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Research Article

Lower Semicontinuity in L^1 of a Class of Functionals Defined on BV with Carathéodory Integrands

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We prove lower semicontinuity in $L^1(\Omega)$ for a class of functionals $\mathscr{G}: BV(\Omega) \longrightarrow \mathbb{R}$ of the form $\mathscr{G}(u) = \int_\Omega g(x,\nabla u)dx + \int_\Omega \psi(x)d|D^s u|$ where $g: \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ is open and bounded, $g(\cdot, p) \in L^1(\Omega)$ for each p, satisfies the linear growth condition $\lim_{|p| \longrightarrow \infty} g(x,p)/|p| = \psi(x) \in C(\Omega) \cap L^\infty(\Omega)$, and is convex in p depending only on $|p|$ for a.e. x. Here, we recall for $u \in BV(\Omega)$; the gradient measure $Du = \nabla u\, dx + d(D^s u)(x)$ is decomposed into mutually singular measures $\nabla u\, dx$ and $d(D^s u)(x)$. As an example, we use this to prove that $\int_\Omega \psi(x)\sqrt{\alpha^2(x) + |\nabla u|^2}\, dx + \int_\Omega \psi(x) d|D^s u|$ is lower semicontinuous in $L^1(\Omega)$ for any bounded continuous ψ and any $\alpha \in L^1(\Omega)$. Under minor addtional assumptions on g, we then have the existence of minimizers of functionals to variational problems of the form $\mathscr{G}(u) + \|u - u_0\|_{L^1}$ for the given $u_0 \in L^1(\Omega)$, due to the compactness of $BV(\Omega)$ in $L^1(\Omega)$.

1. Introduction

We prove an L^1 lower semicontinuity result for convex linear growth functionals $\int_\Omega g(x, Du)$, defined on BV, whose integrands $g: \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}$ are radially symmetric in p for a.e. $x \in \Omega$, that is, $g(x,p)$ depends on $(x, |p|)$, $g(\cdot, p) \in L^1(\Omega)$ for each p, and satisfies a fairly general structure condition. Our results expand the class of integrands from those of the form

$$\varphi(x,p) = \begin{cases} g(x,p), & \text{if } |p| \le \beta, \\ \psi(x)|p| + k(x), & \text{if } |p| > \beta, \end{cases} \tag{1}$$

as presented in [1], for which lower semicontinuity in L^1 holds.

We use the conjugate function g^* of g to prove our main result, Theorem 1 in Section 3. The conjugate function is used, for example, in [2] to approximate $\int_\Omega g(x, Du)$ for the given $u \in BV(\Omega)$ by a sequence $\int_\Omega g(x, \nabla u_n)$ for $u_n \in W^{1,1}(\Omega)$ to prove the existence of the corresponding gradient time flow, although g is assumed to be continuous in x in these cases. Thus, one advantage of Theorem 1 in this paper is that we can also obtain the existence results for time flow by deriving a similar convergence result for our case, but with no continuity assumption in the x variable. In fact for the results presented here, g may contain singularities in x, as in Example 3. In addition, the corresponding convergence results in [3] assume another continuity condition on g in x, similar to (3) which is not covered by our assumptions on g.

We note that the integrands considered in this paper (of the form (1)) have been used in models in applications of image processing [2]. However, as mentioned above, the main result of this paper covers a larger class of integrands.

We assume throughout, unless otherwise stated, that $\Omega \subset \mathbb{R}^N$ is bounded and open and g is radially symmetric in p for a.e. $x \in \Omega$ and g is convex in p for a.e. $x \in \Omega$ so that $g(x, \lambda_1 p_1 + \lambda_2 p_2) \le \lambda_1 g(x, p_1) + \lambda_2 g(x, p_2)$ for each p_1, $p_2 \in \mathbb{R}^N$, $0 \le \lambda_1, \lambda_2 \le 1$, $\lambda_1 + \lambda_2 = 1$, and $g \in L^1(\Omega)$ for each $p \in \mathbb{R}^N$. Since for a.e. x, g is convex and real valued in p, it is well known that g must be continuous in p for a.e. x; hence,

g is a Carathéodory function. Furthermore, we assume the linear growth of g so that

$$\lim_{|p|\longrightarrow\infty}\frac{g(x,p)}{|p|}=\psi(x),\quad \text{for a.e.} x\in\Omega, \tag{2}$$

where $\psi\in C(\Omega)\cap L^{\infty}(\Omega),\psi\geq 0$.

