LAPLACE ADOMIAN SOLUTIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS THAT ARISES IN NATURAL SCIENCE AND ENGINEERING APPLICATIONS

by

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Declaration of Authorship

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Abstract

In this thesis, we investigate the numerical solution of fractional differential equations subject to initial and boundary conditions. For the solution of such equations, we use two iterative approaches. First, we apply the Laplace Decomposition Method, which is a combination of two approaches, namely the Laplace Transform and the Adomian Decomposition Methods. Then, we implement the Differential Transformation Method. Finally, we apply the above mentioned methods to real life problems such as solving complex non-linear Enzyme Inhibitor Reactions Model and a COVID-19 Model.
Keywords: Fractional derivatives, Fractional integration, Caputo fractional derivative, Liouville fractional derivative, Laplace transform, Adomian Decomposition.

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Chapter 1: Introduction

In various areas of science and engineering, differential equations are used to model dynamical systems. Throughout the last two decades, many scholars have focused their attention on fractional differential equations. Most of the numerical methods used to solve ordinary differential equations have been modified to solve fractional differential equations to provide approximate solutions. Fractional Calculus is a branch of mathematics that evolved from the standard definitions of calculus derivative and integral operators just as fractional exponents emerged from integer-value exponents. It investigates the properties of derivatives and integrals of fractional order known as differintegrals. It has a history that spans more than three centuries. Despite its difficult mathematical basis, Fractional Calculus originated from some basic derivation-related questions, such as: what does the half order derivative represent if the first-order derivative represents geometrically the slope of a tangent line? [1] The origins of this question may be traced back to Leibniz's initial suggestion of fractional derivatives in 1695 [2]. Those types of questions have widened the broads for understanding the connection between mathematical modeling and real life problems. Fractional Calculus is used in a broad range of engineering and science domains, including optics, viscoelasticity, fluid mechanics, electromagnetics, signal processing, and biological population models [3]. It is also used as a tool for modeling a variety of complex systems.

The aim of the thesis is three folded. Firstly, we will talk about mathematical modeling. Everything in mathematical biology, mathematical chemistry, physics, and engineering starts with a model. In specific, Fractional Calculus has been used as a tool for modeling a variety of complex systems. Secondly, we will discuss different numerical solutions of fractional derivatives. We will analyze two specific methods. The Laplace Adomian Decomposition method and the Fractional Differential Transformation Method. Lastly, we will highlight some applications of Fractional Derivatives. We will implement the methods discussed onto an Enzyme Inhibitor Reaction Model and a COVID-19 model.

1.1 Mathematical Modeling

Mathematical modeling is the discipline in Mathematics that allows translating and interpreting problems from an application perspective into flexible mathematical formulations which can be analyzed to offer better insight and useful guidance for the originating application. Models express our perceptions about how the world works. By applying mathematical modeling, we transfer these views into the mathematical language to help in formulating concepts and understanding underlying assumptions. Mathematical modeling may be utilized for a variety of purposes. The degree to which a certain goal is met is determined by both the state of information about a system and the quality of the modeling. Scientists use mathematical modeling to enhance their understanding and test the effect of system adjustments to help in their decision making [4]. Mathematical modeling is essential in various applications. It provides precision and allows a thorough understanding of the model. Furthermore, it organizes the model so we can have better control of the system [5].

Mathematical models are classified into various types. For example, a continuous model means the dependent variable is defined over a continuous space time, a stochastic model means some elements are probabilistic, a deterministic model means it is based on cause-effect analysis, while lumped model means the dependent variables are not a function of spatial position. The creation of models such as ours began in the late 19th century. The first book explaining mathematical models was by Brill and Kline from the Royal Technical University in Munich in 1904. In the last century, mathematical modeling has been intensively utilized in natural sciences, engineering, medicine, social sciences, business, military operations, music, and philosophy [6]-[19]. In this thesis, we discuss fractional differential equations which are the new trend in modeling physical phenomena because of the non-locality of the fractional derivatives. This allows us to measure the hereditary properties of the system. Fractional derivatives have been intensively used for modeling the physical properties of various systems. The main advantage of the applications of fractional calculus is that the mathematical models based on fractional-order are usually more accurate than the ones based on integer orders. Furthermore, there are several cases in which integer-order mathematical models, including nonlinear models, fail to perform effectively [20, 21, 22]. Numerous effective analytical and numerical approaches have been established and presented, but further inquiry and research are still required [23]. Understanding the mathematical modeling of different systems is a crucial step in transitioning from a theoretical perspective to mathematical applications.

1.2 Numerical Solutions of Fractional Differential Equations

When analytical approaches fail, numerical methods can be used to get approximate answers. Although they will never be as generic as analytical solutions, they can be just as effective in certain situations. The processes represented in the model are typically replicated via numerical solution of model equations [4]. The numerical solution of difference equations is precise because we can follow the evolution of the system using the principles provided out in the equations. Since mathematical models are usually nonlinear, exact solutions are difficult to acquire. In addition, those kinds of systems do not have exact solutions because they do not have bounded variations. Although numerical methods are widely utilized, they have some disadvantages. In fact, numerical stability is not guaranteed and matching the data to the numerical solution is costly in terms of time. As a result, numerical approaches are used to reach approximate solutions. For instance, Abukhaled solved a nonlinear singular two-point boundary value problem by employing the Variational Iterative Method [24]. Later, his work was modified by Khuri and Wazwaz to solve boundary value problems that arise in electric conducting solids, elliptic BVPs, and Volterra integro-differential equations of Lane-Emden and the Emden-Fowler problems [25, 26, 27, 28]. Kafri and Khuri solved the nonlinear one-dimensional Bratu's problem by coupling Green's Function and fixed-point iterative methods (GFIM) [29]. Abukhaled et al. employed the GFIM to construct a semi-analytical solution of amperometric enzymatic reactions, the one-dimensional curvature equation, strong nonlinear oscillators, and a class of boundary value problems that occur in heat transfer [30, 31, 32, 33]. Last but not least, Khuri et al. successfully solved Toresh's problem, Bratu-like equations originating in electrospinning process equations, and boundary value problems using GFIM [35, 36, 34]. Analytically, solving fractional differential equations can be done in a variety of ways. The efficiency of numerical approaches is traditionally assessed using concepts like convergence, consistency, and stability. The method's consistency order is frequently used as a benchmark for comparing approaches. The order of the method informs the user of the rate at which the error over a fixed interval would decrease when the step length is reduced [37]. In this thesis, we are implementing the Laplace Adomian Decomposition Method and the Differential Transformation Method on different mathematical models.

1.3 Application of Fractional Differential Equations

Fractional Calculus has applications in practically every discipline of scientific and social sciences, economics, finance, health sciences, and engineering [38, 39, 40]. It is been used to simulate physical and technical processes that are best characterized by fractional differential equations. These types of models are utilized for systems that need precise damping modeling. Various analytical and numerical approaches, as well as their applicability to novel problems, have been suggested in recent years. It is believed that the first application of Fractional Calculus was presented by Abel in 1823 with his tautochrone problem. The problem entails establishing a curve shape such that the time taken for an object to slide down the curve under uniform gravity and without friction is independent of the starting position on the curve [41, 42]. The use of the memory effect of fractional derivatives in building simple mathematical models occurs at a significant cost in terms of numerical solvability. Any method of a non-integer derivative must, among other things, care for its non-local structure, which entails a large amount of storage and high total algorithm complexity [43]. Therefore, Fractional Calculus has shown to be a useful tool for describing novel and recent applications in control theory, viscoelasticity, and generalized voltage dividers in the past few decades [44]. To tackle Fractional Initial Value Problems, Abdulaziz et al. [45] adopted the Homotopy Perturbation Method. It is also been modified to solve a variety of fractional differential equations, both linear and nonlinear [46]. Sakar et al. [47] proposed a Legendre reproducing kernel method to solve fractional Bratu-type equations and obtained an extremely accurate approximation. Quasi-Method Newton's [48], and Bezier Curve Method [49] are some of the other approaches for solving fractional differential equations. The harmonic oscillator is also a basic model in classical mechanics that may be used to a wide range of physical, chemical, and engineering applications. Generalizing to fractional derivatives, that entails replacing the second derivative in a classical oscillator equation with a fractional-order derivative, is still under investigation, and the properties of the fractional oscillator for various types of differintegrals are currently being studied by different scholars [1]. There are a variety of other approximate analytical solutions like the Adomian decomposition for solving nonlinear fractional differential equations [50], or by using Differential Transformation Method [51] which we will discuss in detail later on. In this thesis, we will implement the later methods into solving an Enzyme Inhibitor Reaction Model and a COVID-19 model.

Chapter 2: Literature Review

2.1 Gamma Function

In the early 16th century, the Gamma function was first introduced by the Swiss mathematician Leonhard Euler. It is essentially a generalization of the factorial to all non integer values. Due to its significant, it is used in various areas like number theory, definite integration, asymptotic series, and the Riemann zeta function.
The Gamma function is defined as follows

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x \in \mathbb{R}^+. \tag{2.1}$$

The improper integral (2.1) converges for $x > 0$.

