mathbb{R}_+^2$ as $g((x, y), (u, v)) = (F(x, y), u)$. We obtain $g(\uparrow, \downarrow)$ with respect to the $\leq_{se}$. Next, let $X = (x, y)$, $U = (u, v)$, and define the symmetric map $G : \mathbb{R}_+^4 \to \mathbb{R}_+^4$ as

$$G(X, U) = (g(X, U), g(U, X)). \tag{2.1}$$

Then G is non-decreasing in the sense that $(X_1, U_1) \leq_{se} (X_2, U_2)$ implies $G(X_1, U_1) \leq_{se} G(X_2, U_2)$. Observe that we have $G(X, U) \leq_{se} (X, U)$ if and only if $F(x, y) \leq x, u \leq F(u, v), y \leq u$ and $x \leq v$. Also, $(X, U) \leq_{se} G(X, U)$ if and only if $x \leq F(x, y), F(u, v) \leq u, v \leq x$ and $u \leq y$. The iterates of the map G give us a first order difference equation in four dimensions, namely

$$\xi_{n+1} = G(\xi_n)), \quad n \in \mathbb{N}, \;\; \xi_0 \in \mathbb{R}_+^4. \tag{2.2}$$

If $F : [a, b]^2 \to [a, b]$, then for any $(x, y) \in [a, b]^2$, we obtain

$$a \leq F(a, b) \leq F(x, y) \leq F(b, a) \leq b.$$

In this case, consider $A = (a, b), B = (b, a)$ and let $X = (x, y), Y = (y, x) \in [a, b]^2$. We obtain $A \leq_{se} X \leq_{se} B$ and

$$(A, B) \leq_{se} G(A, B) \leq_{se} G(X, Y) \leq_{se} G(B, A) \leq_{se} (B, A).$$

Now, the monotonicity of G and an induction argument give us

$$(A, B) \leq_{se} G^n(A, B) \leq_{se} G^{n+1}(A, B) \leq_{se} G^{n+1}(X, Y) \tag{2.3}$$

and

$$G^{n+1}(Y, X) \leq_{se} G^{n+1}(B, A) \leq_{se} G^n(B, A) \leq_{se} (B, A). \tag{2.4}$$

A fixed point of G is an equilibrium solution of Eq. (2.2) which will be a solution of $G(X,U) = (X,U)$. This means that we must have $(y,x) = U$ and $(F(x,y), F(y,x)) = (x,y)$. It is worth mentioning that fixed points of G are known as *coupled fixed points* [11,18]. Coupled fixed points can be symmetric when $x = y$ or asymmetric when $x \neq y$. The asymmetric ones have been denoted by *artificial fixed points* of F [2]. The concept of artificial cycles plays a significant role in our analysis and, therefore, we embolden and extend it in the following formal definition:

Definition 2.1. *If $F(x,x) = x$, then x is a fixed point of F. If $G(\xi) = \xi$, then ξ is a fixed point of G. If $\xi = ((x,y),(y,x))$ is a fixed point of G and $x \neq y$, then (x,y) is called an artificial fixed point of F. If $T^q(x_1,x_0) = (x_1,x_0)$ and $T^k(x_1,x_0) \neq (x_1,x_0)$ for all $k \leq q-1$, then $\{x_0, x_1, \ldots, x_{q-1}\}$ is a q-cycle of F. If ξ satisfies $G^q(\xi) = \xi$ and $G^k(\xi) \neq \xi$ for all $k \leq q-1$, then $C_q := \{\xi, G(\xi), G^2(\xi), \ldots, G^{q-1}(\xi)\}$ is called a q-cycle of G. We define the diagonal of $\mathbb{R}^4_+$ to be $D := \{((x,y),(x,y)) : \ (x,y) \in \mathbb{R}^2_+\}$, and we define $D_D = \{((x,x),(y,y)) : \ x,y \in \mathbb{R}_+\}$. If $C_q := \{\xi, G(\xi), G^2(\xi), \ldots, G^{q-1}(\xi)\}$ is a q-cycle of G and $\xi \notin D$, then we say F has an artificial q-cycle.*

In general, cycles of G that are not generated by cycles of F produce artificial cycles. For the case of 2-cycles, if $\xi = ((x_1,x_0),(u_1,u_0)$ and $\{\xi, G(\xi)\}$ is a 2-cycle of G, but $\{x_0,x_1\}$ is not a 2-cycle of F, then $[(x_1,x_0),(u_1,u_0)]$ is an artificial 2-cycle of F. To clarify the general case further, suppose $\{\xi_0, \xi_1, \ldots, \xi_{q-1}\}$ is a q-cycle of G for some $q \geq 3$, and let $\xi_0 = (x_0, y_0, u_0, v_0)$. This means that $\xi_{j+1 \bmod q} = G(\xi_{j \bmod q})$, and consequently

$$x_{j+1} = F(x_j, y_j),\ y_{j+1} = u_j,\ u_{j+1} = F(u_j, v_j) \quad \text{and} \quad v_{j+1} = x_j.$$

This gives us $\{(x_0,y_0), \ldots, (x_{q-1}, y_{q-1})\}$ as a q-cycle of

$$\begin{cases} x_{n+1} = F(x_n, y_n) \\ y_{n+3} = F(y_{n+2}, x_n), \end{cases} \tag{2.5}$$

where the initial conditions x_0, y_0, y_1 and y_2 must be given. Observe that if we start our initial conditions $x_0, y_0 = x_{-1}, y_1 = x_0$ and $y_2 = x_1$, then the two equations become identical which gives us Eq. (1.1). Moreover, the diagonal D is invariant under G. Therefore, artificial cycles of F are obtained from cycles of G in which the initial condition $(x_0, y_0, u_0, v_0) \notin D$. The next two propositions and example provide more solid foundations for our notions in Definition 2.1.

Proposition 2.1. *Consider the continuous map of mixed monotonicity $z = F(x,y)$ and the map G as defined in Eq. (2.1). Each of the following holds true:*

(i) Let $x \neq y$. (x,y) and (y,x) are artificial fixed points of F if and only if $(F(x,y), F(y,x)) = (x,y)$.

(ii) If the one dimensional map $y = f(x) = F(x,x)$ has a 2-cycle $\{x_0, f(x_0)\}$, then G has the 2-cycle $\{\xi_0, \xi_1\}$, where

$$\xi_0 := (x_0, x_0, f(x_0), f(x_0)) \quad \textit{and} \quad \xi_1 := (f(x_0), f(x_0), x_0, x_0).$$

Furthermore, $[(x_0,x_0),(f(x_0), f(x_0))]$ is an artificial 2-cycle of F.

(iii) If the two dimensional map $z = F(x, y)$ has a q-cycle $\{x_0, x_1, \ldots, x_{q-1}\}$, then G has the q-cycle $\{\psi_0, \psi_1, \ldots, \psi_{q-1}\}$, where

$$\psi_j := (x_j, x_{j-1}, x_j, x_{j-1}),\ j = 1, \ldots, q-1 \quad \textit{and} \quad \psi_0 := (x_0, x_{q-1}, x_0, x_{q-1}).$$

(iv) Let $\eta = (x, y, u, v)$ and suppose that $G^2(\eta) = \eta$. F has $[(x, y), (v, u)]$ as an artificial 2-cycle if and only if $(u, v) \neq (y, x), (x, y)$.

(v) $[(x, y), (u, v)]$ is an artificial 2-cycle of F if and only if $[(y, x), (v, u)]$ is an artificial 2-cycle.

Proof. Part (i) is becoming trivial (cf. [2, 20]). (ii) Suppose that $\{x_0, f(x_0)\}$ is a 2-cycle of the one dimensional map $y = f(x) = F(x, x)$. By considering $\xi_0 := (x_0, x_0, f(x_0), f(x_0))$, we obtain $G^2(\xi_0) = \xi_0$ and $G(\xi_0) = \xi_1$. To prove Part (iii), Suppose that $C_q := \{x_0, x_1, \ldots, x_{q-1}\}$ is a q-cycle of the two dimensional map $z = F(x, y)$, then we have $F(x_j, x_{j-1}) = x_{j+1}$ for $j = 1, \ldots, q-2$, and $F(x_{q-1}, x_{q-2}) = x_0$. Now, define $\psi_0 := (x_0, x_{q-1}, x_0, x_{q-1})$, we obtain $G^j(\psi_0) = \psi_j$ for all $j = 1, \ldots, q-1$ and $\psi_0 = G^q(\psi_0)$. Next, we show Part (iv). Observe that $G^2(\eta) = \eta$ if and only if

$$x = F(v, u),\ y = F(u, v),\ u = F(y, x) \quad \text{and} \quad v = F(x, y). \tag{2.6}$$

Assume that F has $[(x, y), (u, v)]$ as an artificial 2-cycle. This implies that η is not a fixed point of G, and consequently, $(u, v) \neq (y, x)$. Also, it implies $\{x, y\}$ is not a 2-cycle of F, and consequently $(u, v) \neq (x, y)$ as given in Part (iii). The converse is trivial. Finally, we prove Part (v). Suppose that $[(x, y), (u, v)]$ is an artificial 2-cycle of F. Define $\eta = (x, y, u, v)$, then $\{\eta, G(\eta)\}$ is a 2-cycle of G and $(u, v) \neq (x, y), (y, x)$. Also, we must have the equations in (2.6) satisfied. Now, by considering $\eta^t = (y, x, v, u)$, we obtain

$$\begin{aligned} G(\eta^t) =& (F(y, x), v, F(v, u), y) \\ =& (u, v, x, y) \\ G^2(\eta^t) =& (F(u, v), x, F(x, y), u) \\ =& \eta^t. \end{aligned}$$

Since also $(v, u) \neq (y, x), (x, y)$, we obtain $[(y, x), (v, u)]$ as another artificial 2-cycle of F. The converse is obvious. □

Note that in Part (ii) of Proposition 2.1, we found that a 2-cycle of the one dimensional map $y = F(x, x)$ gives an artificial 2-cycle of F; however, unlike the diagonal D, D_D does not form an invariant set under G. Another fact that is worth emphasizing is that an artificial fixed point cannot exist without the existence of a fixed point. We give this simple fact in the following proposition.