As stated above, we make no continuity assumption for the x variable for g. Additionally, in contrast to the works of [3–8] and [9] we do not assume g to be lower semicontinuous in (x,p). Also, our assumptions on g are not covered by the class of integrands $\mathbf{E}(\Omega;\mathbb{R}^N)$ and $\mathbf{R}(\Omega;\mathbb{R}^N)$ in [10, 11]. The integrand class $\mathbf{E}$ in [10] requires joint continuity of g in x and p, and $\mathbf{R}$ and the integrands in [11] require g for our case be defined on $\bar{\Omega}\times\mathbb{R}^N$ with

$$\lim_{t\longrightarrow\infty,x'\longrightarrow x}\frac{g\left(x',tp\right)}{t}=\psi(x),\quad \text{for each } x\in\bar{\Omega},p\in\mathbb{R}^N, \tag{3}$$

which may not hold if it is only assumed $g\in L^1(\Omega)$ for each p, as g may contain singularities.

2. Mathematical Preliminaries

We recall by definition that $u\in BV(\Omega)$ if and only if $u\in L^1(\Omega)$ and

$$\int_\Omega |Du|:=\sup_{\phi\in\left\{C_0^\infty\left(\Omega,\mathbb{R}^N\right),|\phi(x)|\leq 1 \text{ all } x\in\Omega\right\}}\left\{-\int_\Omega u \operatorname{div}\phi\, dx\right\}<\infty, \tag{4}$$

in which case the total variation measure Du is decomposed in to $Du=\nabla u\,dx+d(D^s u)(x)$ where $\nabla u\,dx<<\mathscr{L}^N$ and $D^s u\perp\mathscr{L}^N$ using the Lebesgue decomposition theorem [12]. Functionals defined for $u\in BV(\Omega)$ with Carathéodory integrands $g(x,p)$ of linear growth (2) and the convex in the p variable are defined [4, 5, 13, 14] by

$$\int_\Omega g(x,Du)=\int_\Omega g(x,\nabla u)dx+\int_\Omega \psi(x)d|D^s u|(x). \tag{5}$$

However, it is not immediate that functionals $\int_\Omega g(x,Du)$ defined by (5) are lower semicontinuous in $L^1(\Omega)$. As noted above, lower semicontinuity was was proven for certain integrands g, but to the best of our knowledge, there is no general L^1 lower semicontinuity result for convex Carathéodory functions g where for each p, $g(\cdot,p)\in L^1(\Omega)$.

We will also use the conjugate function g^* of g where $g^*(x,q):=\sup_{p\in\mathbb{R}^N}\{q\cdot p-g(x,p)\}$ [8]. We note that as g is convex in p and g^* is convex in q.

In [1], L^1 lower semicontinuity of $\int_\Omega\varphi(x,Du)$ for integrands of the form (1) is proved for $\varphi:\Omega\times\mathbb{R}^N\longrightarrow\mathbb{R}$ convex in $p,\varphi(x,p)$ radially symmetric in p for a.e. $x\in\Omega$, $\varphi(\cdot,p)\in L^1(\Omega)$ and a fairly general structure condition on φ which does not assume continuity in x. The proof is based on proving that

$$\begin{aligned}\int_\Omega\varphi(x,\nabla u)dx+\int_\Omega\psi(x)d|D^s u| &= \sup_{\left\{\phi\in C_0^1\left(\Omega,\mathbb{R}^N\right):\,|\phi(x)|\leq\psi(x) \text{ for all } x\in\Omega\right\}} \\ &\cdot\left\{\int_\Omega \nabla u\cdot\phi(x)-\varphi^*(x,\phi(x))\,dx\right\}=\sup_{\left\{\phi\in C_0^1\left(\Omega,\mathbb{R}^N\right):\,|\phi(x)|\leq\psi(x) \text{ for all } x\in\Omega\right\}} \\ &\cdot\left\{-\int_\Omega u div\phi(x)+\varphi^*(x,\phi(x))\,dx\right\},\end{aligned} \tag{6}$$

where φ^* is the conjugate function of φ and the last equality follows from integration by parts for $u\in BV(\Omega)$ [12]. Lower semicontinuity in $L^1(\Omega)$ immediately follows as the final equality is the supremum of functionals, each L^1 continuous in u. In the next section, we use the method above to prove our main result, Theorem 1.