Proof. we start by dividing the integral as a sum of two terms

$$\Gamma(x) = \int_0^1 t^{x-1} e^{-t} dt + \int_1^\infty t^{x-1} e^{-t} dt$$

For the first term, since the function e^{-t} is decreasing, it attains its maximum on the interval $[0, 1]$ at $t = 0$, so

$$\begin{aligned}\Gamma(x) = \int_0^1 t^{x-1} e^{-t} dt &< \int_0^1 t^{x-1} dt \\ &= \left. \frac{t^x}{x} \right|_0^1 \\ &= \frac{1}{x}\end{aligned} \tag{2.2}$$

Since $x > 0$, by Direct Comparison Test we conclude that $\int_0^1 t^{x-1} e^{-t} dt$ converges. As for the second term, the exponential grows faster than any polynomial, for every x so we can find $N \in \mathbb{N}$, big enough so that $e^{\frac{t}{2}} \geq t^{x-1}$, for $t \in [N, +\infty)$. Thus

$$\begin{aligned}\int_1^\infty t^{x-1} e^{-t} dt &= \int_1^N t^{x-1} e^{-t} dt + \int_N^\infty t^{x-1} e^{-t} dt \\ &\leq \int_1^N t^{x-1} e^{-t} dt + \int_N^\infty e^{\frac{t}{2}} e^{-t} dt \\ &= \int_1^N t^{x-1} e^{-t} dt + \int_N^\infty e^{\frac{-t}{2}} dt\end{aligned}$$

The first term $\int_1^N t^{x-1} e^{-t} dt$ is finite real number because the function $t^{x-1} e^t$ is continuous on $[1, N]$, $\int_N^\infty e^{\frac{-t}{2}} dt = -\left. \frac{1}{2} e^{\frac{-t}{2}} \right|_N^\infty = \frac{1}{2} e^{\frac{-N}{2}}$ is convergent. Hence, $\int_1^\infty t^{x-1} e^{-t} dt < \infty$

□

The Gamma functions holds some fundamental properties. From (2.1) we can conclude that

$$\begin{aligned}\Gamma(n) &= \int_0^\infty x^{n-1}e^{-x}dx = (n-1)!, \text{ for n} \in \mathbb{N} \text{ and}\\ \Gamma(n+1) &= \int_0^\infty x^n e^{-x}dx = n!.\end{aligned} \tag{2.3}$$

Also, the Gamma function satisfies the recursive property

$$\Gamma(n+1) = n\,\Gamma(n). \tag{2.4}$$

Proof. Using (2.3) we have [52]:

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}dx = n!. \tag{2.5}$$

Now, let $u = x^n,\ dv = e^{-x}dx,\ du = nx^{n-1}dx$ and $v = -e^{-x}$ we get

$$\begin{aligned}\Gamma(n+1) &= -\ x^n e^{-x}\big|_0^\infty - \int_0^\infty x^{n-1}n\left(-e^{-x}\right)dx,\\ &= n\int_0^\infty x^{n-1}e^{-x}dx = n\Gamma(n).\end{aligned} \tag{2.6}$$

Thus,

$$\Gamma(n+1) = n\Gamma(n). \tag{2.7}$$

□

We know that $\Gamma(n+1) = n!$ where n is a non-negative integer. We will introduce some remarkable values of the Gamma function that are important in probability and engineering. One of the most useful values $\Gamma(1) = 1$

Proof. Using the definition (2.1)

$$\begin{aligned}\Gamma(1) &= \int_0^\infty x^{1-1}e^{-x}dx,\\ \Gamma(1) &= \int_0^\infty e^{-x}dx,\\ \Gamma(1) &= -\frac{1}{e^{-x}}\bigg|_0^\infty,\\ \Gamma(1) &= -\frac{1}{e^{-\infty}} - (-e)^{-0} = 0 - (-1) = 1.\end{aligned} \tag{2.8}$$

□

Another special value of Gamma function that is frequently used is

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{2.9}$$

Proof. [53]

$$\begin{aligned} \Gamma\left(\frac{1}{2}\right) &= \int_0^\infty t^{\frac{1}{2}-1} e^{-x} dx, \\ \Gamma\left(\frac{1}{2}\right) &= \int_0^\infty \frac{1}{\sqrt{t}} e^{-x} dx. \end{aligned} \tag{2.10}$$

We let $x = y^2,\ dx = 2ydy$ and we obtain

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{1}{y} e^{-y^2} 2ydy = 2\int_0^\infty e^{-y^2} dy, \tag{2.11}$$

Simplifying, we obtain

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy \tag{2.12}$$

Using polar coordinates, with $r^2 = x^2 + y^2$, (2.12) becomes

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} rdrd\theta = 4\left[\theta\right]_0^{\frac{\pi}{2}} \left[\frac{e^{-r^2}}{-2}\right]_0^\infty = 4(\frac{\pi}{2})(\frac{1}{2}) = \pi. \tag{2.13}$$

□

As for the negative numbers z, $\Gamma(z)$ is not defined. Therefore, we need to use the recursion relation to define it as follow:

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1) \text{ for } z < 0. \tag{2.14}$$

Example 2.1.

$$\begin{aligned} \Gamma(-0.2) &= \frac{1}{-0.2}\Gamma(0.8), \\ \Gamma(-1.2) &= \frac{1}{(-0.2)(-1.2)}\Gamma(0.8). \end{aligned} \tag{2.15}$$

2.2 Mittag Leffler Function

The Mittag-Leffler function was introduced to address a classical complex analysis problem, specifically, how to represent the technique of analytic continuation of power series outside of their convergence disk. Its significance was rediscovered when the connection between the Mittag-Leffler function and Fractional Calculus was completely established. It is considered a generalization of the exponential function e^x. It also has a significant role in solving fractional differential equations and fractional integral equations of integer order known as differintegrals. Its significance lies in its direct involvement in physics, chemistry, biology, and engineering [54].

The Mittag-Leffler function of one parameter and two parameter equation are the most common and extensively applicable functions. The generalization of the Mittag-Leffler function of one parameter is defined as follows:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}, \quad where \quad \alpha, z \in \mathbb{C}\; with\; Re(\alpha) > 0$$

The Mittag-Leffler function of two parameters $\alpha, \beta \in \mathbb{C}$ with $Re(\alpha) > 0$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)}.$$

We obtain some well known functions by choosing specific values for α and β [54].

Example 2.2.

$$\begin{aligned} E_{1,1}(z) &= \mathrm{e}^z, \\ E_{1,2}(z) &= \frac{\mathrm{e}^z - 1}{z}, \\ E_{2,1}\left(z^2\right) &= \cosh(z), \\ E_{2,2}\left(z^2\right) &= \frac{\sinh(z)}{z}. \end{aligned} \tag{2.16}$$

2.3 Laplace Transform

Laplace Transform is an integral transform commonly used in physics and engineering. It was developed by Abel, Heaviside, Lerch, and Bromwich in the 19th century [55]. It is named after the mathematician and astronomer Pierre-Simon Laplace. Although Laplace Transform was known since the 19th century, the current use of it came about after World War II.

Laplace Transform can be seen as an operator acting on functions. We shall see later that when we apply Laplace Transform to differential equations, the later become algebraic equations, and hence easier to handle. ,

Definition 2.3.1. For a function f defined on [0,$+\infty$] and for $s \in \mathbb{C}$, we define the Laplace Transform of f at s b:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt. \tag{2.17}$$

For this integral to exist, we must have $e^{-\alpha t}|f(t)| \leq M$ for $all\ t \geq T$, where α, M and T are positive constants.

Example 2.3. *If* $f(t) = 1$ *for* $t \geq 0$, *for all* $s > 0$*, we have*

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} dt \\
&= \lim_{\tau \to \infty} \left(\left. \frac{e^{-st}}{-s} \right|_0^\tau \right) \\
&= \lim_{\tau \to \infty} \left(\frac{e^{-s\tau}}{-s} + \frac{1}{s} \right) \\
&= \frac{1}{s}.
\end{aligned} \tag{2.18}$$

Therefore, $\mathcal{L}\{1\} = \frac{1}{s}$ *for* $s > 0$

The inverse Laplace transform is the transformation into a function of time. If $f(t)$ is the inverse Laplace Transform of $F(s)$, then $\mathcal{L}\{f(t)\} = F(s)$ and inverse can be written as $f(t) = \mathcal{L}^{-1}\{F(s)\}$

Example 2.4. *Let* $f(t) = e^{at}$ *for some* $a \in \mathbb{R}$. *Then*

$$\begin{aligned}
\mathcal{L}\left\{e^{at}\right\} &= \int_0^\infty e^{-st} e^{at} \, \mathrm{d}t \\
&= \int_0^\infty e^{-(s-a)t} \, \mathrm{d}t \\
&= -\left. \frac{1}{s-a} e^{-(s-a)t} \right|_0^\infty, \; when \; s > a. \\
&= \frac{1}{s-a}.
\end{aligned} \tag{2.19}$$

Example 2.5. *Let* $f(t) = sint$. *Using Euler's formula* $\mathrm{e}^{\mathrm{i}t} = \cos t + \mathrm{i} \sin t$*, we write*

$$\sin t = \frac{\mathrm{e}^{\mathrm{i}t} - e^{-\mathrm{i}t}}{2\mathrm{i}}.$$

Then

$$\begin{aligned}
\mathcal{L}\{\sin t\} =& \frac{1}{2\mathrm{i}} \int_0^\infty \left(e^{\mathrm{i}t} - e^{-\mathrm{i}t}\right) e^{-st} \, \mathrm{d}t \\
=& \frac{1}{2\mathrm{i}} \left(\frac{1}{s-\mathrm{i}} - \frac{1}{s+\mathrm{i}} \right) \\
=& \frac{1}{s^2+1}.
\end{aligned} \tag{2.20}$$

2.3.1 Solving Differential Equations Laplace Transformation is a powerful tool for solving ordinary linear differential equations algebraically.