Proposition 2.2. *Let the map F of Eq. (1.1) be continuous and $F(\uparrow, \downarrow)$. If F has an artificial fixed point (a, b), then F has a fixed point between a and b.*

Proof. Suppose (a, b) is an artificial fixed point of F, i.e., $(F(a, b), F(b, a)) = (a, b)$ where $a \neq b$. Without loss of generality, we consider $a < b$. Define $g(x) = F(x, x) - x$, then we obtain

$$g(a) = F(a, a) - a > F(a, b) - a = 0 \quad \text{and} \quad g(b) = F(b, b) - b < F(b, a) - b = 0.$$

Now, the continuity of g and the Intermediate Value Theorem gives us a point $\bar{x}$ between a and b such that $g(\bar{x}) = 0$. Therefore, $\bar{x}$ is a fixed point of F and the proof is complete. □

Example 2.1. *(i) Consider*

$$F(x,y) = xe^{a-\frac{y}{b}}, \quad x,y \geq 0 \quad and \quad a,b>0$$

Clearly, F is continuous, it satisfies $F(\uparrow,\downarrow)$ and $F:\ \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$. The fixed points $\bar{x}_1 = 0$ and $\bar{x}_2 = ab$ of F create two fixed points of G given by $\bar{\xi}_i = (\bar{x}_i,\bar{x}_i,\bar{x}_i,\bar{x}_i)$, $i=1,2$. Since $(F(x,y),F(y,x)) = (x,y)$ has no solution for $x \neq y$, then G has no other fixed points and F has no artificial fixed points. Next, fix $a=5$ and let b be the unique positive solution of $e^{-5+\frac{50}{b}}(b-5) = 5$, then $[50, 10b-50]$ is a 2-cycle of the one dimensional map $y = f(x) = F(x,x)$. The 2-cycle of the map G is given by

$$\{\xi_0,\xi_1\} = \{(50,50,10b-50,10b-50),(10b-50,10b-50,50,50)\}.$$

Finally, it can be easily shown that $(F(x,y),F(y,x)) = (y,x)$ has no solution for $x \neq y$, and therefore, $z = F(x,y)$ has no 2-cycles. Also, by investigating solutions of $G^2(\xi) = \xi$, it can be shown that F has no artificial 2-cycles.

(ii) Consider

$$F(x,y) = xe^{-ay} + \frac{b}{1+y^2}, \quad x,y \geq 0.$$

Again here, F is continuous, it satisfies $F(\uparrow,\downarrow)$ and $F:\ \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$. For $a=4$ and $b=1$, it can be shown that $F(\bar{x},\bar{x}) = \bar{x}$ has a unique solution, $(F(x,y),F(y,x)) = (y,x)$ has no solution while $(F(x,y),F(y,x)) = (x,y)$ has two solutions. Based on our definition, F has a fixed point, no 2-cycles and has two artificial fixed points. Therefore, G has three fixed points, one coming from the fixed point of F and two coming from the two artificial fixed points. Also, $(F(x,x),F(y,y)) = (y,x)$ has no solution for $x \neq y$, which means $y = F(x,x)$ has no 2-cycles. Thus, G has no 2-cycles of the type in Part (ii) of Proposition 2.1. Finally, it can be shown graphically that F has no artificial 2-cycles by showing $G^2(\xi) = \xi$ has no solution for which $G(\xi) \neq \xi$. Next, we keep $a=4$ and increase b to 2. In this case, it is a computational matter to find that G has a 2-cycle $\{\xi_0,\xi_1\}$ derived from a 2-cycle of $y = f(x) = F(x,x)$, namely

$$\begin{aligned} \xi_0 &\simeq (0.6264, 0.6264, 1.4875, 1.4875) \\ \xi_1 = G(\eta_0) &\simeq (1.4875, 1.4875, 0.6264, 0.6264) \end{aligned}$$

and the two 2-cycles $\{\eta_0,\eta_1\}, \{\eta_0^t,\eta_1^t\}$, where

$$\begin{aligned} \eta_0 &\simeq (0.8352, 0.5479, 1.1975, 1.6315) \\ \eta_1 = G(\eta_0) &\simeq (1.6315, 1.1975, 0.5479, 0.8352) \end{aligned}$$

and

$$\begin{aligned} \eta_0^t &\simeq (0.5479, 0.8352, 1.6315, 1.1975) \\ \eta_1^t = G(\eta_0^t) &\simeq (1.1975, 1.6315, 0.8352, 0.5479). \end{aligned}$$

Those two 2-cycles of G correspond to two artificial 2-cycles of F, namely

$$[(0.8352, 0.5479), (1.6315, 1.1975)] \quad and \quad [(0.5479, 0.8352), (1.1975, 1.6315)], \quad respectively.$$

After illustrating the embedding technique and the notion of artificial cycles, we cite some results on globally attracting cycles that are particularly relevant to our work here. The next result can be found in [17].

Theorem 2.1. *Suppose that $F: [a,b]^2 \to [a,b]$ is continuous and $F(\uparrow,\downarrow)$. If $(F(x,y),F(y,x)) = (x,y)$ has a unique solution in $[a,b]^2$, then Eq. (1.1) has a unique equilibrium solution which forms a global attractor with respect to $[a,b]^2$.*

The method of Kulenovic and Merino in proving Theorem 2.1 was based on establishing two monotonic sequences which converge to the unique solution of $(F(x,y),F(y,x)) = (x,y)$, and the two sequences are used to squeeze the orbits of Eq. (1.1). Under the umbrella of the embedding technique [19, 20], Theorem 2.1 becomes simple since F has no artificial fixed points and the fact that the domain is a box makes $F(a,b) < b$ while $F(b,a) > a$. For initial conditions x_{-1}, x_0 of Eq. (1.1), we have

$$G((x_0,x_{-1}),(x_0,x_{-1})) = (F(x_0,x_{-1}),x_0,F(x_0,x_{-1}),x_0) = (x_1,x_0,x_1,x_0)$$

and in general

$$G^n((x_0,x_{-1}),(x_0,x_{-1})) = (x_n,x_{n-1},x_n,x_{n-1}). \tag{2.7}$$

From the inequalities in (2.3), (2.4) and Eq. (2.7), the orbit of G through $\xi_0 = (x_0,x_{-1},x_0,x_{-1})$ can be squeezed between the orbits of G through (A,B) and (B,A). The order convergence of $G^n(A,B)$ and $G^n(B,A)$ to a unique fixed point leads to a topological convergence of orbits of Eq. (1.1) to a unique equilibrium solution.

Plenty of examples can be found in which all components of Theorem 2.1 are satisfied except the invariant domain is not a box [2]. In this case, a less restrictive condition is to have a specific orbit of Eq. (1.1) with initial conditions $x_{-1} < x_0$ that satisfy $F(x_0,x_{-1}) < x_0$ and $F(x_{-1},x_0) > x_{-1}$. Then an argument similar to inequalities (2.3) and (2.4) can still give us a globally attracting fixed point with respect to a certain invariant domain. We extract the following result from [20].

Theorem 2.2. *Let Ω be an ordered metric space with a closed order relation $\leq_C$, and let the map $F:\Omega\times\Omega\to\Omega$ of Eq. (1.1) be continuous and $F(\uparrow,\downarrow)$. Suppose there exists $a,b\in\Omega$ such that $a\leq b$, $[a,b]\subset\Omega$, $a\leq F(a,b)$, $F(b,a)<b$ and $F([a,b]^2)$ has compact closure in Ω. If G has no artificial fixed points in $[a,b]^4$, then all orbits of Eq. (1.1) with $x_0,x_{-1}\in[a,b]$ converge to a fixed point of F.*

Whether in Theorem 2.1 or Theorem 2.2, it is crucial to have a square region as an invariant domain, and a typical rectangular invariant domain for Eq. (1.1) is the positive orthant. A setback can be faced when neither the positive orthant nor a rectangular domain can be found as an invariant domain [2]. As a remedy, the authors in [2] gave an approach to extend the domain of F without destroying the continuity of F or the monotonicity in its arguments. If an extension $\widetilde{F}$ is found in which continuity and monotonicity is preserved, and the range of $\widetilde{F}$ coincides with the range of F, the extension is called a *nice extension*. Now, we extract the next theorem from [2].