3. Main Results

We first define

$$\mathscr{V}=\left\{\phi\in C_0^1\left(\Omega,\mathbb{R}^N\right):\,|\phi(x)|\leq\psi(x),\quad \text{for all } x\in\Omega\right\}. \tag{7}$$

Theorem 1. *Assume* $g:\Omega\times\mathbb{R}^N\longrightarrow\mathbb{R}$ *with*

$$=\psi(x)\in C(\Omega)\cap L^\infty(\Omega). \tag{8}$$

$g(x,p)$ are both radially symmetric and convex in p for a.e. x, and if for each $M>0$,

$$\varphi_M(x,p):=\begin{cases} g(x,p), & \text{if } |p|\leq M,\\ \psi(x)|p|+g(x,M)-\psi(x)M, & \text{if } |p|>M,\end{cases} \tag{9}$$

is convex in p and there exists $f_M\in L^1(\Omega)$ such that $|g(x,p)-\varphi_M(x,p)|\leq f_M(x)$ a.e. x, for all $|p|\geq M$, where $\int_\Omega f_M(x)dx\longrightarrow 0$ as $M\longrightarrow\infty$. Additionally, assume the following structure condition on g : that is, for some G, we have $g(x,p)=G(r_1(x),\cdots,r_k(x),p)$ for all p where $G(z_1,\cdots,z_K,p)=g(z_1,\cdots,z_k,p)$ and where g is C^1 in the variable $z=(z_1\cdots,z_K)\in U\subset\mathbb{R}^K$, U open,$r_i\in L^1(\Omega)$ each i, $(r_1(x),\cdots,r_K(x))\in U$ a.e. x, and $|(\nabla_z g)(z,p)|\leq C$,$C$ independent of (z,p), $|p|\leq M$ for each M. Then, for $\int_\Omega g(x,Du)$ defined by

$$\int_\Omega g(x,Du):=\sup_{\mathscr{V}}\left\{\int_\Omega\nabla u\cdot\phi(x)-g^*(x,\phi(x))dx\right\}, \tag{10}$$

we have in fact

$$\int_\Omega g(x,Du)=\int_\Omega g(x,\nabla u)dx+\int_\Omega\psi(x)d|D^s u|. \tag{11}$$

Thus, the functional

$$\mathcal{G}(u) = \int_\Omega g(x,\nabla u)dx + \int_\Omega \psi(x)d|D^s u|, \tag{12}$$

defined on $BV(\Omega)$, is lower semicontinuous in $L^1(\Omega)$, that is, if $u_n \longrightarrow u$ in $L^1(\Omega)$, then, $\mathcal{G}(u) \le \liminf_{n\longrightarrow\infty}\mathcal{G}(u_n)$. Moreover, if $\partial\Omega$ is Lipschitz, then, for the given $h \in L^1(\partial\Omega)$,

$$\begin{aligned}&\int_\Omega g(x,\nabla u) + \int_\Omega \psi(x)d|D^s u|(x) + \int_{\partial\Omega}\psi(x)|u-h|d\mathcal{H}^{N-1}\\ &\quad= \sup_{\left\{\phi\in C^1\left(\Omega,\mathbb{R}^N\right):\,|\phi|\le\psi(x)\right\}}\left\{-\int_\Omega u\,div\phi + g^*(x,\phi(x))\,dx + \int_{\partial\Omega}\phi\cdot\hat{n}hd\mathcal{H}^{N-1}\right\},\end{aligned} \tag{13}$$

and hence, the functional

$$\mathcal{G}_h(u) = \int_\Omega g(x,\nabla u) + \int_\Omega \psi(x)d|D^s u|(x) + \int_{\partial\Omega}\psi(x)|u-h|d\mathcal{H}^{N-1}, \tag{14}$$

defined on $BV(\Omega)$, is lower semicontinuous in $L^1(\Omega)$. Here, u is defined on $\partial\Omega$ in the sense of trace [12].

Proof. From the above assumptions on g, we have

$$\begin{aligned}g^*(x,q) &= \sup_{p\in\mathbb{R}^N}\{p\cdot q - g(x,p)\} \le \sup_{p\in\mathbb{R}^N}\{p\cdot q - \varphi_M(x,p)\}\\ &\quad+ \sup_{p\in\mathbb{R}^N}\{|\varphi_M(x,p) - g(x,p)|\}\\ &= \sup_{p\in\mathbb{R}^N}\{p\cdot q - \varphi_M(x,p)\} + \sup_{|p|\ge M}\{|\varphi_M(x,p) - g(x,p)|\}\\ &\le \varphi_M^*(x,q) + f_M(x).\end{aligned} \tag{15}$$

Similarly, we have $\varphi_M^*(x,q) \le g^*(x,q) + f_M(x)$ giving $|g^*(x,q) - \varphi_M^*(x,q)| \le f_M(x)$ for all $|q| \le \psi(x)$.