Theorem 2.3.1. *Suppose* f *is of integer order, and that* f *is continuous and* f' *is piecewise continuous on any interval* $0 \leq t \leq A$. *The Laplace Transform of integer order derivatives*

$$\mathcal{L}\left\{f'(t)\right\} = s\mathcal{L}\{f(t)\} - f(0).$$

Generalizing the theorem for any integer order derivative by applying the theorem multiple times we get:

$$\begin{aligned}\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0),\\ - \ldots - s^2 f^{(n-3)}(0) - sf^{(n-2)}(0) - f^{(n-1)}(0).\end{aligned} \tag{2.21}$$

Example 2.6. *We shall solve the following IVP $y'(t) = 5 - 2t, y(0) = 1$ using Laplace Transform. We start by applying Laplace Transform to both sides of the DE and we obtain*

$$\begin{aligned}\mathcal{L}\{y'(t)\} &= \mathcal{L}\{5 - 2t\}\\ &= \frac{5}{s} - \frac{2}{s^2}.\end{aligned}$$

Using Theorem 2.3.1, we have

$$s\mathcal{L}\{y(t)\} - y(0) = \frac{5}{s} - \frac{2}{s^{2.}}$$

Since $y(0) = 1$, we divide both sides by s and we obtain

$$\mathcal{L}(y(t)) = \frac{1}{s} + \frac{5}{s^2} - \frac{2}{s^3}.$$

Finally, we apply the inverse Laplace Transform to conclude that

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\{\frac{1}{s}\} + \mathcal{L}^{-1}\{\frac{5}{s^2}\} - \mathcal{L}^{-1}\{\frac{2}{s^3}\}\\ &= 1 + 5t - t^2.\end{aligned}$$

Example 2.7. *Solve the initial value problem $y' + 2y = 4te^{-2t}$, $y(0) = -3$ using Laplace Transformation method [53]. We start by applying Laplace Transform to both sides of the DE and we obtain*

$$\mathcal{L}\{y'\} + \mathcal{L}\{2y\} = \mathcal{L}\left\{4te^{-2t}\right\}. \tag{2.22}$$

Using Theorem 2.3.1, we have

$$s\mathcal{L}\{y\} - y(0) + 2\mathcal{L}\{y\} = \frac{4}{(s+2)^2}. \tag{2.23}$$

Simplifying

$$\begin{aligned}(s\mathcal{L}\{y\} - (-3)) + 2\mathcal{L}\{y\} &= \frac{4}{(s+2)^2},\\ \mathcal{L}\{y\}(s+2) + 3 &= \frac{4}{(s+2)^2},\\ \mathcal{L}\{y\}(s+2) &= \frac{4}{(s+2)^2} - 3. \textit{Hence}\end{aligned} \tag{2.24}$$

$$\mathcal{L}\{y\} = \frac{4}{(s+2)^3} - \frac{3}{(s+2)} = \frac{4 - 3(s+2)^2}{(s+2)^3} = \frac{-3s^2 - 12s - 8}{(s+2)^3}.$$

Using partial fractions decomposition

$$
\begin{aligned}
\mathcal{L}\{y\} &= \frac{-3s^2-12s-8}{(s+2)^3} = \frac{a}{(s+2)^3} + \frac{b}{(s+2)^2} + \frac{c}{(s+2)} \\
&= \frac{-3s^2-12s-8}{(s+2)^3} = \frac{a}{(s+2)^3} + \frac{b(s+2)}{(s+2)^2} + \frac{c(s+2)^2}{(s+2)} \\
&= \frac{-3s^2-12s-8}{(s+2)^3} = \frac{a+bs+2b+cs^2+4cs+4c}{(s+2)^3} \\
&= \frac{-3s^2-12s-8}{(s+2)^3} = \frac{cs^2+(b+4c)s+(a+2b+4c)}{(s+2)^3}.
\end{aligned}
\tag{2.25}
$$

Equating the numerators, we obtain

$$c = -3,\ a = 4,\ b = 0.$$

Substituting the values back in $\mathcal{L}\{y\}$, we have

$$\mathcal{L}\{y\} = \frac{-3s^2-12s-8}{(s+2)^3} = \frac{4}{(s+2)^3} - \frac{3}{(s+2)}. \tag{2.26}$$

Using inverse Laplace, we deduce that

$$
\begin{aligned}
y(t) &= 4\mathcal{L}^{-1}\left(\frac{1}{(s+2)^3}\right) - 3\mathcal{L}^{-1}\left(\frac{1}{(s+2)}\right) \\
&= 2t^2e^{-2t} - 3e^{-2t}.
\end{aligned}
\tag{2.27}
$$

Chapter 3: Fractional Calculus and Fractional Differential Equations

Fractional Calculus is the theory of having integrals and derivatives of arbitrary order, which generalizes the concepts of integer-order differentiation and integration. The history of Fractional Calculus dates back to 1695. It first appeared in a letter L'Hôpital wrote to Leibniz asking about the n$^{\text{th}}$ derivative of the function

$$\frac{D^n f(x)}{Dx^n} \tag{3.1}$$

and what would the result be if $n = \frac{1}{2}$. Leibniz responded that it would be "an apparent paradox, from which one day useful consequences will be drawn" [56]. Later, Lacroix mentioned fractional derivative in a paper published in 1819. He found the n$^{\text{th}}$ derivative for $y = x^m$ when m is a positive integer [56]. His formula is given by

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n}, \ m \geq n$$

The idea behind this lies in generalizing the power rule. In fact, we have that

$$\begin{aligned} &\frac{d}{dx}(x^n) = nx^{n-1}, \\ &\frac{d^2}{dx^2}(x^n) = n(n-1)x^{n-2}, \\ &\frac{d^3}{dx^3}(x^n) = n(n-1)(n-2)x^{n-3}. \end{aligned} \tag{3.2}$$

By identifying the pattern, $\frac{d^3}{dx^3}(x^n)$ can be written as

$$\frac{d^3}{dx^3}(x^n) = \frac{n(n-1)(n-2)(n-3)(n-4)\ldots 3\cdot 2\cdot 1}{(n-3)(n-4)\ldots 3\cdot 2\cdot 1} x^{n-3}. \tag{3.3}$$

Using the factorial notation, we obtain that

$$\frac{d^n y}{dx^n}(x^m) = \frac{m!}{(m-n)!} x^{m-n}, \ m \geq n$$

Then, Lacroix generalized the factorial using the Gamma function to extend the factorial to any real number. As a result, we can write the derivative in terms of the Gamma function

$$\frac{d^n y}{dx^n}(x^m) = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}. \tag{3.4}$$

Moreover, Lacroix gave an example where $y = x$ and $n = \frac{1}{2}$ [56], and he obtained

$$\frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} = \frac{2\sqrt{x}}{\sqrt{\pi}}. \tag{3.5}$$

Nevertheless, the first mathematician who used the applications of Fractional Calculus was Abel, not Lacroix. Abel used fractional operations for the first time in 1823 when he applied Fractional Calculus to solve an integral equation that appeared in the formulation of the tautochrone problem.

Example 3.1. *Let us find the half derivative of* $f(x) = x^5$*. Using* (3.4) *we have*

$$\begin{aligned}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left(x^5\right) &= \frac{\Gamma(5+1)}{\Gamma\left(5+1-\frac{1}{2}\right)}x^{5-\frac{1}{2}} \\ &= \frac{\Gamma(6)}{\Gamma\left(\frac{11}{2}\right)}x^{\frac{9}{2}} = \frac{5!}{\frac{945\sqrt{\pi}}{32}}x^{\frac{9}{2}}.\end{aligned} \tag{3.6}$$

Example 3.2. *To find the half derivative of a constant* $f(x) = 1$*, we observe that* $x^0 = 1$ *for* $x \neq 0$*. Then* (3.4) *implies*

$$\begin{aligned}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}(1) &= \frac{d^{\frac{1}{2}}\left(x^0\right)}{dx^{\frac{1}{2}}} \\ &= \frac{\Gamma(0+1)}{\Gamma\left(0+1-\frac{1}{2}\right)}x^{0-\frac{1}{2}} \\ &= \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)}x^{-\frac{1}{2}} \\ &= \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)}\frac{1}{\sqrt{x}}.\end{aligned} \tag{3.7}$$

3.1 The Grünwald-Letnikov Derivative Operator

We will derive the Grünwald-Letnikov Fractional Derivative . The proof is based on the backwards difference definition given by