Theorem 2.3. *Let F of Eq. (1.1) be $F(\uparrow,\downarrow)$ and continuous on a compact and convex domain Ω in the plane. If the boundary of Ω ($\partial\Omega$) is a piecewise smooth Jordan curve, circumscribed on a rectangle R and has a parametrization $r: [0,1]\to\partial\Omega$ so that $F\circ r$ is differentiable on $[0,1]$ in which the derivative has only finitely many zeros, then F admits a nice extension $\widetilde{F}$ on R.*

According to Theorem 2.3, if the invariant domain is not a box, it may be extended into one, and if the extension $\widetilde{F}$ of F is a nice extension, the embedding technique may be used in conjunction with Theorem 2.1. The essence of this conclusion is given in the next theorem which appears in [2].

Theorem 2.4. *Let Ω be a compact subset of $\mathbb{R}^2$. Suppose the map F of Eq. (1.1) is continuous, $F(\uparrow,\downarrow)$ and Ω is invariant under F. If F has a nice extension $\widetilde{F}$ over a rectangular domain containing Ω such that $\widetilde{F}$ has no artificial fixed points and $Fix(F) = Fix(\widetilde{F})$, then for all $(x_0, x_{-1}) \in \Omega$, the orbits of Eq. (1.1) must converge to a fixed point of F.*

Thus, the critical concern becomes whether it is possible to create artificial fixed points of $\widetilde{F}$ or whether it is possible to create fixed points of $\widetilde{F}$ that are not fixed points of the original map F. In fact, the contrapositive of Theorem 2.4 can be useful to conclude the following corollary.

Corollary 2.1. *Let Ω be a compact subset of $\mathbb{R}^2$. Suppose the map F of Eq. (1.1) is continuous, $F(\uparrow,\downarrow)$ and Ω is invariant under F. If F has a unique fixed point in Ω that is not a global attractor, then any nice extension $\widetilde{F}$ of F over a rectangular domain containing Ω must have either artificial fixed points or $Fix(\widetilde{F}) \neq Fix(F)$.*

3 Embedding periodic maps of mixed monotonicity

In this section, we focus on Eq. (1.3), in which $F_j : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a p-periodic sequence of continuous maps that satisfy $F_j(\uparrow,\downarrow)$ for each j. A region $\Omega \subseteq \mathbb{R}^2_+$ is called an invariant region of Eq. (1.3) if $T_j(\Omega) \subseteq \Omega$ for all j. As for the autonomous case, we define $g_n((x,y),(u,v)) = (F_n(x,y),u)$ and

$$G_n(X,U) = (g_n(X,U), g_n(U,X)) = ((F_n(x,y),u),(F_n(u,v),x)). \tag{3.1}$$

This gives a p-periodic system in 4-dimensions, namely

$$\xi_{n+1} = G_{n \bmod p}(\xi_n), \quad \text{where} \quad \xi_n = (X_n, U_n) = ((x_n,y_n),(u_n,v_n)). \tag{3.2}$$

An orbit of Eq. (3.2) is given by

$$\mathcal{O}_{G_n}(\xi) = \{\xi, G_0(\xi), G_1(G_0(\xi)), G_2(G_1(G_0(\xi))), \dots\}. \tag{3.3}$$

The orbits of Eq. (3.3) can be partitioned based on the orbits of p autonomous systems, namely

$$\xi_{n+1} = \widehat{G}_j(\xi_n) \quad \text{where} \quad \widehat{G}_j = G_{j-1} \circ \cdots \circ G_0 \circ G_{p-1} \circ \cdots \circ G_j \tag{3.4}$$

and $j = 0, 1, \dots, p-1$. The connection between the orbits of Eq. (1.3) and Eq. (1.4) is obvious. Also, the connection between the orbits of Eq. (3.2) and the orbits of the equations in (3.4) is easy to observe. For instance, when $p = 2$ and n starts at 0, the even terms of the orbits in Eq. (3.2) give the orbits of $\xi_{n+1} = \widehat{G}_0(\xi_n)$, and the odd terms give the orbits of $\xi_{n+1} = \widehat{G}_1(\xi_n)$. It remains to establish the connection between the orbits of Eq. (1.3) and Eq. (3.2). Equilibrium points and cycles in first order nonautonomous periodic systems have been explored in the past two decades. We refer the interested reader to [1,3,5,6]. Here, the second order and the embedded maps G_n in Eq. (3.2) add another factor of complexity to the structure of cycles and the notion of artificial cycles. We begin by extending Definition 2.1 to periodic systems.

Definition 3.1. *If $F_j(x,x)=x$ for all $j=0,\dots,p-1$ then x is an equilibrium solution (equilibrium point) of Eq. (1.3). Similarly, if $G_j(\xi)=\xi$ for all $j=0,\dots,p-1$ then ξ is an equilibrium solution (equilibrium point) of Eq. (3.2). If $\xi=((x,y),(y,x))$ is an equilibrium point of Eq. (3.2) and $x\neq y$, then (x,y) is called an artificial equilibrium point of Eq. (1.3). If there exists $x_0,\dots,x_{q-1}$ such that q is the smallest positive integer for which $F_{n+1 \bmod p}(x_{n+1 \bmod q},x_{n \bmod q})=x_{n+2 \bmod q}$ for all $n\in\mathbb{N}$, then $[x_0,x_1,\dots,x_{q-1}]$ is a q-cycle of Eq. (1.3). Similarly, $C_q:=[\xi_0,\xi_1,\dots,\xi_{q-1}]$ is a q-cycle of Eq. (3.2) if q is the smallest positive integer for which $G_{n \bmod p}(\xi_{n \bmod q})=\xi_{n+1 \bmod q}$ for all $n\in\mathbb{N}$. If $C_q:=[\xi_0,\xi_1,\dots,\xi_{q-1}]$ is a q-cycle of Eq. (3.2) and $\xi_0\notin D$, then we say that Eq. (1.3) has an artificial q-cycle.*

As in the autonomous case, an equilibrium point $\bar{\xi}=(\bar{x},\bar{y},\bar{u},\bar{v})$ of Eq. (3.2) means $(\bar{u},\bar{v})=(\bar{y},\bar{x})$ and $(F_j(\bar{x},\bar{y}),\bar{F}_j(\bar{y},\bar{x}))=(\bar{x},\bar{y})$ for all j. Therefore, an equilibrium point of Eq. (3.2) comes as a consequence of an equilibrium point or an artificial equilibrium point of Eq. (1.3). On the other hand, if $\xi=((x,y),(u,v)$ and $\{\xi,G_0(\xi)\}$ is a 2-cycle of Eq. (3.2), but $[x,y]$ is not a 2-cycle of Eq. (1.3), then $[(x,y),(u,v)]$ is an artificial 2-cycle of Eq. (1.3). Now, we give the analog of Proposition 2.1 in the following lemma:

Lemma 3.1. *Consider the p-periodic difference equations in (1.3) and (3.2). Each of the following holds true:*

(i) An equilibrium solution $\bar{\xi}$ of Eq. (3.2) is a common fixed point for all maps G_j. Furthermore, $\bar{\xi}$ must be of the form $\bar{\xi}=(\bar{x},\bar{y},\bar{y},\bar{x})$.

(ii) Suppose that $\bar{\xi}=(\bar{x},\bar{y},\bar{y},\bar{x})$ is an equilibrium solution of Eq. (3.2). If $\bar{x}=\bar{y}$, then $\bar{x}$ is a common fixed point for all maps F_j; otherwise, $(\bar{x},\bar{y})$ and $(\bar{y},\bar{x})$ are two common artificial fixed points for all maps F_j.