From the above estimate for $|g^* - \varphi^*|$, we have $g^*(x,q) = \infty$ if and only if $\varphi_M^*(x,q) = \infty$ if and only if $|q \le \psi(x)$, and hence,

$$g(x,p) = \sup_{|q|\le\psi(x)}\{p\cdot q - g^*(x,q)\}. \tag{16}$$

□ □

Now,

$$\begin{aligned}\int_\Omega g(x,Du) &= \sup_{\mathcal{V}}\left\{\int_\Omega \nabla u\cdot\phi(x) - g^*(x,\phi(x))\,dx\right\}\\ &= \sup_{\mathcal{V}}\left\{\int_\Omega \nabla u\cdot\phi(x) - \varphi_M^*(x,\phi(x))\,dx\right\}\\ &\quad+ \sup_{\mathcal{V}}\left\{\int_\Omega \nabla u\cdot\phi(x) - \varphi_M^*(x,\phi(x)) + \varepsilon_1(x,\phi(x))dx\right\}\\ &\quad- \sup_{\mathcal{V}}\left\{\int_\Omega \nabla u\cdot\phi(x) - \varphi_M^*(x,\phi(x))dx\right\}\\ &= \int_\Omega \varphi_M(x,\nabla u)dx + \int_\Omega \psi(x)d|D^s u| + \varepsilon_2,\end{aligned} \tag{17}$$

with $\varepsilon_1(x,\phi(x)) \coloneqq \varphi_M^*(x,\phi(x)) - g^*(x,\phi(x))$ and $|\varepsilon_1(x,\phi(x))| \le f_M(x)$ from the above and ε_2 is defined by

$$\begin{aligned}\varepsilon_2 \coloneqq &\sup_{\mathcal{V}}\left\{\int_\Omega \nabla u\cdot\phi(x) - \varphi_M^*(x,\phi(x)) + \varepsilon_1(x,\phi(x))dx\right\}\\ &- \sup_{\mathcal{V}}\left\{\int_\Omega \nabla u\cdot\phi(x) - \varphi_M^*(x,\phi(x))dx\right\}.\end{aligned} \tag{18}$$

We now show $\varepsilon_2 \longrightarrow 0$ as $M \longrightarrow \infty$. In fact,

$$\begin{aligned}&\int_\Omega \nabla u\cdot\phi(x) - \varphi_M^*(x,\phi(x)) + \varepsilon_1(x,\phi(x))dx\\ &\quad\le \int_\Omega \nabla u\cdot\phi(x) - \varphi_M^*(x,\phi(x))dx + \int_\Omega f_M(x)dx,\end{aligned} \tag{19}$$

from $|\varepsilon_1(x,\phi(x))| \le f_M(x)$. Taking supremum over both sides gives

$$\begin{aligned}\varepsilon_2 = &\sup_{\mathcal{V}}\int_\Omega \nabla u\cdot\phi(x) - \varphi_M^*(x,\phi(x)) + \varepsilon_1(x,\phi(x))dx\\ &- \sup_{\mathcal{V}}\int_\Omega \nabla u\cdot\phi(x) - \varphi_M^*(x,\phi(x))dx \le \int_\Omega f_M(x)dx.\end{aligned} \tag{20}$$

Similarly, we have $-\varepsilon_2 \le \int_M f_M(x)dx$ giving

$$|\varepsilon_2| \le \int_\Omega f_M(x)dx \longrightarrow 0 \text{ as } M \longrightarrow \infty. \tag{21}$$

Now, let $M \longrightarrow \infty$ in (17) to get

$$\int_\Omega g(x,Du) = \int_\Omega g(x,\nabla u)dx + \int_\Omega \psi(x)d|D^s u|. \tag{22}$$

For the second claim in the theorem, as $\partial\Omega$ is Lipschitz, the continuous trace operator [12])$T: BV(\Omega) \longrightarrow L^1(\partial\Omega,\mathcal{H}^{N-1})$ exists and (13) follows as in the proof above. Finally, lower semicontinuity of $\mathcal{G}_h$ follows from Theorem 5 in [1].