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x)-f(x-h)}{h}. \tag{3.8}$$

Using the definition (3.8), the second derivative is

$$\begin{aligned}\frac{d^2}{dx^2}f(x) =& \frac{d}{dx}f'(x) \\ =& \lim_{h\to 0}\frac{f'(x)-f'(x-h)}{h} \\ =& \lim_{h\to 0}\frac{\frac{f(x)-f(x-h)}{h} - \frac{f(x-h)-f(x-2h)}{h}}{h}.\end{aligned} \tag{3.9}$$

Thus,

$$\frac{d^2}{dx^2}f(x) = \lim_{h\to 0}\frac{1}{h^2}[f(x)-2f(x-h)+f(x-2h)]. \tag{3.10}$$

Similarly, the limit definition of the third derivative using the backwards difference is given by

$$\frac{d^3}{dx^3}f(x) = \lim_{h\to 0}\frac{1}{h^3}[f(x) - 3f(x-h) + 3f(x-2h) - f(x-3h)]. \tag{3.11}$$

Hence, we deduce the general case of the limit definition of the nth derivative using the backwards difference as

$$\frac{d^k}{dx^k}f(x) = \lim_{h\to 0}\left(\frac{1}{h}\right)^k \sum_{l=0}^{k}(-1)^l \begin{pmatrix} k \\ l \end{pmatrix} f(x-lh), \text{where} k \in \mathbb{N} \text{and} \begin{pmatrix} k \\ l \end{pmatrix} = \frac{k!}{l!(k-l)!} \tag{3.12}$$

In order to extend the natural numbers in (3.12) to all real numbers $\alpha \in \mathbb{R}$, we use the fact that $\Gamma(n+1) = n!$ and we obtain

$$\frac{d^k}{dx^k}f(x) = \lim_{n\to 0}\left(\frac{1}{h}\right)^k \sum_{l=0}^{k}(-1)^l \frac{\Gamma(k+1)}{l!\Gamma(k+1-l)} f(x-lh). \tag{3.13}$$

Mapping k to α , where $\alpha \in \mathbb{R}$, we get

$$\frac{d^\alpha}{dx^\alpha}f(x) = \lim_{h\to 0}\left(\frac{1}{h}\right)^\alpha \sum_{l=0}^{k}(-1)^l \frac{\Gamma(\alpha+1)}{l!\Gamma(\alpha+1-l)} f\left(x-lh\right), \tag{3.14}$$

At this stage, we introduce the change of variables $kh = x-a$ where a is a constant. Then, $k = \frac{x-a}{h}$ and $\frac{1}{h} = \frac{k}{x-a}$. Thus, our equation becomes

$$\frac{d^\alpha}{dx^\alpha}f(x) = \lim_{k\to\infty}\left(\frac{k}{x-a}\right) \sum_{l=0}^{k}(-1)^l \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-l)l!} f\left(x-\frac{l}{k}(x-a)\right). \tag{3.15}$$

Finally, the Grünwald-Letnikov Differential Operator is given by

$${}_aD_x^\alpha f(x) = \lim_{n\to\infty}\left(\frac{n}{x-a}\right)^\alpha \sum_{l=0}^{n}(-1)^l \frac{\Gamma(\alpha+1)}{l!\Gamma(\alpha+1-l)} f\left(x-\frac{l}{n}(x-a)\right). \tag{3.16}$$

Similarly, we define the Grünwald-Letnikov Differential Operator for negative base points by

$${}_aD_x^{-\alpha}(f) = \lim_{n\to\infty}\left(h^\alpha\right) \sum_{l=0}^{\infty}\frac{\Gamma(a+l)}{l!\Gamma(\alpha)} \cdot f\left(x-\frac{l}{n}(x-a)\right), \text{where } \alpha \geq 0 \tag{3.17}$$

3.2 Connection between Derivatives and Integrals

In this section, we derive the integral operator of fractional order and investigates its connection to the fractional derivative operator. Take $\alpha = 1$. Then

$$\begin{aligned} {}_aD_x^{-1}(f) &= \lim_{n\to\infty}\left(h^1\right) \sum_{l=0}^{n}\frac{\Gamma(1+l)}{l!\Gamma(1)} f(x-lh) \\ &= \lim_{n\to\infty}\sum_{l=0}^{n} f(x-lh)h. \end{aligned} \tag{3.18}$$

We notice that the right hand side is a Riemann sum, and the equation becomes

$$ {}_aD_x^{-1}(f) = \int_0^{x-a} f(x-t)dt. \tag{3.19} $$

Applying the substitution $u = x - t$, we obtain

$$ {}_aD_x^{-1}(f) = -\int_x^a f(u)(-du) \quad = \int_a^x f(t)dt. \tag{3.20} $$

Now, consider $\alpha = 2$ in the equation (3.17). Then,

$$ \begin{aligned} {}_aD_a^{-2}(f) &= \lim_{n\to\infty} \left(h^2\right) \sum_{l=0}^{n} \frac{\Gamma(l+2)}{l!\Gamma(2)} f(x-lh) \\ &= \lim_{n\to\infty} \left(h^2\right) \sum_{l=0}^{n} (l+1) f(x-lh) \\ &= \frac{1}{2} \lim_{n\to\infty} \sum_{l=1}^{n+1} (lh) f(x-lh)h. \end{aligned} \tag{3.21} $$

We notice that the right hand side is half of the Riemann sum, and the equation becomes

$$ {}_aD_x^{-2}(f) = \int_a^x (x-t)f(t). \tag{3.22} $$

Similarly, for $\alpha = 3$

$$ {}_aD_x^{-3} = \frac{1}{2!} \int_a^x (x-t)^2 f(t)dt. \tag{3.23} $$

When $\alpha = 4$, we get

$$ {}_aD_x^{-4} = \frac{1}{3!} \int_a^x (x-t)^3 f(t)dt. \tag{3.24} $$

Generally, we have

$$ {}_aD_x^{-(k+1)}(f) = \frac{1}{k!} \cdot \int_a^x (x-t)^k f(t)dt. \tag{3.25} $$

Lastly, in order to extend the connection between derivatives and integrals to the fractional order, we use the Gamma function to obtain the fractional integral operator which is referred to as Riemann-Liouville fractional integral.

Definition 3.2.1. Let f be a continuous function, $0 < \alpha \leq 1$, and $t, x \in \mathbb{R}^+$. The fractional integral of order α is defined as

$$ {}_aD_x^{-\alpha}(f) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt. \tag{3.26} $$

Definition 3.2.2. Let $f \in C^n[a, b], \alpha \geq 0$, and $n - 1 < a \leq n$. The Riemann-Liouville fractional derivative is given by

$$D_L^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha+n-1}} dx, \quad a < t < b. \tag{3.27}$$

Example 3.3. *Let $f(x) = x$. Then,*

$$D_L^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-x)^{n-\alpha-1} f(x) dx$$

For $n = 1$ and $\alpha = \frac{1}{2}$, we get

$$D_L^{\frac{1}{2}} f(t) = \frac{1}{\Gamma\left(1-\frac{1}{2}\right)} \frac{d}{dt} \int_0^t (t-x)^{1-\frac{1}{2}-1} x dx,$$

By substitution, we obtain

$$\begin{aligned} D_L^{\frac{1}{2}} f(t) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{d}{dt} \left(\frac{4t^{\frac{3}{2}}}{3} \right) \\ &= \frac{2\sqrt{t}}{\sqrt{\pi}}. \end{aligned} \tag{3.28}$$

Another integral operator that is an alternative to the Riemann-Liouville operator was introduced in 1967, by an Italian Mathematician Caputo, who adjusted the definition of the Riemann–Liouville operator.

Definition 3.2.3. Let $f \in C^n[a,b],\ \alpha \geq 0$, and $n-1 < a \leq n.$ The Caputo Fractional Derivative is given by

$$\begin{aligned} D^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} \frac{d^n f(x)}{dx^n} dx, \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha+1-n}} dx, \quad a \leq t < b. \end{aligned} \tag{3.29}$$

Example 3.4. *Let $f(x) = x^3$. Using Definition* (3.2.3)*, the half derivative of f is*

$$\begin{aligned} D^{\frac{1}{2}} f(t) &= \frac{1}{\Gamma\left(3-\frac{1}{2}\right)} \int_0^t (t-x)^{3-\frac{1}{2}-1} \frac{d^3 f(x)}{dx^3} dx \\ &= \frac{1}{\Gamma\left(\frac{5}{2}\right)} \int_0^t (t-x)^{\frac{3}{2}} 6 dx \\ &= \frac{6}{\Gamma\left(\frac{5}{2}\right)} \int_0^t (t-x)^{\frac{3}{2}} dx \\ &= \frac{12}{5\Gamma\left(2+\frac{1}{2}\right)} \left[(t-x)^{\frac{3}{2}+1} \right]_0^t \\ &= \frac{12}{5\frac{3}{4}\Gamma\left(\frac{1}{2}\right)} (-t)^{\frac{5}{2}} = \frac{8}{15\sqrt{\pi}}. \end{aligned} \tag{3.30}$$

3.3 Riemann-Liouville Fractional Integral

Definition 3.3.1. Let f be a continuous function, $0 < \alpha \leq 1$, and $t, x \in \mathbb{R}^+$. The fractional integral of order α is defined as

$$ {}_aD_x^{-\alpha}(f) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt. \tag{3.31}$$

There are some alternate forms of the Riemann-Liouville fractional derivative.