(iii) Let $p=2$. If $[\xi_0,\xi_1]$ is a 2-cycle of Eq. (3.2), then we must have $\xi_0:=(x_0,y_0,y_1,x_1)$ and $\xi_1:=(x_1,y_1,y_0,x_0)$, where

$$F_0(x_0,y_0)=x_1,\ F_0(y_1,x_1)=y_0,\ F_1(x_1,y_1)=x_0 \quad \textit{and} \quad F_1(y_0,x_0)=y_1.$$

(iv) Let $f_j(x)=F_j(x,x)$ for all $j=0,\dots,p-1$. If the equation $x_{n+1}=f_n(x_n)$ has $[x_0,x_1]$ and $[x_1,x_0]$ as 2-cycles, then Eq. (3.2) has the 2-cycles $[\xi_0,\xi_1]$ and $[\xi_1,\xi_0]$, where

$$\xi_0:=(x_0,x_0,x_1,x_1) \quad \textit{and} \quad \xi_1:=(x_1,x_1,x_0,x_0).$$

(v) Let $f_j(x)=F_j(x,x)$ for all $j=0,\dots,p-1$. If $\{x_0,x_1\}$ is a common 2-cycle for all maps f_j, then Eq. (3.2) has the 2-cycles $[\xi_0,\xi_1]$ and $[\xi_1,\xi_0]$, where

$$\xi_0:=(x_0,x_0,x_1,x_1) \quad \textit{and} \quad \xi_1:=(x_1,x_1,x_0,x_0).$$

(vi) If Eq. (1.3) has a p-cycle $[x_0,x_1,\dots,x_{p-1}]$, then Eq. (3.2) has the p-cycle $[\psi_0,\psi_1,\dots,\psi_{p-1}]$, where

$$\psi_j:=(x_{j+1},x_j,x_{j+1},x_j),\ j=0,\dots,p-2 \quad \textit{and} \quad \psi_{p-1}:=(x_0,x_1,x_0,x_{p-1}).$$

(vii) Let $p=2$, $\eta=(x,y,u,v)$ and suppose that $G_1(G_0(\eta))=\eta$. Eq. (1.3) has $[(x,y),(u,v)]$ as an artificial 2-cycle if and only if $(u,v)\neq(y,x),(x,y)$.

(viii) Let $p = 2$. $[(x,y),(u,v)]$ and $[(y,x),(v,u)]$ are two artificial 2-cycles of Eq. (1.3) if and only if they are common artificial 2-cycles for both maps F_0 and F_1.

Proof. (i) From the orbit in Eq. (3.3), a fixed point $\bar{\xi}$ must satisfy $G_j(\bar{\xi}) = \bar{\xi}$ for all j. Now, Unfold $G_j(\xi) = \xi$ to obtain

$$G_j(\xi) = (g_j(X,U), g_j(U,X)) = ((F_j(x,y),u),(F_j(u,v),x)) = ((x,y),(u,v)).$$

Thus, we need $(u,v) = (y,x)$ which means $\bar{\xi}$ must be of the form $\bar{\xi} = (\bar{x},\bar{y},\bar{y},\bar{x})$. Part (ii) follows from Part (i) and by unfolding $G_j(\xi) = \xi$. To prove Part (iii), let $\xi_j = (x_j, y_j, u_j, v_j)$ for $j = 0,1$. Since $[\xi_0, \xi_1]$ is a 2-cycle of Eq. (3.2), we must have $G_0(\xi_0) = \xi_1$ and $G_1(\xi_1) = \xi_0$. Unfold the two equations to obtain the result. Next, we prove Part (iv), If $f_i = f_j$ for all $0 \le i, j \le p-1$, then the equation $x_{n+1} = f_n(x_n)$ becomes autonomous and $[x_0, x_1], [x_1, x_0]$ become the same cycle, i.e., $\{x_0, x_1\}$. However, there is no loss of generality if we consider $x_{n+1} = f_n(x_n)$ to be non-autonomous. In this case, having $[x_0, x_1], [x_1, x_0]$ as 2-cycles means $\{x_0, x_1\}$ is a common 2-cycle for all maps f_j. Now, define ξ_0 as given and observe that

$$G_0(\xi_0) = (f_0(x_0), x_1, f_0(x_1), x_0) = (x_1, x_1, x_0, x_0) = \xi_1.$$

Then use induction on $G_j(\xi_{j \bmod 2}) = \xi_{j+1 \bmod 2}$ to obtain the 2-cycle $[\xi_0, \xi_1]$. Similarly for the other 2-cycle. Part (v) becomes obvious from Part (iv) since having $[x_0, x_1], [x_1, x_0]$ as 2-cycles of $x_{n+1} = f_n(x_n)$ is equivalent to having $\{x_0, x_1\}$ as a common cycle for both maps f_0 and f_1. Next, we prove Part (vi), assume Eq. (1.3) has the 2-cycle $[x_0, x_1]$. This means $x_0 \neq x_1$ and $(F_0(x_1, x_0), F_1(x_0, x_1)) = (x_0, x_1)$. On the other hand, $G_1(G_0(\psi)) = \psi = (x,y,u,v)$ is equivalent to

$$F_0(x,y) = v,\ F_0(u,v) = y,\ F_1(y,x) = u \quad \text{and} \quad F_1(v,u) = x. \tag{3.5}$$

Now, consider $(x,y) = (x_1, x_0)$, then define ψ_0 and ψ_1 as given to obtain the 2-cycle $[\psi_0, \psi_1]$ of Eq. (3.2). To prove Part (vii), we depend on Eqs. (3.5) since they are obtainable by unfolding $G_1(G_0(\eta)) = \eta$, then assume that Eq. (1.3) has $[(x,y),(u,v)]$ as an artificial 2-cycle. This implies η is not an equilibrium point of Eq. (3.2), and consequently, $(u,v) \neq (y,x)$. It also implies that $[x,y]$ is not a 2-cycle of Eq. (1.3), and consequently $(u,v) \neq (x,y)$ as given in Part (vi). The converse is obvious. Finally, we prove Part (viii). Consider $[(x,y),(u,v)]$ to be an artificial 2-cycle of Eq. (1.3). We have $(u,v) \neq (x,y),(y,x)$ and Eqs. (3.5) are valid. To have $[(y,x),(v,u)]$ as another artificial 2-cycle, we need

$$F_1(x,y) = v,\ F_1(u,v) = y,\ F_0(y,x) = u \quad \text{and} \quad F_0(v,u) = x.$$

Now, the rest of the proof becomes obvious. □

In part (iv) of Lemma 3.1, the two 2-cycles $[x_0, x_1]$, $[x_1, x_0]$ of the equation $x_{n+1} = f_n(x_n)$ were sufficient to give two 2-cycles for Eq. (3.2). This fact raises the question whether the existence of the two cycles is necessary. Our next example settles this question.

Example 3.1. *Consider the 2-periodic difference equation $x_{n+1} = F_n(x_n, x_{n-1})$, where*

$$F_0(x,y) = \frac{p_0 + qx}{1 + x + y} \qquad \textit{and} \qquad F_1(x,y) = \frac{p_1 + qx}{1 + x + y}.$$

Since the difference equation is 2-periodic, must have $p_0 \neq p_1$. If we fix $q = 2p_1$, then we obtain

$$f_0(x) = F_0(x,x) = \frac{p_0 + 2p_1 x}{1+2x} \qquad \text{and} \qquad f_1(x) = F_1(x,x) = \frac{p_1 + 2p_1 x}{1+2x} = p_1.$$

The 2-periodic system $x_{n+1} = f_n(x_n)$ has the unique 2-cycle $[x_0, x_1]$, where $x_0 := p_1$ and $x_1 := \frac{p_0+2p_1^2}{1+2p_1}$. In fact, every orbit converges in finite time to this 2-cycle. Since $[x_1, x_0]$ is not a 2-cycle of $x_{n+1} = f_n(x_n)$, it is easy to observe that Eq. (3.2) has no 2-cycles of the type given in parts (iv) and (v) of Lemma 3.1.

4 Global stability

This section focuses on the asymptotic behavior of orbits of Eq. (3.2) and their effect on the orbits of Eq. (1.3). As shown by Lemma 3.1 and the partitioned orbits in Eqs (3.4), an equilibrium solution $\bar{\xi}$ of Eq. (3.2) requires all maps G_j to agree on $\bar{\xi}$, which is rather restrictive. Therefore, rather than an equilibrium solution, we consider a q-cycle for some q that divides p. In fact, to have less restriction on the intersections between the maps G_j, it is natural to consider $q = p$. This means the folded map

$$\widehat{G}_0 := G_{p-1} \circ \cdots \circ G_1 \circ G_0 : \ [a,b]^4 \ \to \ [a,b]^4$$

has a fixed point, say ξ_0, which forms the first element of a p-cycle. The second element of the p-cycle will be $\xi_1 := G_0(\xi_0)$ which must be a fixed point of the folded map $\widehat{G}_1$. In general, an element ξ_j in the p-cycle $C_r := [\xi_0, \xi_1, \ldots, \xi_j, \ldots, \xi_{p-1}]$ implies the folded map $\widehat{G}_j$ has a fixed point at ξ_j. Observe that the continuity of all maps F_j leads to the continuity of all maps G_j. Thus, if $\psi_{n+1} = \widehat{G}_0(\psi_n)$ converges to the fixed point ξ_0 for some initial condition ψ_0, we obtain convergence in $\psi_{n+1} = \widehat{G}_j(\psi_n)$ to ξ_j for each j. Consequently, the orbit of Eq. (3.2) through ψ_0 converges to the p-cycle C_r. Therefore, it is sufficient to focus on the orbits of $\psi_{n+1} = \widehat{G}_0(\psi_n)$. Now, we give the following global stability result:

Theorem 4.1. *Consider the p-periodic difference equation in Eq. (1.3). Suppose that $\Omega := [a,b]^2$ is an invariant region and the folded map $\widehat{G}_0 = G_{p-1} \circ \cdots \circ G_1 \circ G_0$ has a unique fixed point. Then Eq. (1.3) has a globally attracting q-cycle for some q that divides p. In particular, if the q-cycle is locally stable, then it is globally stable.*

Proof. Consider Eq. (1.4) which is the vector form of Eq. (1.3). The map $T := T_{p-1} \circ \cdots \circ T_1 \circ T_0$ maps Ω into itself. By Brouwer fixed-point theorem, T has a fixed point in Ω, say $X_0 = (x_1, x_0)$. Unfold the map T into its T_j components to obtain a q-cycle for some q that divides p. The cycle will be

$$[(x_1, x_0), (x_2, x_1), \ldots, (x_{q-1}, x_{q-2}), (x_0, x_{q-1})],$$

which gives a q-cycle of Eq. (1.3), namely $C_q := [x_0, x_1, \ldots, x_{q-1}]$. This cycle C_q creates a q-cycle for Eq. (3.2), namely

$$[(X_0, X_0), \ldots (X_{q-1}, X_{q-1})], \quad \text{where} \quad X_{j-1} = (x_j, x_{j-1}),\ j = 1, \ldots, q-1$$

and $X_{q-1} = (x_0, x_{q-1})$. Since the folded map $\widehat{G}_0$ has a unique fixed point, then it must be the one derived from C_q, namely $\xi_0 = (X_0, X_0)$. Furthermore, the cycle C_q of Eq. (1.3) must be unique. Now, it remains to show that the obtained q-cycle is globally attracting. From the inequalities in

(2.3) and (2.4), consider G to be the folded map $\widehat{G}_0$, and consider an arbitrary orbit of Eq. (1.3) through $(y_0, y_{-1}) \in \Omega$. If $y_0 \le y_{-1}$, consider $X = (y_0, y_{-1})$ and $Y = (y_{-1}, y_0)$, then $X \le_{se} Y$ and we obtain

$$\widehat{G}_0^n(A,B) \le_{se} \widehat{G}_0^n(X,Y) \le_{se} \widehat{G}_0^n(X,X) \le_{se} \widehat{G}_0^n(Y,X) \le_{se} \widehat{G}_0^n(B,A),$$

where $A = (a,b)$ and $B = (b,a)$. On the other hand, if $y_0 \ge y_{-1}$, consider $Y = (y_0, y_{-1})$ and $X = (y_{-1}, y_0)$, then $X \le_{se} Y$ and we obtain

$$\widehat{G}_0^n(A,B) \le_{se} \widehat{G}_0^n(X,Y) \le_{se} \widehat{G}_0^n(Y,Y) \le_{se} \widehat{G}_0^n(Y,X) \le_{se} \widehat{G}_0^n(B,A).$$

Since $\{\widehat{G}_0^n(A,B)\}$ must converge to a fixed point of $\widehat{G}_0$, the uniqueness condition makes

$$\lim \widehat{G}_0^n(A,B) = \lim \widehat{G}_0^n(X,X) = \lim \widehat{G}_0^n(B,A)$$

in the first case and

$$\lim \widehat{G}_0^n(A,B) = \lim \widehat{G}_0^n(Y,Y) = \lim \widehat{G}_0^n(B,A)$$

in the second case. Therefore, in both cases, we obtain (y_0, y_{-1}) attracted to the first element of the cycle C_q through the folded map $\widehat{G}_0$. Similarly, each initial point $(y_0, y_{-1}) \in \Omega$ is attracted to the j^{th} element of the cycle C_q through the folded map $\widehat{G}_j$. Hence, Eq. (1.3) has a globally attracting q-cycle for some q that divides p. Finally, if the attracting cycle is locally stable, then it becomes globally stable with respect to Ω. □

Next, if the invariant region Ω of Theorem 4.1 is just a compact region (not necessarily a box), we assume the positive orthant $\mathbb{R}_+^2$ to be invariant, and we assume the existence of two points $A := (a,b), B := (c,d) \in \mathbb{R}_+^2$ such that $a \le c, d \le b$ and $(A,B) \le_{se} \widehat{G}_0(A,B)$. Define the set $\widetilde{\Omega}$ to be the union of all orbits that emanate from $[a,c] \times [d,b]$, i.e.,

$$\widetilde{\Omega} := \{X : \ X \in \mathcal{O}_{F_n}^+(X_0), \ A \le_{se} X_0 \le_{se} B\}. \tag{4.1}$$

This aids us in obtaining the following result.

Theorem 4.2. *Consider the p-periodic difference equation in Eq. (1.3). Suppose that $\mathbb{R}_+^2$ is an invariant region, $\widetilde{\Omega}$ is precompact, and there exists a point $\xi_0 = ((a,b),(c,d))$ such that $a < c, d < b$ and $\xi_0 \le_{se} \widehat{G}_0(\xi_0)$. If the folded map $\widehat{G}_0 = G_{p-1} \circ \cdots \circ G_1 \circ G_0$ has a unique fixed point in $[a,c] \times [d,b]$, then Eq. (1.3) has a globally attracting q-cycle with respect to $\widetilde{\Omega}$. Furthermore, q is a divisor of p, and if the q-cycle is locally stable, then it is globally stable with respect to $\widetilde{\Omega}$.*

Proof. Suppose there is a point $\xi_0 = ((a,b),(c,d))$ such that $a \le c, d \le b$ and $\xi_0 \le_{se} \widehat{G}_0(\xi_0)$. In this case, all initial conditions $(y_0, y_{-1}) \in [a,c] \times [d,b]$ lead us to

$$\xi_0 \le_{se} \widehat{G}_0(\xi_0) \le_{se} \widehat{G}_0(X,X) \le_{se} \widehat{G}_0(\xi_0^t) \le_{se} \xi_0^t,$$

where $\xi_0^t = ((c,d),(a,b))$, $X = (y_0, y_{-1})$. Thus, $\widehat{G}_0^n(\xi_0)$ and $\widehat{G}_0^n(\xi_0^t)$ converge to the unique fixed point. This forces $\widehat{G}_0^n(X,X) = (y_{np}, y_{np-1}, y_{np}, y_{np-1})$ to converge to the unique fixed point. Therefore, we obtained a subsequence of the orbit of Eq. (1.3) through (y_0, y_{-1}) to converge to a point. On the other hand, the unique fixed point of the folded map $\widehat{G}_0$ gives a q-cycle of Eq. (1.3) for some q that divides p. This shows that the orbit of Eq. (1.3) through (y_0, y_{-1}) is partitioned into q subsequences, each of which converges to an element of the q-cycle. Also, the region $[a,c] \times [d,b]$ is attracted to the q-cycle. Moreover, if an orbit leaves $[a,c] \times [d,b]$, then it comes back to the set in finite time. This means $\tilde{\Omega}$ as defined in 4.1 is attracted to the q-cycle. □

It is worth noting that the condition $\xi_0 \leq_{se} \widehat{G}_0(\xi_0)$ can be replaced by the more tempting condition $a \leq F_j(a,b) \leq F_j(b,a) \leq b$ for all j. In this case, $\xi_0 = ((a,b),(b,a))$. However, this will be strong, sufficient and computationally manageable but unnecessary, and could be vacuous.

5 Applications

In this section, we analyze two examples to demonstrate applicability of our developed approach. The first is a periodic rational difference equation, whereas the second is a population model with periodic stocking. Because our goal here is to demonstrate the application of our theory, we give ourself the liberty to restrict the periodicity.

5.1 A rational equation

Consider the periodic difference equation

$$x_{n+1} = F_n(x_n, x_{n-1}) = \frac{p_n + q x_n}{1 + x_n + x_{n-1}}, \qquad x_0, x_{-1} \geq 0,\ n \in \mathbb{Z}^+, \tag{5.1}$$

where $\{p_n\}$ is a p-periodic sequence of non-negative real numbers. Since our intention here is to show the applicability of the developed theory, we limit our attention to $p = 2$. The periodicity can be forced on both parameters, or either one of them, but here, we consider it on $\{p_n\}$. An autonomous version ($p_n = p$ and $F_n = F$) of Eq. (5.1) has been widely discussed in the literature [7–9, 15, 16]. Different choices of the parameters p and q give various scenarios of stability. When $0 < p \leq q$, it can be easily shown that the rectangular region $[0,q] \times [0,q]$ is invariant, F is increasing in its first component and decreasing in its second one. Therefore, Theorem 2.1 is applicable here [2]. Under the forced periodicity that we have in Eq. (5.1), we assume $0 < p_j \leq q$ for $j = 0, 1$, then appeal to the developed theory in sections 3 and 4. For the reader's convenience, we summarize the following facts.