Remark 2. We note that the condition $|(\nabla_z g)(z,p)| \le C$ may be modified if in the expression $g(z_1,\cdots,z_K,p)$, one of the $z_i's$ corresponds to $\psi(x)$. That is, if, e.g., $z_K = \psi(x)$, we may only require that $|(\nabla_{(z_1,\cdots,z_{K-1})}g)(z,p)| \le C$ as each $x\in\Omega$ is a Lebesgue point of ψ by continuity. In fact, we have, noting that $g^*(x,q) = g^*(r_1(x),\cdots,r_{K-1}(x),\psi(x),q)$,

$$\begin{aligned}
&\frac{1}{|B_\rho|}\int_{B_\rho(x)} |g^*(r_1(x),\cdots,r_{K-1}(x),\psi(x),p)\\
&\quad - g^*(r_1(y),\cdots,r_{K-1}(y),\psi(y),p)|dy\\
&\quad \le \frac{1}{|B_\rho|}\int_{B_\rho(x)} |g^*(r_1(x),\cdots,r_{K-1}(x),\psi(x),p)\\
&\quad - g^*(r_1(x),\cdots,r_{K-1}(x),\psi(y),p)|dy\\
&\quad + \frac{1}{|B_\rho|}\int_{B_\rho(x)} |g^*(r_1(x),\cdots,r_{K-1}(x),\psi(y),p)\\
&\quad - g^*(r_1(y),\cdots,r_{K-1}(y),\psi(y),p)|dy\\
&\quad \le \frac{1}{|B_\rho|}\int_{B_\rho(x)} |g^*(r_1(x),\cdots,r_{K-1}(x),\psi(x),p)\\
&\quad - g^*(r_1(x),\cdots,r_{K-1}(x),\psi(y),p)|dy\\
&\quad + \frac{1}{|B_\rho|}\int_{B_\rho(x)} \sup_{(z,p)}\left|\nabla_{(z_1,\cdots,z_{K-1})} g^*(z,p)\right| \cdot |(r_1(x),\cdots,r_{K-1}(x))\\
&\quad - (r_1(y),\cdots,r_{K-1}(y))|dy.
\end{aligned} \tag{23}$$

The last term is bounded by $((1/|B_\rho|)1/|B_\rho|)\int_{B_\rho(x)} C|(r_1(x),\cdots,r_{K-1}(x)) - (r_1(y),\cdots,r_{K-1}(y))|dx$ which approaches 0 as $\rho \longrightarrow 0$ on the common Lebesgue set of $r_1,\cdots,r_{k-1}$. The next to last term approaches 0 a.e. x as $\rho \longrightarrow 0$ since $g^*(r_1(x),\cdots,r_{K-1}(x),\psi(y),p)$ is continuous in y wherever $r_1(x),\cdots,r_{K-1}(x)$ are defined. The Lebesgue set of g^* thus contains the Lebesgue set of $r_1,\cdots,r_k$ independent of p. The rest follows exactly as in the proof of Theorem 4 in [15], but with $\nabla_{(z_1,\cdots,z_{K-1})}G_\varepsilon(z,p)$ and $\nabla_{(z_1,\cdots,z_{K-1})}G^*_\varepsilon(z,p)$ replacing $\nabla_z G_\varepsilon(z,p)$ and $\nabla_z G^*_\varepsilon(z,p)$ and Remark 2 in [1].

Example 3. For $\alpha \in L^1(\Omega)$, $u \in BV(\Omega)$, $\psi \in C(\Omega)\cap L^\infty(\Omega)$, the functional,

$$\mathcal{F}(u) = \int_\Omega \psi(x)\sqrt{\alpha^2(x) + |\nabla u|^2}\,dx + \int_\Omega \psi(x)d|D^s u|, \tag{24}$$

is lower semicontinuous on $L^1(\Omega)$.