$$ {}_aI_x^{\alpha}(f(x)) = \frac{1}{\Gamma(x)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ when\ x > a, \tag{3.32}$$

$$ {}_aI_x^{\alpha}(f(x)) = \frac{1}{\Gamma(\alpha)} \int_x^a (x-t)^{\alpha-1} f(t)dt, \ when\ x < a. \tag{3.33}$$

Example 3.5. *Let $f(x) = x^3$. Using Definition* (3.3.1)*, we have*

$$ {}_aI_x^{1/2}\left(x^3\right) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^x (x-t)^{-\frac{1}{2}} t^3 dt. \tag{3.34}$$

Taking $x = 1$, we obtain

$$ {}_aI_1^{1/2}\left(x^3\right) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^1 (1-t)^{\frac{-1}{2}} t^3 dt. \tag{3.35}$$

Using the Beta function definition

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt,$$

we deduce that

$$\begin{aligned} {}_aI_1^{1/2}\left(x^3\right) &= \frac{1}{\Gamma(\frac{1}{2})} \int_0^1 t^{4-1}(1-t)^{\frac{1}{2}-1}dt \\ &= \frac{1}{\Gamma(\frac{1}{2})} B(4, \frac{1}{2}) \end{aligned} \tag{3.36}$$

Since $B(4, \frac{1}{2}) = \frac{\Gamma(4)\Gamma(\frac{1}{2})}{\Gamma(4+\frac{1}{2})}$, we conclude that

$$\begin{aligned} {}_aI_1^{1/2}\left(x^3\right) &= \frac{1}{\Gamma(\frac{1}{2})} \frac{\Gamma(4)\Gamma(\frac{1}{2})}{\Gamma(4+\frac{1}{2})} \\ &= \frac{3!}{\Gamma(\frac{1}{2})} \\ &= \frac{32}{35\sqrt{\pi}}. \end{aligned} \tag{3.37}$$

The Riemann-Liouville fractional integral holds some essential properties.

Properties

1. Identity operator: ${}_aI_xI^0f(x) = f(x)$.
2. Constant multiple: ${}_aI_x^\alpha(cf(x)) = c \cdot_a I_x^\alpha(f(x))$.
3. Linear operator: ${}_aI_x^\alpha(f(x) \pm g(x)) =_a I_x^\alpha(f(x)) \pm_a I_x^\alpha(g(x))$.
4. Composition: ${}_aI_x^\alpha({}_aI_x^\beta(f(x))) =_a I_x^{\alpha+\beta}(f(x))$.

Proof. (of property "4"). We choose to prove (4) since the other properties are straightforward consequences of the definition.

$$\begin{aligned}{}_aI_x^\alpha({}_aI_x^\beta(f(x))) =& \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[{}_aI_t^\beta(f(t))\right]dt,\\ =& \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[\frac{1}{\Gamma(\beta)}\int_a^t (t-\xi)^{\beta-1}f(\xi)d\xi\right]dt,\end{aligned} \tag{3.38}$$

Rearranging the integral:

$$\begin{aligned}{}_aI_x^\alpha({}_aI_x^\beta(f(x))) =& \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_a^t (x-t)^{\alpha-1}(t-\xi)^{\beta-1}f(\xi)d\xi dt\\ =& \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_0^1 (x-\xi)^{\alpha+\beta-1}(1-u)^{\alpha-1}u^{\beta-1}f(\xi)dud\xi\end{aligned} \tag{3.39}$$

Using Fubini's theorem, we obtain

$$\begin{aligned}{}_aI_x^\alpha({}_aI_x^\beta(f(x))) =& \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left(\int_0^1 (1-u)^{\alpha-1}u^{\beta-1}du\right)\left(\int_a^x (x-\xi)^{\alpha-\beta-1}f(\xi)d\xi\right)\\ =& \frac{1}{\Gamma(\alpha)\Gamma(\beta)}B(\alpha,\beta)\int_a^x (x-\xi)^{\alpha+\beta-1}f(\xi)d\xi.\end{aligned} \tag{3.40}$$

Since $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, we conclude that

$$\begin{aligned}{}_aI_x^\alpha({}_aI_x^\beta(f(x))) =& \frac{1}{\Gamma(\alpha,\beta)}\int_a^x (x-t)^{\alpha+\beta+1}f(t)dt\\ =& {}_aI_x^{\alpha+\beta}(f(x)).\end{aligned} \tag{3.41}$$

□

3.4 Fractional Integrals of Exponential Functions

Consider $f(x) = e^x$. Then

$$ {}_aI_0^\alpha\left(e^x\right) = \frac{1}{\Gamma(x)}\int_a^x (x-t)^{\alpha-1}e^t dt. \tag{3.42}$$

Let $u = x - t$. Then $du = -dt$. When $t = a = -\infty$, $u = x - a = \infty$ and when $t = x$, $u = 0$. Thus

$$\begin{aligned} {}_aI_0^{\alpha}\left(e^x\right) =& \frac{1}{\Gamma(a)} \int_{\infty}^{0} u^{\alpha-1} e^{x-u} (-du) \\ =& \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} u^{\alpha-1} e^{x} e^{-u} du \\ =& e^x \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} u^{\alpha-1} e^{-u} du. \end{aligned} \tag{3.43}$$

Since $\int_0^\infty u^{\alpha-1} e^{-u} du = \Gamma(\alpha)$, we get

$${}_aI_0^{\alpha}\left(e^x\right) = e^x. \tag{3.44}$$

We also notice that

$$D^{\alpha}\left(e^x\right) = e^x, \ \forall\ \alpha > 0.$$

Now, we can generalize the definition of the integral of e^x as follows

$${}_aI_0^{\alpha}\left(e^{ax}\right) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-t)^{\alpha-1} e^{at} dt. \tag{3.45}$$

Similarly, as in the previous examples, we obtain

$$\begin{aligned} {}_aI_0^{\alpha}\left(e^{ax}\right) =& \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} u^{\alpha-1} e^{a(x-u)} du \\ =& \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} u^{\alpha-1} e^{ax} e^{-au} du \\ =& e^{ax} \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} u^{2-1} e^{-au} du. \end{aligned} \tag{3.46}$$

Using substitution, $w = au$, we obtain that

$$\begin{aligned} {}_aI_0^{\alpha}\left(e^{ax}\right) =& e^{ax} \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} \left(\frac{w}{a}\right)^{\alpha-1} e^{-w} \frac{1}{a} dw \\ =& e^{ax} \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} \frac{w^{\alpha-1}}{a^{\alpha-1}} e^{-w} \frac{1}{a} dw \\ =& \frac{e^{ax}}{a^{\alpha}} \frac{1}{\Gamma(a)} \int_{0}^{\infty} w^{\alpha-1} e^{-w} dw. \end{aligned} \tag{3.47}$$

Notice that $\Gamma(w) = \int_0^\infty w^{\alpha-1} e^{-w} dw$. Therefore

$$I^{\alpha}\left(e^{ax}\right) = \frac{e^{ax}}{a^{\alpha}}. \tag{3.48}$$

3.5 Fractional Integrals of Sine and Cosine

We will start by using the Riemann-Liouville definition

$$ {}_aI_x^\alpha(\sin x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \sin t dt. \tag{3.49}$$

Then, applying the Taylor series expansion of $\sin x$ yields

$$ {}_aI_x^\alpha(\sin x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}(-1)^k dt. \tag{3.50}$$

Interchanging the summation and the integral,

$$ {}_aI_x^\alpha(\sin x) = \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{(2k+1)!} \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} t^{2k+1} dt \right]. \tag{3.51}$$

Notice that $\frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} t^{2k+1} dt = {}_a I_x^\alpha \left(x^{2k+1}\right)$. Let $v = 2k+1$. When $a = 0$, we have

$$ \begin{aligned} {}_aI_x^\alpha\left(x^{2k+1}\right) &= \frac{\Gamma(v+1)}{\Gamma(v+1+\alpha)} x^{v+\alpha} \\ &= \frac{\Gamma(2k+\alpha)}{\Gamma(2k+2+\alpha)} x^{2k+1+\alpha}. \end{aligned} \tag{3.52}$$

Therefore, equation (3.51) becomes:

$$ \begin{aligned} {}_0I_x^\alpha(\sin x) &= \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{(2k+1)!} \frac{\Gamma(2k+2)}{T(2k+2+\alpha)} x^{2k+1+\alpha} \right] \\ &= \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{\Gamma(2k+2+\alpha)} x^{2k+1+\alpha} \right]. \end{aligned} \tag{3.53}$$

Example 3.6. *When $a = 0$ and $\alpha = -1$, we have*

$$ \begin{aligned} I_x^{-1}(\sin x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(2k+1)} x^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \cos x. \end{aligned} \tag{3.54}$$

Example 3.7. *When $a = 0$ and $\alpha = 1$, we have*

$$
\begin{aligned}
{}_0I_x^1(\sin x) &= \sum_{x=0}^{\infty} \frac{(-1)^k}{\Gamma(2k+3)} x^{2k+2} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+2)!} x^{2k+2} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k)!} x^{2k} \\
&= -\sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \\
&= -(\cos x - 1) \\
&= 1 - \cos x
\end{aligned} \tag{3.55}
$$

Let

$$
\begin{aligned}
G_a(x) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_a^x (x-t)^{\frac{1}{2}-1} (\sin t) dt \\
&=_a I_x^{\frac{1}{2}} (\sin x),
\end{aligned} \tag{3.56}
$$

Comparing the function $G_a(x)$, for various values of a, with the function $g(x) = \sin(x - \frac{\pi}{4})$, we notice that

$$
\lim_{a \to -\infty} G_a(x) = g(x). \tag{3.57}
$$

As a result, we can define the fractional integral of sine and cosine as

$$
\begin{aligned}
I^\alpha(\sin x) &=_{-\infty} I_x^\alpha(\sin x) = \sin\left(x + \frac{\alpha\pi}{2}\right), \\
I^\alpha(\cos x) &=_{-\infty} I_x^\alpha(\cos x) = \cos\left(x + \frac{\alpha\pi}{2}\right).
\end{aligned} \tag{3.58}
$$

Chapter 4: Analytical solutions of Fractional Differential Equations

4.1 Laplace Adomian Decomposition Method

The Laplace Adomian Decomposition Method is a technique that generates an efficient approximate analytical solution for nonlinear systems of ordinary and partial differential equations [70].