Proposition 5.1. *Consider F_n as defined in Eq. (5.1) and G_n as defined in Eq. (3.1). Suppose $p = 2$ and $p_0 \neq p_1$, then each of the following holds true:*

(i) *Eq. (3.2) has no equilibrium solutions.*

(ii) *Eq. (5.1) has neither equilibrium solutions nor artificial equilibrium solutions.*

(iii) *We have $F_0 : [0,q]^2 \to [0,q]$, $F_1 : [0,q]^2 \to [0,q]$ and $[0,q]^4$ is an invariant region for Eq. (3.2).*

(iv) *Eq. (5.1) has a unique 2-cycle in the region $[0,q]^2$.*

(v) *Eq. (5.1) has no artificial 2-cycles.*

Proof. (i) Based on parts (i) and (ii) of Lemma 3.1, Eq. (3.2) has an equilibrium solution $\bar{\xi}$ if $\bar{\xi} = (x,y,y,x)$ and $(F_j(x,y), F_j(y,x)) = (x,y)$ for both $j = 0$ and $j = 1$. However, it is elementary computations to show that $p_0 \neq p_1$ forces the nonexistence of solutions. Since equilibrium solutions of Eq. (3.2) give either equilibrium solutions or artificial equilibrium solutions of Eq. (5.1), the proof of Part (i) makes Part (ii) obvious. Part (iii) follows from the fact that $x_{-1}, x_0 \geq 0$, $p_0, p_1, q > 0$ and

$$F_j(x_n, x_{n-1}) \leq q \frac{1 + x_n}{1 + x_n + x_{n-1}} \leq q, \quad \text{for} \quad j = 0, 1.$$

To prove Part (iv), we depend on Part (vi) of Lemma 3.1 and equations (3.5) in its proof. We need to solve $(F_0(x,y), F_1(y,x)) = (y,x)$ for x and y. This gives us

$$y = -x - q - 1 + \frac{q^2 + q - p_1}{q - x} \qquad \text{and} \qquad x = -y - q - 1 + \frac{q^2 + q - p_0}{q - y}. \tag{5.2}$$

The curves of both equations intersect at a unique point in the region $[0,q]^2$ (Fig. 1) gives an illustration). In fact, the 2-cycle can be given explicitly, but we are not concerned with a formidable expression here. Finally, we rely on Part (vii) of Lemma 3.1 to prove Part (v). Indeed, Eqs. (3.5) lead us to

$$F_0(F_1(y,x), F_0(x,y)) = y \qquad \text{and} \qquad F_1(F_0(x,y), F_1(y,x)) = x,$$

which give us same solution as $(F_0(x,y), F_1(y,x)) = (y,x)$. This completes the proof. $\square$

It is worth mentioning that if we take $f_0(x) = F_0(x,x)$ and $f_1(x) = F_1(x,x)$, then the system $x_{n+1} = f_n(x_n)$ has a unique 2-cycle. A special case was given in Example 3.1. However, since the 2-cycle is not for each individual map f_j, then it does not contribute to creating artificial cycles of Eq. (5.1). Now, we are ready to apply Theorem 4.1 and show that the 2-cycle is globally attracting.

Corollary 5.1. *Consider Eq. (5.1) in which $x_{-1}, x_0 \geq 0$, and $0 < p_0, p_1 \leq q$. The 2-cycle assured by Part (iv) of Proposition 5.1 is globally attracting with respect to $\mathbb{R}_+^2$.*

Proof. It is clear that orbits of Eq. (5.1) enter the region $[0,q]^2$ in finite time. Now, inside the region $[0,q]^2$, orbits are attracted to the 2-cycle by Theorem 4.1. $\square$

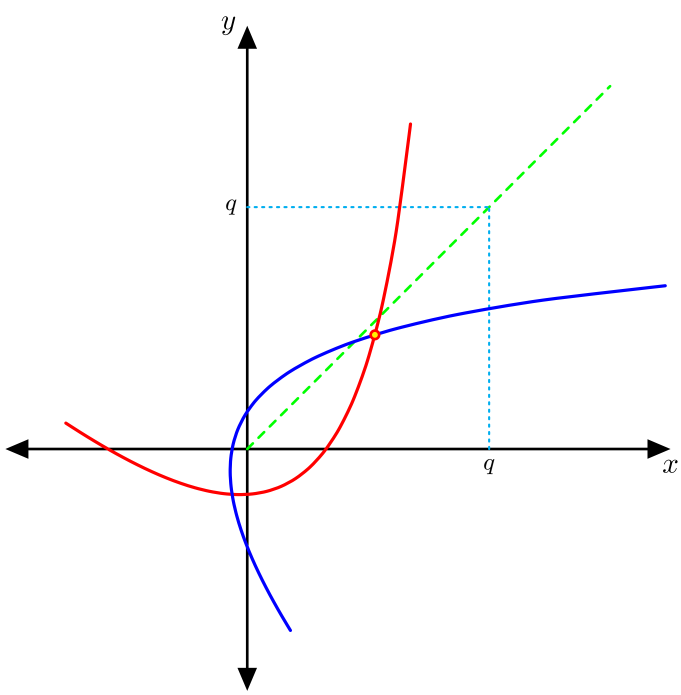

Figure 1: This figure shows the intersection between the curves of equations (5.2). The curves were captured at $p_0 = 1, p_1 = 3$ and $q = 4$. In this case, the unique 2-cycle of Eq. (5.1) is $\approx [2.111, 1.889]$. Based on Part (vi) of Lemma 3.1, the corresponding 2-cycle of Eq. (3.2) is $[\psi_0, \psi_1]$, where $\psi_0 \approx (1.889, 2.111, 1.889, 2.111)$ and $\psi_1 \approx (2.111, 1.889, 2.111, 1.889)$.

5.2 A population model

Equations of the form $x_{n+1} = x_n f(x_n)$ are used in mathematical ecology to model single species with non-overlapping generations [13, 14]. In this case, x_n represents sexually mature individuals at discrete time n, while $f(x_n)$ represents the density dependent growth rate. Biological or environmental factors are commonly used to determine the nature of the function f. A prototype of $f(x)$ is $\frac{\beta}{c+x}$ or βe^{-cx}, which motivate us to proceed with the assumption that $f:\ \mathbb{R}_+ \to \mathbb{R}_+$, $f(0) = \beta > 1$, f is continuous and decreasing. When a significant amount of time is required for sexual maturation, a delay effect must be included in the density function f, and in this case, we can consider $x_{n+1} = x_n f(x_{n-1})$. Furthermore, when a species is subject to constant stocking as a result of refuge or immigration, the equation becomes

$$x_{n+1} = F(x_n, x_{n-1}) = x_n f(x_{n-1}) + h, \quad \text{where} \quad h > 0, n \in \mathbb{N}, x_0, x_{-1} \in \mathbb{R}_+. \tag{5.3}$$

By forcing seasonal stocking, we obtain the p-periodic equation

$$x_{n+1} = F_n(x_n, x_{n-1}) = x_n f(x_{n-1}) + h_n, \quad \text{where} \quad n \in \mathbb{N}, h_n > 0, x_0, x_{-1} \in \mathbb{R}_+. \tag{5.4}$$

Both equations were considered in [4] in which the emphasis was on global stability, and the general case was left open. Here, we show how to implement our developed approach. We focus on the stocking parameter as the controlled or regulated parameter. Observe that a vertical segment (γ, t) of the domain is mapped to $(T(\gamma, t), \gamma) = (\gamma f(t) + h, \gamma)$, and since $f(0) = \beta > 1$, $\gamma f(t) + h$ cannot be smaller than γ at $t = 0$. This implies that we cannot obtain an invariant domain of the form $\Omega = [0, \gamma]^2$. However, we appeal to Theorem 2.2. We look for a point (a, b), $a < b$ such that

$$a < F(a, b) = af(b) + h < F(b, a) = bf(a) + h < b. \tag{5.5}$$

Or alternatively, (a, b), $b < a$ such that

$$b < F(b, a) = bf(a) + h < F(a, b) = af(b) + h < a. \tag{5.6}$$

We continue by considering a specific form of f which is $f(t) = \frac{\beta}{1+t}$. In this case, Eq. (5.3) has a unique positive equilibrium solution given by $\bar{x} = \frac{1}{2}\left(\beta + h - 1 + \sqrt{(\beta + h + 1)^2 - 4\beta}\right)$. When $h \neq \beta - 1$, F in Eq. (5.3) has no artificial fixed points, but when $h = \beta - 1$, we obtain an uncountable set of artificial fixed points given by

$$\left\{\left(t, \frac{(\beta - 1)(t + 1)}{t - (\beta - 1)}\right) :\ \beta > 1,\ t > \beta - 1,\ t \neq \bar{x}\right\}.$$

The inequalities in (5.5) give us

$$1 - \frac{h}{a} < f(b) < f(a) < 1 - \frac{h}{b}$$

and the ones in (5.6) give us

$$1 - \frac{h}{b} < f(a) < f(b) < 1 - \frac{h}{a}.$$

The first inequality has a feasible region when $h > \beta - 1$, and the second one has a feasible region when $h < \beta - 1$. We plot the feasible regions of both cases in Fig. 2. It has been shown in [4] (Proposition 1) that the equilibrium solution $\bar{x}$ is locally asymptotically stable, and a $\liminf, \limsup$ argument were used to show that it is globally attracting. Here, the same problem can be tackled using our approach. For the reader's convenience, we give the details in the following proposition:

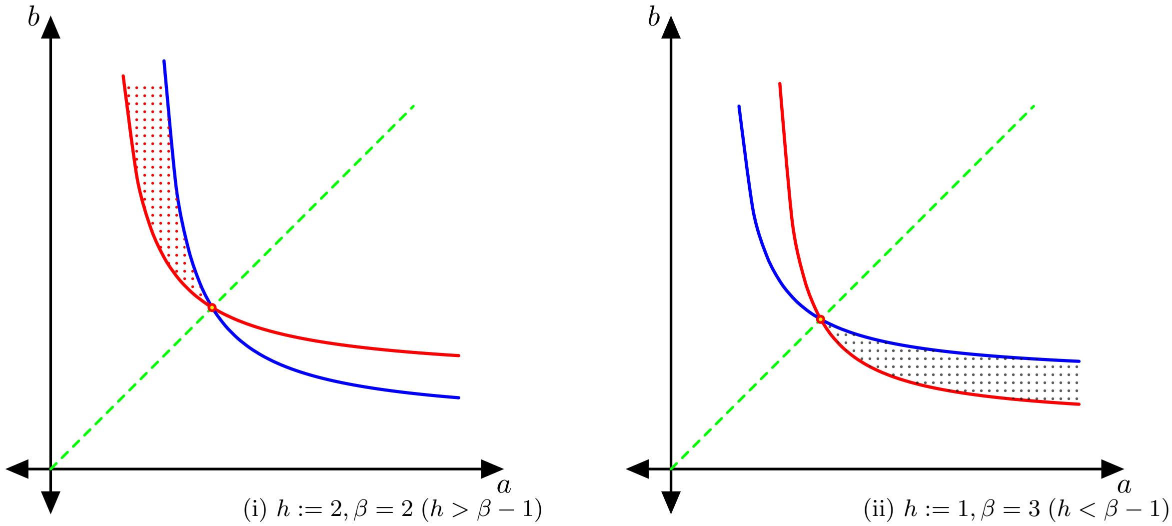

Figure 2: The shaded region in Figure (i) shows the points (a, b), $a < b$ that satisfy the inequalities in (5.5), while the shaded region in Figure (ii) shows the points (a, b), $b < a$ that satisfy the inequalities in (5.6). Figures (i) and (ii) have been captured at $b = h = 2$ and $b = 2, h = \frac{1}{2}$, respectively.

Proposition 5.2. *Consider Eq. (5.3) in which* $f(t) = \frac{\beta}{1+t}$, $\beta > 1$. *If* $h \neq \beta - 1$, *then the equilibrium solution* $\bar{x} = \frac{1}{2}\left(\beta + h - 1 + \sqrt{(\beta + h + 1)^2 - 4\beta}\right)$ *is globally attracting.*

Proof. For all $(x_0, x_{-1}) \in \mathbb{R}_+^2$, we have $x_1 > h$. Thus, we can focus on the invariant region $[h, \infty)^2$. Consider the case $h > \beta - 1$. For each $(x_0, x_{-1}) \in [h, \infty)^2$, we find (a, b) such that $x_0 \geq a, x_{-1} \leq b$ and $a < F(a, b) < F(b, a) < b$. Indeed, we can take $\beta - 1 < a < h$ and $b > x_{-1}$ such that (a, b) belongs to the feasible region of Fig. 2 (i). Now, depend on Theorem 4.2 to obtain that (x_0, x_{-1}) is attracted to $(\bar{x}, \bar{x})$. The case $h < \beta - 1$ can be handled in a similar way based on Fig. 2 (ii) and Theorem 4.2. $\square$

Theorem 4.2 does not cover the case $h = \beta - 1$ because the artificial fixed points prevent the iterates of G from squeezing the orbits of Eq. (5.3). Now, we turn our attention to the periodic case as given in Eq. (5.4), and we focus on the case $p = 2$.

Lemma 5.1. *Consider Eq. (5.4) with* $p = 2, \beta > 1$ *and* $h_0, h_1 > 0$. *Define* $\beta_0 := \sqrt{(h_0 + 1)(h_1 + 1)}$, *then each of the following holds true:*

(i) Eq. (5.4) has neither equilibrium solutions nor artificial equilibrium solutions.

(ii) If $\beta = \beta_0$, *then the* 2*-periodic equation has a unique* 2*-cycle and infinity many artificial* 2*-cycles.*

(iii) If $\beta \neq \beta_0$, *then the* 2*-periodic equation has a unique* 2*-cycle and no artificial* 2*-cycles.*

Proof. Part (i) follows from the fact that $h_0 \neq h_1$. The existence of a 2-cycle in Part (ii) and Part (iii) can be obtained by Brouwer fixed-point theorem (cf. [4]); however, the uniqueness can be established from the solution of the system $F_0(x, y) = y$ and $F_1(y, x) = x$. This gives us

$$y = \frac{1}{\beta}(x - h_1)(x + 1) \quad \text{and} \quad x = \frac{1}{\beta}(y - h_0)(1 + y). \tag{5.7}$$

The intersection between the two curves shows the uniqueness in both cases. Next, we proceed to test the existence artificial 2-cycles. Solve

$$G_1(G_0(\xi)) = \xi, \quad \text{where} \quad \xi = ((x,y),(u,v)), G_j(\xi) = (F_j(x,y), u, F_j(u,v), x), \; j = 0, 1.$$

This leads us to the system

$$v = F_0(x,y), \quad u = F_1(y,x), \quad x = F_1(v,u) \quad \text{and} \quad y = F_0(u,v).$$

Substitute u and v from the first two equations in the latter ones to obtain $x = F_1(F_0(x,y), F_1(y,x))$ and $y = F_0(F_1(y,x), F_0(x,y))$. Then eliminate y to obtain a single equation. Factor the obtained equation, and ignore the factors that generate the 2-cycle, i.e., the factor that appear when eliminating y from the two equations in (5.7). The elimination and factoring can be done using the resultant command in MAPLE. This gives us

$$x(x+1)^2(\beta^2 - h_0h_1 - h_0 - h_1 - 1)^2 = 0.$$

Thus, $\beta = \beta_0$ makes the equation valid regardless of the x value, which means that we have uncountable set of artificial 2-cycles. The other factors are positive and can be ignored. Hence, $\beta = \beta_0$ gives a unique 2-cycle and infinity many artificial 2-cycles. On the other hand, $\beta \neq \beta_0$ gives a unique 2-cycle and no artificial 2-cycles. □

Next, we gear towards proving that the existed 2-cycle is globally attracting. Recall the solution of $(F_1(y,x), F_0(x,y)) = (x,y)$ gives the 2-cycle, say $[x_0, x_1]$. Let (x,y) be chosen so that $F_1(y,x) > x$ and $F_0(x,y) < y$. Obviously, this is possible since the two inequalities are

$$y > \frac{1}{\beta}(x - h_1)(1+x) \quad \text{and} \quad x < \frac{1}{\beta}(y - h_0)(1+y),$$

whenever $x > h_0$ and $y > h_0$. In fact, the feasible region contains the region $\{(t,s) : \; t < x_1, s > x_0\}$. Our next result becomes straightforward and simplifies our last result.

Lemma 5.2. *Let $y > \frac{1}{\beta}(x-h_1)(1+x)$, $x < \frac{1}{\beta}(y-h_0)(1+y)$, $X := (x,y)$ and define $u := F_1(y,x)$ and $v := F_0(x,y)$. We obtain $x < u, v < y$ and*

$$\xi := ((x,y),(u,v)) \leq_{se} (X,X) \leq_{se} \xi^t = ((u,v),(x,y)).$$

Furthermore, if $y \geq F_0(u,v)$ and $x \leq F_1(v,u)$, then

$$\xi \leq_{se} G_1(G_0(\xi)) \leq_{se} G_1(G_0(X,X)) \leq_{se} G_1(G_0(\xi^t)) \leq_{se} \xi^t.$$

Finally, we reached the point where we can give the main result of this section.