Proof. Letting $g_\delta(x,p) = \psi(x)\sqrt{\alpha^2(x)+\delta+|p|^2}$, $\alpha \in L^1(\Omega)$, $\delta > 0$ we have

$$\varphi_M(x,p) = \begin{cases} \psi(x)\sqrt{\alpha^2(x)+\delta+|p|^2}, & \text{if } |p| \le M,\\ \psi(x)|p| + k_M(x), & \text{if } |p| > M, \end{cases} \tag{25}$$

where

$$k_M(x) = \psi(x)\frac{\alpha(x)+\delta}{\sqrt{\alpha^2(x)+\delta+M^2}+M}. \tag{26}$$

Letting $\alpha_\delta = \alpha + \delta$, we have for $|p| \ge M$

$$\begin{aligned}
|g_\delta(x,p) - \varphi_M(x,p)| &= \psi(x)\left|\frac{\alpha(x)_\delta}{\sqrt{\alpha^2(x)_\delta + |p|^2}+|p|} - \frac{\alpha(x)_\delta}{\sqrt{\alpha^2(x)_\delta + M^2}+M}\right|\\
&\le 2\psi(x)\frac{\alpha(x)_\delta}{\sqrt{\alpha^2(x)_\delta + M^2}+M} := f_M(x) \longrightarrow 0 \text{ as } M \longrightarrow \infty,
\end{aligned} \tag{27}$$

and hence, $\int_\Omega f_M\,dx \longrightarrow 0$ as $M \longrightarrow \infty$ by Lebesgue's dominated convergence theorem, as $f_M(x) \le 2\psi(x)$ a.e. Note that φ_M is convex in t since by defining $\tilde{\varphi}_M : \Omega \times 0,\infty) \longrightarrow \mathbb{R}$

$$\tilde{\varphi}_M(x,t) = \begin{cases} \psi(x)\sqrt{\alpha^2(x)+\delta+t^2}, & \text{if } 0 \le t \le M,\\ \psi(x)t + k_M(x), & \text{if } t > M. \end{cases} \tag{28}$$

We see the left derivative $(\partial/\partial t)\tilde{\varphi}_M(x,t)$ at $t = M$ is

$$\tilde{\varphi}'_M(x, M-) = \psi(x)\frac{t}{\sqrt{\alpha^2_\delta(x)+t^2}}, \tag{29}$$

while the right derivative at $t = M$ is $\tilde{\varphi}'_M(x,M+) = \psi(x) \ge \psi(x)(t/\sqrt{\alpha^2_\delta(x)+t^2}) = \tilde{\varphi}'_M(x,M-)$. Thus, $\tilde{\varphi}_M$ is convex in t. As $\tilde{\varphi}_M$ is also increasing in t, we have that $\varphi_M(x,p) = \tilde{\varphi}_M(x,|p|)$ is convex in p.□ □

The conditions for Theorem 1 are thus satisfied for g_δ and φ_M with

$$\begin{gathered}
G(z_1,z_2,p) = g_\delta(z_1,z_2,p) = z_2\sqrt{z_1^2+\delta+|p|^2},\\
U = \mathbb{R}^2,
\end{gathered}$$

$$\begin{aligned}
\left|\frac{\partial g_\delta}{\partial z_1}(z_1,z_2,p)\right| &= \left|z_2\frac{z_1}{\sqrt{z_1^2+\delta+|p|^2}}\right| \le |z_2|\\
&\le \|\psi\|, \quad \text{for all } (z_1,z_2,p) \in \mathbb{R}^2\times\mathbb{R}^N,
\end{aligned} \tag{30}$$

noting Remark 2. Hence,

$$\int_\Omega g_\delta(x,Du) = \int_\Omega g_\delta(x,\nabla u)dx + \int_\Omega \psi(x)d|D^s u|. \tag{31}$$

For the case $g(x,p) = \psi(x)\sqrt{\alpha^2(x)+|p|^2}$, $\alpha \in L^1(\Omega)$, we note that

$$|g_\delta(x,p) - g(x,p)| \le \psi(x)\frac{\delta}{\sqrt{\alpha^2(x)+\delta+|p|^2}+\sqrt{\alpha^2(x)+|p|^2}} \le \sqrt{\delta}, \tag{32}$$

for a.e. $x \in \Omega$ and for each $p \in \mathbb{R}^N$. As in the proof of Lemma 2 in [16] and Theorem 1 above, we have for a.e x, all $q \in \mathbb{R}^N$,

$$|g^*(x,q) - g^*_\delta(x,q)| \le |g_\delta(x,p) - g(x,p)| \le \sqrt{\delta}, \tag{33}$$

and similar to the above estimates (17), for each $\delta > 0$, we have

$$\left| \int_\Omega g(x, Du) - \left(\int_\Omega g_\delta(x,\nabla u)dx + \int_\Omega \psi(x)d|D^s u| \right) \right| \le \int_\Omega \sqrt{\delta} dx. \tag{34}$$

Letting $\delta \longrightarrow 0$ gives

$$\begin{aligned} \int_\Omega g(x, Du) &:= \sup_{\mathscr{V}} \left\{ \int_\Omega \nabla u \cdot \phi(x) - g^*(x, \phi(x))dx \right\} \\ &= \int_\Omega g(x,\nabla u)dx + \int_\Omega \psi(x)d|D^s u|. \end{aligned} \tag{35}$$

Lower semicontinuity of $\mathscr{F}$ immediately follows.