Consider the following equation

$$D^{\alpha} y + f(y) = g(x), \quad \text{where } 1 < \alpha \leq 2, \tag{4.1}$$

with the boundary conditions

$$y(0) = c, \quad y(1) = d.$$

The solution can be expressed as an infinite series of the form

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{4.2}$$

The nonlinear term can be written as an infinite series of polynomials in y_n of the form

$$[f(y)] = \sum_{n=0}^{\infty} A_n. \tag{4.3}$$

and the A_n terms are called the Adomian polynomials and are given by the formula

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{i=0}^{\infty} \lambda^i y_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots \tag{4.4}$$

Applying Laplace transform $\mathcal{L}$ to both sides of equation (4.1), we obtain

$$\{L\} [D^{\alpha} y] + \mathcal{L}\{f(y)\} = \mathcal{L}\{g(x)\}. \tag{4.5}$$

Then, using Caputo's definition of fractional derivative and rearranging, we acquire the following

$$\frac{s^2 \mathcal{L}\{y\} - sy(0) - y'(0)}{s^{2-\alpha}} = \mathcal{L}\{g(x)\} - \mathcal{L}\{f(y)\}. \tag{4.6}$$

Substituting the boundary conditions $y(0) = c,\ y(1) = d$, we obtain the following

$$\frac{s^2 \mathcal{L}[y] - cs - k}{s^{2-\alpha}} + \mathcal{L}[f(y)] = \mathcal{L}[g(x)]. \tag{4.7}$$

Rearranging the equation in terms of $\mathcal{L}\{y\}$ implies

$$\mathcal{L}\{y\} = \frac{c}{s} + \frac{k}{s^2} - \frac{1}{s^\alpha}\mathcal{L}\{f(y)\} + \frac{1}{s^\alpha}\mathcal{L}\{g(x)\}. \tag{4.8}$$

Substituting Equations (4.2) and (4.3) into (4.8) yields

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} y_n\right\} = \frac{c}{s} + \frac{k}{s^2} - \frac{1}{s^\alpha}\mathcal{L}\left\{\sum_{n=0}^{\infty} A_n\right\} + \frac{1}{s^\alpha}\mathcal{L}\{g(x)\}. \tag{4.9}$$

By applying the inverse Laplace, the recurrence relation is defined by

$$\begin{aligned}
y_0 &= \mathcal{L}^{-1}\left\{\frac{k}{s^2} + \frac{1}{s^\alpha}\mathcal{L}\{g(x)\}\right\}, \\
y_1 &= \mathcal{L}^{-1}\left\{-\frac{1}{s^\alpha}\mathcal{L}\left\{A_0\right\}\right\}, \\
&\vdots \\
y_{n+1} &= \mathcal{L}^{-1}\left\{-\frac{1}{s^\alpha}\mathcal{L}\left\{A_n\right\}\right\}.
\end{aligned} \tag{4.10}$$

In order to obtain the Adomian polynomials, we first write Taylor series expansion of the nonlinear function $f(y)$ about y_0 to get

$$f(y) = f\left(y_0\right) + f'\left(y_0\right)\left(y - y_0\right) + \frac{1}{2!}f''\left(y_0\right)\left(y - y_0\right)^2 + \frac{1}{3!}f'''\left(y_0\right)\left(y - y_0\right)^3 + \ldots \tag{4.11}$$

Substituting an expansion of Equation (4.2) into Equation (4.11) and simplifying imply

$$f(y) = f\left(y_0\right) + f'\left(y_0\right)\left(y_1 + y_2 + \ldots\right) + \frac{1}{2!}f''\left(y_0\right)\left(y_1 + y_2 + \ldots\right)^2 + \frac{1}{3!}f'''\left(y_0\right)\left(y_1 + y_2 + \ldots\right)^3 + \ldots \tag{4.12}$$

Expanding Equation (4.12)

$$\begin{aligned}
f(y) = {} & f\left(y_0\right) + f'\left(y_0\right)y_1 + f'\left(y_0\right)y_2 + \frac{1}{2!}f''\left(y_0\right)y_1^2 + f'\left(y_0\right)y_3 \\
& + \frac{2}{2!}f''\left(y_0\right)y_1y_2 + \frac{1}{3!}f'''\left(y_0\right)y_1^3 + f'\left(y_0\right)y_4 + \frac{1}{2!}f''\left(y_0\right)y_2^2 \\
& + \frac{2}{2!}f''\left(y_0\right)y_1y_3 + \frac{3}{3!}f'''\left(y_0\right)y_1^2y_2 + \frac{1}{4!}f''''\left(y_0\right)y_1^4 + f'\left(y_0\right)y_4 \\
& + \frac{2}{2!}f''\left(y_0\right)y_2y_3 + \frac{2}{2!}f''\left(y_0\right)y_1y_4 + \frac{3}{3!}f'''\left(y_0\right)y_1y_2^2 \\
& + \frac{3}{3!}f'''\left(y_0\right)y_1^2y_3 + \frac{4}{4!}f''''\left(y_0\right)y_1^3y_2 + \frac{1}{5!}f'\left(y_0\right)y_1^5 + \ldots
\end{aligned} \tag{4.13}$$

Rearranging the terms based on their order will help extract the Adomian polynomials. The subscript of the Adomian polynomial needs to match all the terms with the same order

[53] . So A_n includes all the terms in the series of order n. Therefore, the first six terms of the Adomian polynomials are as follows [57]

$$\begin{aligned}
&A_0 = f\left(y_0\right), \\
&A_1 = f'\left(y_0\right) y_1, \\
&A_2 = f'\left(y_0\right) y_2 + \tfrac{1}{2!} f''\left(y_0\right) y_1^2, \\
&A_3 = f'\left(y_0\right) y_3 + \tfrac{2}{2!} f''\left(y_0\right) y_1 y_2 + \tfrac{1}{3!} f'''\left(y_0\right) y_1^3, \\
&A_4 = f'\left(y_0\right) y_4 + \tfrac{1}{2!} f''\left(y_0\right)\left(2 y_1 y_3 + y_2^2\right) + \tfrac{3}{3!} f'''\left(y_0\right) y_1^2 y_2 + \tfrac{1}{4!} f''''\left(y_0\right) y_1^4, \\
&A_5 = f'\left(y_0\right) y_5 + \tfrac{1}{2!} f''\left(y_0\right)\left(2 y_1 y_4 + 2 y_2 y_3\right) + \tfrac{1}{3!} f'''\left(y_0\right)\left(3 y_1^2 y_3 + 3 y_1 y_2^2\right), \\
&\quad + \tfrac{4}{4!} f^{(4)}\left(y_0\right) y_1^3 y_2 + \tfrac{1}{5!} f^{(5)}\left(y_0\right) y_1^5,
\end{aligned} \tag{4.14}$$

The general formula of the Adomian polynomials is given by

$$A_n\left(y_0, y_1, \ldots, y_n\right) = \frac{1}{n!} \frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{\infty} y_k \lambda^k\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots \tag{4.15}$$

The Adomian Polynomial are obtained using the general formula as follows [57]

$$\begin{aligned}
&A_0 = N\left(y_0\right), \\
&A_1 = \tfrac{d}{d\lambda} N\left(y_0 + y_1 \lambda\right)\big|_{\lambda=0} = N\left(y_0\right) y_1, \\
&A_2 = \tfrac{1}{2!} \tfrac{d}{d\lambda}\left(\left(y_1 + 2 y_2 \lambda\right) N''\left(y_0 + y_1 \lambda\right)\right)\big|_{\lambda=0} = N'\left(y_0\right) y_2 + \tfrac{1}{2!} N''\left(y_0\right) y_1^2, \\
&\vdots
\end{aligned} \tag{4.16}$$

4.2 Fractional Differential Transformation Method

The Fractional Differential Transformation method is used to obtain an approximate analytical solution of linear and nonlinear ordinary differential equations of fractional order. It is based on Taylor series expansion that generates polynomials as the analytical solutions [64].