Theorem 5.1. *Under the assumptions of Lemma 5.1, if $\beta \neq \beta_0$, then Eq. (5.4) has a globally attracting 2-cycle with respect to $[0,\infty)^2$.*

Proof. We give the proof for $\beta < \beta_0$, and the case $\beta > \beta_0$ can be handled in a similar way. By Lemma 5.1, there exists a unique 2-cycle, and no equilibrium solutions nor artificial 2-cycles. Let the 2-cycle be $[x_0, x_1]$, where $F_0(x_1, x_0) = x_0$ and $F_1(x_0, x_1) = x_1$. We need to utilize Lemma 5.2, and then apply Theorem 4.2. Therefore, we need to find two points $(a,b), (c,d)$ in which

$b > \frac{1}{\beta}(a-h_1)(1+a)$, $a < \frac{1}{\beta}(b-h_0)(1+b)$, $c = F_1(b,a)$, $d = F_0(a,b)$ and $b \geq F_0(c,d)$, $a \leq F_1(d,c)$. To obtain $b \geq F_0(c,d)$ and $a \leq F_1(d,c)$, we need to investigate the feasible region of the inequalities

$$\alpha(x,h_0)y^2 + \beta(x,h_0,h_1)y - \gamma(x,h_0,h_1) > 0 \tag{5.8}$$

and

$$\alpha(y,h_1)x^2 + \beta(y,h_1,h_0)x - \gamma(y,h_1,h_0) < 0 \tag{5.9}$$

where

$$\begin{aligned}
\alpha(t,r) =& (t+1)(r+1) - \beta^2 \\
\beta(t,r,s) =& \beta t^2 + (\beta + 1 - r^2 - \beta s)t - \beta^2 - \beta s - r^2 + 1 \\
\gamma(t,r,s) =& (t+1)(\beta r t + \beta s + r^2 + r).
\end{aligned}$$

Focus on the elliptic curves $y = F_0\left(F_1(y,x), F_0(x,y)\right)$ and $x = F_1\left(F_0(x,y), F_1(y,x)\right)$, which form the boundary of the feasible region. The first equation has a vertical asymptote at $x = \frac{b^2}{h_0+1} - 1$ and horizontal asymptote at $y = h_0$, while the second one has a vertical asymptote at $x = h_1$ and a horizontal asymptote at $y = \frac{b^2}{h_1+1} - 1$. Furthermore, the intersection between the two curves occur at the point (x_1, x_0), which generates the 2-cycle $[x_0, x_1]$. Also, at $\beta = \beta_0$, the two curves overlap. This description is enough to sketch the feasible region for $y > x$; however, since we don't need to identify all the feasible region, we fix $x = h_1$. Inequality (5.8) becomes

$$\beta(h_1+1)(\beta h_1 + h_0 y + h_0) > 0$$

and Inequality (5.9) becomes

$$P(y) = \alpha(h_0,h_1)y^2 + \left((h_1+1)(1-h_0^2) - \beta^2\right)y - (h_1+1)(h_0+1)(\beta h_1 + h_0) > 0.$$

Thus any y value larger than the positive root (say $y = b_0$) of $P(y) = 0$ makes (x,y) within the feasible region. By this choice, the inequalities $y > \frac{1}{\beta}(x-h_1)(1+x)$ and $x < \frac{1}{\beta}(y-h_0)(1+y)$ are already satisfied. Now, we consider $a = h_1, b > b_0, c = F_1(b,a)$ and $d = F_0(a,b)$. In this case, the region $\Omega = \left[h_1, \dfrac{\beta b}{1+h_1} + h_1\right] \times \left[\dfrac{\beta h_1}{1+b} + h_0, b\right]$ is attracted to the fixed point $\bar{\xi} = ((x_0,x_1),(x_0,x_1))$ of $\widehat{G}_0 = G_1 \circ G_0$. Observe that we are free to select b as large as we wish, but we know that Eq. (5.4) has a compact region that attracts the positive orthant (cf. [4]). This makes the 2-cycle globally attracting with respect to the positive orthant. $\square$

We close this section by an illustrative example. Consider $h_0 = 1, h_1 = 5$ and $\beta = 3$, then we have $\beta < \beta_0$. The 2-cycle is $[x_0, x_1] \approx [6.781, 4.620]$. Now, consider $a = 5$, $b > b_0 = \frac{1}{2}(3 + \sqrt{265}) \approx 9.639$, $c = \frac{1}{2}b + 5$ and $d = \frac{15}{1+b} + 1$. The region $\Omega := [a,c] \times [d,b]$ is attracted to the point (x_0, x_1) under the subsequences $\{(x_{2n}, x_{2n-1})\}$ that result from the folded map $\widehat{G}_0 = G_1 \circ G_0$. When $b = 20$, the regions Ω and $\widetilde{\Omega}$ are shown in Figure 3.

6 Conclusion

In this paper, we considered the p-periodic difference equation

$$x_{n+1} = F_n(x_n, x_{n-1}), \quad n \in \mathbb{N} = \{0\} \cup \mathbb{Z}^+, \quad x_{-1}, x_0 \geq 0, \tag{6.1}$$

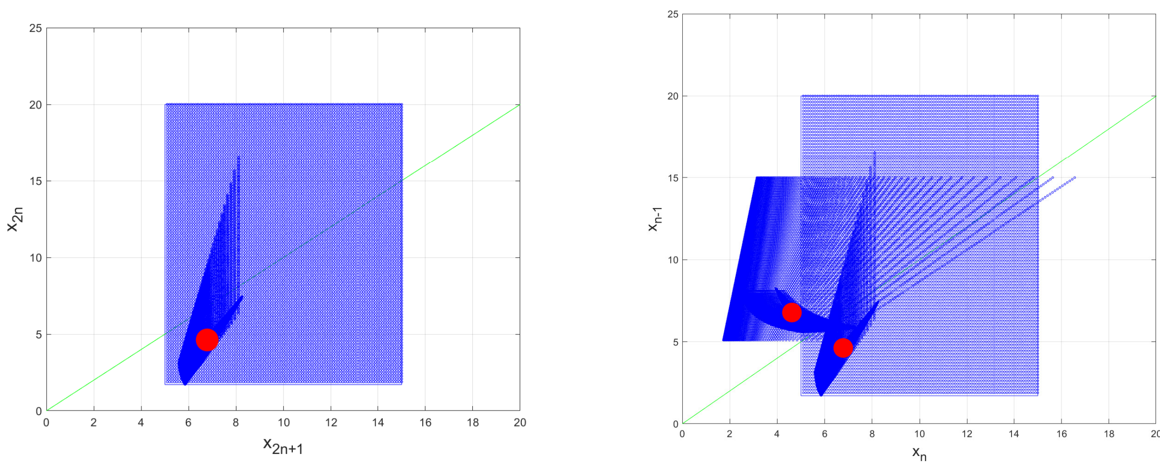

Figure 3: In both figures, $h_1 = 1, h_1 = 5, \beta = 3, a = 5; b = 20, c = F_1(20, 5) = 15, d = F_0(5, 20) = \frac{12}{7}$. The figure to the left shows the rectangular region assured by Theorem 4.2 through the folded map $\widehat{G} = G_1 \circ G_0$, while the figure to the right shows the region $\widetilde{\Omega}$ that is attracted to the 2-cycle.

where $F_j : \mathbb{R}_+^2 \to \mathbb{R}_+$ is continuous and $F_j(\uparrow, \downarrow)$ for each $j = 0, \dots, p-1$. The embedding strategy is based on the creation of a higher dimensional symmetric dynamical system that has the advantages of establishing a comparison principle that may aid in obtaining globally attracting cycles. We extended the embedding technique to cover Eq. (6.1). We found it convenient to introduce the concept of artificial cycles. Artificial cycles are obtained from the cycles of the embedded symmetric system, but are not cycles of Eq. (6.1). We illustrated the connection between normal cycles and artificial cycles and gave some characterization results. The nonexistence of artificial cycles have the advantage of making the notion of globally attracting cycles a manageable task. A global stability result was given in Theorem 4.2 in addition to some illustrative examples that clarify the developed theory. In particular, a global attracting 2-cycle has been obtained for each of the 2-periodic difference equations

$$x_{n+1} = F_n(x_n, x_{n-1}) = \frac{p_n + q x_n}{1 + x_n + x_{n-1}}, \qquad x_0, x_{-1} \geq 0, \ n \in \mathbb{Z}^+ \tag{6.2}$$

and

$$x_{n+1} = F_n(x_n, x_{n-1}) = x_n f(x_{n-1}) + h_n, \quad \text{where} \quad n \in \mathbb{N}, h_n > 0, x_0, x_{-1} \in \mathbb{R}_+. \tag{6.3}$$

The latter equation represents a population model with periodic stocking, and the characterization was done when $f(t) = \frac{\beta}{1+t}$. As expected in scientific research, any scientific approach can address certain problems, but at the same time, raises some new questions that need further investigation. For instance, it is interesting to investigate the bifurcation of artificial cycles in both the autonomous and nonautonomous cases. The cycles of the one dimensional map $z = F(x, x)$ seem to play a role in the existence of artificial cycles of $z = F(x, y)$. Is it true that if the autonomous map $z = F(x, x)$ has no k-cycle, then $z = F(x, y)$ has no artificial k-cycle? The answer was positive for the 1-cycle (a fixed point). A positive answer means we can go one step further to conclude that if $z = F(x, x)$ has no k-cycle, then $z = F(x, y)$ has no artificial q-cycles for all q to the left of k in the Sharkovsy's ordering of the positive integers. Finally, it is worth noting that in both of Eq. (6.2) and Eq. (6.3), our analysis was confined to the period $p = 2$, but more technical computations can be used to extend the analysis for larger values of p.

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