We finally note that a version of Theorem 1, along with Remark 2, holds for nonradially symmetric integrands g, but with the additional smoothness assumption that for a.e. $x \in \Omega, g(x, \cdot) \in C^2(\mathbb{R}^N)$.

Theorem 4. *Assume that* $g : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}$ *with*

$$\frac{\lim_{|p| \longrightarrow \infty} g(x, p)}{|p|} = \psi(x) \in C(\Omega) \cap L^\infty(\Omega), \tag{36}$$

where $g(x, p)$ is convex and C^2 in p for a.e. x, and if for each $M > 0$,

$$\varphi_M(x, p) := \begin{cases} g(x, p), & \text{if } |p| \le M, \\ \psi(x)|p| + g(x, M) - \psi(x)M, & \text{if } |p| > M, \end{cases} \tag{37}$$

is both convex and C^1 in p and there exists $f_M \in L^1(\Omega)$ such that $|g(x, p) - \varphi_M(x, p)| \le f_M(x)$ a.e. x, for all $|p| \ge M$, where $\int_\Omega f_M(x)\,dx \longrightarrow 0$ as $M \longrightarrow \infty$. Additionally assume the following structure condition on g : that is, for some G we have $g(x, p) = G(r_1(x), \cdots, r_k(x), p)$ for all p where $G(z_1, \cdots, z_K, p) = g(z_1, \cdots, z_k, p)$ and where g is C^1 in the variable $z = (z_1 \cdots, z_K) \in U \subset \mathbb{R}^K$, U open,$r_i \in L^1(\Omega)$ each i,$(r_1(x), \cdots, r_K(x)) \in U$ a.e. x, and $|(\nabla_z g)(z, p)| \le C$,$C$ independent of (z, p), $|p| \le M$ for each M. Then

$$\begin{aligned} \int_\Omega g(x, Du) &:= \sup_{\mathscr{V}} \left\{ \int_\Omega \nabla u \cdot \phi(x) - g^*(x, \phi(x))dx \right\} \\ &= \int_\Omega g(x,\nabla u)dx + \int_\Omega \psi(x)d|D^s u|. \end{aligned} \tag{38}$$

Thus, the functional

$$\mathscr{G}(u) = \int_\Omega g(x,\nabla u)dx + \int_\Omega \psi(x)d|D^s u|, \tag{39}$$

defined on $BV(\Omega)$, is lower semicontinuous in $L^1(\Omega)$. Moreover, if $\partial\Omega$ is Lipschitz, then, for the given $h \in L^1(\partial\Omega)$,

$$\begin{aligned} &\int_\Omega g(x,\nabla u) + \int_\Omega \psi(x)d|D^s u|(x) + \int_{\partial\Omega} \psi(x)|u - h|d\mathscr{H}^{N-1} \\ &= \sup_{\left\{\phi \in C^1\left(\bar{\Omega}, \mathbb{R}^N\right) : \, |\phi| \le \psi(x)\right\}} \left\{ -\int_\Omega u div\phi + g^*(x, \phi(x))\, dx \right. \\ &\quad \left. + \int_{\partial\Omega} \phi \cdot \hat{n} h d\mathscr{H}^{N-1} \right\}, \end{aligned} \tag{40}$$

and hence, the functional

$$\mathscr{G}_h(u) = \int_\Omega g(x,\nabla u) + \int_\Omega \psi(x)d|D^s u|(x) + \int_{\partial\Omega} \psi(x)|u - h|d\mathscr{H}^{N-1}, \tag{41}$$

defined on $BV(\Omega)$, is lower semicontinuous in $L^1(\Omega)$.