We recall that the fractional derivative of Reimann-Liouville is defined by

$$D_{x_0}^q f(x) = \frac{1}{\Gamma(m-q)} \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\int_{x_0}^x \frac{f(t)}{(x-t)^{1+q-m}} \mathrm{d}t\right], \tag{4.17}$$

where $m - 1 \leqslant q < m, m \in Z^+$ and $x > x_0$. By expanding the continuous analytical function f(x) as fractional power series we obtain the following [64]

$$f(x) = \sum_{k=0}^{\infty} F(k)\left(x - x_0\right)^{k/\alpha}, \tag{4.18}$$

Where α is the fractional order and F(k) is representing the fractional differential transformation of f(x).

The fractional derivative in the Caputo sense is defined to avoid the fractional initial and boundary conditions. The Riemann–Liouville fractional derivative and Caputo's fractional derivative have the following relationship

$$D_{*x_0}^{q} f(x) = D_{x_0}^{q}\left[f(x) - \sum_{k=0}^{m-1} \frac{1}{k!}\left(x - x_0\right)^k f^{(k)}\left(x_0\right)\right]. \tag{4.19}$$

Let

$$f(x) = f(x) - \sum_{k=0}^{m-1} \frac{1}{k!}\left(x - x_0\right)^k f^{(k)}\left(x_0\right). \tag{4.20}$$

Substituting (4.20) into (4.17) and using (4.19), we obtain the Caputo's fractional derivative as follows

$$D_{*x_0}^{q} f(x) = \frac{1}{\Gamma(m-q)} \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left\{\int_{x_0}^{x}\left[\frac{f(t) - \sum_{k=0}^{m-1}(1/k!)\left(t - x_0\right)^k f^{(k)}\left(x_0\right)}{(x-t)^{1+q-m}}\right]\mathrm{d}t\right\}. \tag{4.21}$$

The initial conditions, which are originally given as integer derivatives, are transformed as follows

$$F(k) = \begin{cases} \dfrac{1}{(k/\alpha)!}\left[\dfrac{\mathrm{d}^{k/\alpha} f(x)}{\mathrm{d}x^{k/\alpha}}\right]_{x=x_0}, & \text{for } k = 0, 1, 2, \ldots, (q\alpha - 1), \text{ if } \dfrac{k}{\alpha} \in Z^+ \\ 0 & \text{if } \dfrac{k}{\alpha} \notin Z^+ \end{cases} \tag{4.22}$$

where q is the fractional order derivative. Using (4.17) and (4.18), we have the following results [64]

Theorem 4.2.1. *If* $f(x) = g(x) \pm h(x)$*, then* $F(k) = G(k) \pm H(k)$*.*

Theorem 4.2.2. *If* $f(x) = g(x)h(x)$*, then* $F(k) = \sum_{l=0}^{k} G(l)H(k-l)$*.*

Theorem 4.2.3. *If* $f(x) = g_1(x)g_2(x)\ldots g_{n-1}(x)g_n(x)$*, then*

$F(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_1\left(k_1\right) G_2\left(k_2 - k_1\right) \ldots G_{n-1}\left(k_{n-1} - k_{n-2}\right) G_n\left(k - k_{n-1}\right)$

Theorem 4.2.4. *If* $f(x) = (x - x_0)^p$*, then* $F(k) = \delta(k - \alpha p)$ *where* $\delta(k) = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{if } k \neq 0. \end{cases}$

Theorem 4.2.5. *If* $f(x) = D_{x_0}^{q}[g(x)]$*, then* $F(k) = \frac{\Gamma(q+1+k/\alpha)}{\Gamma(1+k/\alpha)} G(k + \alpha q)$*.*

Theorem 4.2.6. *If the product of the general form of fractional derivatives,*

$$f(x) = \frac{\mathrm{d}^{q_1}}{\mathrm{d}x^{q_1}}\left[g_1(x)\right] \frac{\mathrm{d}^{q_2}}{\mathrm{d}x^{q_2}}\left[g_2(x)\right] \cdots \frac{\mathrm{d}^{q_{n-1}}}{\mathrm{d}x^{q_{n-1}}}\left[g_{n-1}(x)\right] \frac{\mathrm{d}^{q_n}}{\mathrm{d}x^{q_n}}\left[g_n(x)\right],$$

then

$$F(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\frac{\Gamma\left(q_1+1+k_1/\alpha\right)}{\Gamma\left(1+k_1/\alpha\right)}\frac{\Gamma\left[q_2+1+\left(k_2-k_1\right)/\alpha\right]}{\Gamma\left[1+\left(k_2-k_1\right)/\alpha\right]}$$
$$\cdots\frac{\Gamma\left[q_{n-1}+1+\left(k_{n-1}-k_{n-2}\right)/\alpha\right]}{\Gamma\left[1+\left(k_{n-1}-k_{n-2}\right)/\alpha\right]}\frac{\Gamma\left[q_n+1+\left(k-k_{n-1}\right)/\alpha\right]}{\Gamma\left[1+\left(k-k_{n-1}\right)/\alpha\right]}G_1\left(k_1+\alpha q_1\right)$$
$$\times G_2\left(k_2-k_1+\alpha q_2\right)\cdots G_{n-1}\left(k_{n-1}-k_{n-2}+\alpha q_{n-1}\right)$$
$$\times G_n\left(k-k_{n-1}+\alpha q_n\right). \tag{4.23}$$

where $\alpha q_i \in Z^+$ *for* $i=1,2,3,\dots n$.

Example 4.1. *We consider the system of fractional differential equations [64]*

$$\begin{aligned} D_*^{\beta}x(t) &= x(t)+y(t) \\ D_*^{\gamma}y(t) &= -x(t)+y(t), \end{aligned} \tag{4.24}$$

subject to the initial conditions

$$x(0)=0, \quad y(0)=1. \tag{4.25}$$

Applying Theorem 1 and Theorem 5, we transform the system (4.24)

$$\begin{aligned} X\left(k+\beta\alpha_1\right) &= \tfrac{\Gamma(1+k/\alpha_1)}{\Gamma(\beta+1+k/\alpha_1)}[X(k)+Y(k)], \\ Y\left(k+\gamma\alpha_2\right) &= \tfrac{\Gamma(1+k/\alpha_2)}{\Gamma(\gamma+1+k/\alpha_2)}[-X(k)+Y(k)], \end{aligned} \tag{4.26}$$

where α_1 is the unknown values of the fraction β and α_2 is the unknown values of the fraction γ. Transforming the initial conditions in (4.25), we get

$$\begin{aligned} &X(k)=0 \quad \text{for } k=0,1,\dots,\beta\alpha_1-1, \\ &Y(k)=0 \quad \text{for } k=1,\dots,\gamma\alpha_2-1, \\ &Y(0)=1. \end{aligned} \tag{4.27}$$

We calculate $X(k)$ and $Y(k)$ using (4.26) and (4.27) up to $k=10$, with $\beta=1$ and $\gamma=1$. Then, using (4.18), we obtain $x(t)$ and $y(t)$ as follows

$$\begin{aligned} x(t) =& t+t^2+\frac{t^3}{3}-\frac{t^5}{30}-\frac{t^6}{90}-\frac{t^7}{630}+\frac{t^9}{22680}+\frac{t^{10}}{113400}+\cdots \\ y(t) =& 1+t-\frac{t^3}{3}-\frac{t^4}{6}-\frac{t^5}{30}+\frac{t^7}{630}+\frac{t^8}{2520}+\frac{t^9}{22680}+\cdots \end{aligned} \tag{4.28}$$

Example 4.2. *We consider the following system [64]*

$$\begin{aligned} D_*^{1.3}y_1 &= y_1+y_2^2, \\ D_*^{2.4}y_2 &= y_1+5y_2, \end{aligned} \tag{4.29}$$

subject to the initial conditions

$$y_1(0)=0, \quad y_1'(0)=1, \quad y_2(0)=0, \quad y_2'(0)=1, \quad y_2''(0)=1. \tag{4.30}$$

Transforming the system (4.29), we obtain

$$\begin{aligned} Y_1(k+13) &= \tfrac{\Gamma(1+k/10)}{\Gamma(1.3+1+k/10)}\left[Y_1(k)+\textstyle\sum_{k_1=0}^{k} Y_2(k_1)\,Y_2(k-k_1)\right], \\ Y_2(k+24) &= \tfrac{\Gamma(1+k/10)}{\Gamma(2.4+1+k/10)}\left[Y_1(k)+5Y_2(k)\right]. \end{aligned} \tag{4.31}$$

Transforming the initial conditions in (4.30), we have

$$\begin{aligned} &Y_1(k)=0, \quad \text{for } k=0,1,\ldots,9,11,12 \\ &Y_1(10)=1, \\ &Y_2(k)=0, \quad \text{for } k=0,\ldots,9,11,\ldots,19,21,22,23, \\ &Y_2(10)=1, \quad Y_2(20)=\tfrac{1}{2}. \end{aligned} \tag{4.32}$$