Proof. The proof is the same as the proof of Theorem 1, noting Theorem 4 in [15] and Remark 2 in [1].□ □

We immediately have from standard theory the following existence result:

$$\inf_\Omega \psi(x) = c_1 > 0, \tag{42}$$

$$g(x, p) \ge c_2|p|, \quad \text{for some } c_2 > 0. \tag{43}$$

Corollary 5. *Let g satisfy the assumptions of Theorems 1 and 4 or Remark 2. If in addition we have*

Then, for the given $u_0 \in L^1(\Omega)$, the functionals

$$\begin{aligned} \Phi(u) &= \int_\Omega g(x, Du) + \|u - u_0\|_{L^1(\Omega)}, \\ \Phi_h(u) &= \int_\Omega g(x, Du) + \int_{\partial\Omega} \psi(x)|u - h|d\mathscr{H}^{N-1} + \|u - u_0\|_{L^1(\Omega)}, \end{aligned} \tag{44}$$

have a minimizer in $BV(\Omega)$. Furthermore, the minimizer is unique if g is strictly convex in p.

Proof. For Φ, this follows from lower semicontinuity of Φ in L^1 and standard compactness results for BV, noting that assumptions (42) and (43) imply using (5) that

$$\int_\Omega |Du| \le \min\,(c_1, c_2) \int_\Omega g(x, Du). \tag{45}$$

Thus, minimizing sequences $\{u_n\}$ of Φ are bounded in the BV norm $\|u\|_{BV(\Omega)} := \int_\Omega |Du| + \|u\|_{L^1(\Omega)}$ so that there is $u \in L^1(\Omega)$ with $u_n \longrightarrow u$ in $L^1(\Omega)$ [12], and hence,

$$\Phi(u) \le \lim_{n \longrightarrow \infty} \inf \Phi(u_n) = \min_{v \in L^1(\Omega)} \Phi(v). \tag{46}$$

The proof is essentially the same for Φ_h using (13). Finally, if g is strictly convex, then, so is Φ and Φ_h. Thus, if there are minimizers $u_1 \neq u_2$, then, we have

$$\Phi\left(\frac{(u_1+u_2)}{2}\right) < \min_{v\in L^1(\Omega)} \Phi(v),$$
$$\Phi_h\left(\frac{(u_1+u_2)}{2}\right) < \min_{v\in L^1(\Omega)} \Phi_h(v), \tag{47}$$

a contradiction.□ □

We finally remark, as noted in [1], that Theorems 1 and 4 of this paper may be extended to vector-valued functions $\mathbf{u}(x) = (u_1(x), \cdots, u_M(x))$ where $D\mathbf{u}$ is an $M \times N$ matrix with $Du_i \in BV(\Omega)$ for each i and $\int_\Omega g(x, D\mathbf{u})$ is defined by writing $D\mathbf{u}$ as a vector of length NM with $g : \Omega \times \mathbb{R}^{NM} \longrightarrow \mathbb{R}$ and g depending on $(x, |D\mathbf{u}|)$ for the case of Theorem 1. We may also consider integrands $\int_\Omega g(x, \mathbf{u}, D\mathbf{u})$ with appropriate assumptions on $g(x, z, p)$, such as Lipschitz continuity in z, using similar methods as presented here and in [1, 15, 16].

4. Conclusion

In this paper, we have expanded the class of functionals $\int_\Omega g(x, Du)$ defined on the BV space which are $L^1(\Omega)$ lower semicontinuous to include certain integrands $g(x, p)$ which, for each $p \in \mathbb{R}^N$, are only assumed to be in $L^1(\Omega)$. The structure condition for which lower semicontinuity holds is fairly general and is for many cases not difficult to verify. Furthermore, as mentioned above, using the method presented here, we may expand the main theorem of this paper to include functionals of the form $\int_\Omega g(x, \mathbf{u}, D\mathbf{u})$ for vector-valued functions $\mathbf{u} : \Omega \longrightarrow \mathbb{R}^M$. Finally, as noted above, lower semicontinuity is essential to proving the existence of minimizers of functionals of the form $\Phi(u) = \int_\Omega g(x, Du) + \|u - u_0\|_{L^1(\Omega)}$ over $BV(\Omega)$.

Data Availability

As this is a theoretical paper, there is no data but all references used throughout the manuscript are found in the bibliography at the end of the paper (see below) and can be easily accessed, although I do not believe that they are publicly archived. The references list all the necessary information for accessing the papers. Also, I have the full papers recently published by the author (Thomas Wunderli) as listed in the bibliography and will be provided by the author upon request.

Conflicts of Interest

The author(s) declare(s) that they have no conflicts of interest.

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