Hence we obtain the series solution $y_1(k)$ and $y_2(k)$ using (4.30) and (4.32) with iterations up to $k=50$ as follows

$$\begin{aligned} y_1(t) &= t+\tfrac{t^{23/10}}{\Gamma(\frac{33}{10})}+\tfrac{2t^{33/10}}{\Gamma(\frac{43}{10})}+\tfrac{t^{18/5}}{\Gamma(\frac{23}{5})}+\tfrac{6t^{43/10}}{\Gamma(\frac{53}{10})}+\tfrac{2t^{23/5}}{\Gamma(\frac{28}{5})}+\cdots, \\ y_2(t) &= t+\tfrac{t^2}{2}+\tfrac{6t^{17/15}}{\Gamma(\frac{22}{5})}+\tfrac{5t^{22/5}}{\Gamma(\frac{27}{5})}+\tfrac{t^{47/10}}{\Gamma(\frac{57}{10})}+\cdots \end{aligned} \tag{4.33}$$

4.3 Padé Approximation

Padé approximation $P_M^N(x) = [M/N]$ is a procedure that expands a function as a ratio of two power series. It provides a better approximation of a function than Taylor series especially when it contain poles.

$$P_M^N(x) = \frac{\sum_{n=0}^{N} a_n x^n}{\sum_{n=0}^{M} b_n x^n}. \tag{4.34}$$

We can normalize the approximation by using $b_0 = 1$ to generalize the Taylor series expansion. Hence

$$T_{M+N}(x) = \sum_{n=0}^{M+N} c_n x^n. \tag{4.35}$$

The coefficients of Padé approximation can be found from Taylor series expansion

$$c_0 + c_1 x + c_2 x^2 + \ldots = \frac{a_0 + a_1 x + a_2 x^2 + \ldots}{1 + b_1 x + b_2 x^2 + \ldots}. \tag{4.36}$$

Equivalently, we have

$$\begin{aligned}
a_0 &= c_0, \\
a_1 &= c_1 + c_0 b_1, \\
a_2 &= c_2 + c_1 b_1 + c_0 b_2, \\
a_3 &= c_3 + c_2 b_1 + c_1 b_2 + c_0 b_3, \\
&\vdots
\end{aligned} \tag{4.37}$$

In order to solve the system, we need to specify the degree of the numerator and the denominator to be N and M, respectively. Moreover, the degree of the truncated Taylor series expansion is determined to be $M + N$.

Example 4.3. *The [2/2] Padé approximation of* $\sin x$ *is*

$$\sin x \approx \frac{A_0 + A_1 x + A_2 x^2}{B_0 + B_1 x + B_2 x^2} = x - \frac{x^3}{6} + O\left(x^5\right). \tag{4.38}$$

Then, by letting $B_0 = 1$, we have

$$\begin{aligned}
A_0 + A_1 x + A_2 x^2 &= \left(1 + B_1 x + B_2 x^2\right)\left(x - \frac{x^3}{6} + O\left(x^5\right)\right) \\
&= x + B_1 x^2 + B_2 x^3 - \frac{x^3}{6} - B_1 \frac{x^4}{6} + O\left(x^5\right).
\end{aligned} \tag{4.39}$$

Equating the coefficients, we obtain that

$$A_0 = 0, \quad A_1 = 1, \quad A_2 = B_1,$$

$$B_2 = \frac{1}{6}, \quad B_1 = 0 = A_2,$$

Substituting the values back in the [2/2] Padé approximation yields

$$P_2^2(x) = \frac{x}{1 + \frac{x^2}{6}}. \tag{4.40}$$

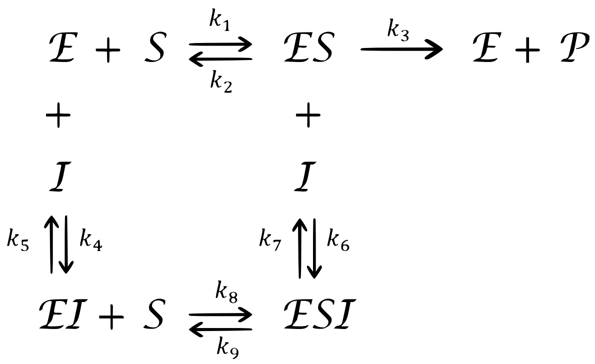
This technique helps in minimizing the error and enables finding a better approximation than Taylor approximation. Laplace Decomposition Method and the Differential Transformation Method generate accurate solutions only over small domains. Therefore, when dealing with large domains, combining the two methods with Padé approximation prevents obtaining a divergent series solution [62].

Chapter 5: Applications

5.1 Enzyme Inhibitor Reaction Model

Mathematical modeling for enzyme inhibitor systems plays an important role in system biology. The model consists of seven chemical components and thirteen chemical kinetic constants [58].

Figure 5.1: A complex Enzyme Inhibitor Reaction Model



The components S stands for substrate, E for enzyme, P for product and I for inhibition. In addition, EI, ES and ESI represent the complex intermediate species. The seven variables will be denoted by S=[S], E=[E], P=[P], I=[I], C_1=[ES], C_2=[EI] and C_3=[ESI]. The system of the nonlinear differential equations based on the mass action law can be written as:

$$
\begin{aligned}
\frac{dE}{dt} &= -k_1ES + k_2C_1 + k_3C_1 - k_4EI + k_5C_2S, \\
\frac{dS}{dt} &= -k_1ES + k_2C_1 + k_4EI - k_5C_2S - k_8C_2S + k_9c_3 \\
\frac{dC_1}{dt} &= k_1ES - k_2C_1 - k_3C_1 - k_6C_1I + k_7C_3, \\
\frac{dP}{dt} &= k_3C_1, \\
\frac{dI}{dt} &= -k_4EI + k_5C_2S - k_6C_1I + k_7C_3, \\
\frac{dC_2}{dt} &= k_4EI - k_5C_2S - k_8C_2S + k_9C_3, \\
\frac{dC_3}{dt} &= k_6C_1I - k_7C_3 + k_8C_2S - k_9C_3,
\end{aligned}
\tag{5.1}
$$

We will take the complex enzyme inhibitor reaction model (5.1) and apply fractional order system since it would be more realistic to take memory and hereditary properties into account. It will result in having the following fractional differential equations.

$$
\begin{aligned}
D_t^{\alpha} E &= -k_1 ES + k_2 C_1 + k_3 C_1 - k_4 EI + k_5 C_2 S, \\
D_t^{\alpha} S &= -k_1 ES + k_2 C_1 + k_4 EI - k_5 C_2 S - k_8 C_2 S + k_9 c_3, \\
D_t^{\alpha} I &= -k_4 EI + k_5 C_2 S - k_6 C_1 I + k_7 C_3, \\
D_t^{\alpha} P &= k_3 C_1 \\
D_t^{\alpha} C_1 &= k_1 ES - k_2 C_1 - k_3 C_1 - k_6 C_1 I + k_7 C_3, \\
D_t^{\alpha} C_2 &= k_4 EI - k_5 C_2 S - k_8 C_2 S + k_9 C_3, \\
D_t^{\alpha} C_3 &= k_6 C_1 I - k_7 C_3 + k_8 C_2 S - k_9 C_3,
\end{aligned}
\tag{5.2}
$$

where $0 < \alpha \leq 1$.

5.1.1 Solution Via Laplace Decomposition Method The system (5.2) will be solved analytically using Laplace Adomian Decomposition Method subject to the following initial conditions

$$
E(0) = e_0,\ S(0) = s_0,\ I(0) = i_0 \ and \ C_1(0) = C_2(0) = C_3(0) = P(0) = 0, \tag{5.3}
$$

and the parameters $K_1, K_2, \ldots, K_9$ represent the rate constants. We start by applying Laplace to both sides of equation (5.2), we obtain

$$
\begin{aligned}
\mathcal{L}[D_t^{\alpha} E] &= \mathcal{L}[-k_1 ES + k_2 C_1 + k_3 C_1 - k_4 EI + k_5 C_2 S], \\
\mathcal{L}[D_t^{\alpha} S] &= \mathcal{L}[-k_1 ES + k_2 C_1 + k_4 EI - k_5 C_2 S - k_8 C_2 S + k_9 c_3], \\
\mathcal{L}[D_t^{\alpha} I] &= \mathcal{L}[-k_4 EI + k_5 C_2 S - k_6 C_1 I + k_7 C_3], \\
\mathcal{L}[D_t^{\alpha} P] &= \mathcal{L}[k_3 C_1], \\
\mathcal{L}[D_t^{\alpha} C_1] &= \mathcal{L}[k_1 ES - k_2 C_1 - k_3 C_1 - k_6 C_1 I + k_7 C_3], \\
\mathcal{L}[D_t^{\alpha} C_2] &= \mathcal{L}[k_4 EI - k_5 C_2 S - k_8 C_2 S + k_9 C_3], \\
\mathcal{L}[D_t^{\alpha} C_3] &= \mathcal{L}[k_6 C_1 I - k_7 C_3 + k_8 C_2 S - k_9 C_3],
\end{aligned}
\tag{5.4}
$$

which implies that for $\alpha = 1$

$$
\begin{aligned}
s^{\alpha}\mathcal{L}[E] - s^{\alpha-1}E(0) &= \mathcal{L}[-k_1 ES + k_2 C_1 + k_3 C_1 - k_4 EI + k_5 C_2 S], \\
s^{\alpha}\mathcal{L}S] - s^{\alpha-1}S(0) &= \mathcal{L}[-k_1 ES + k_2 C_1 + k_4 EI - k_5 C_2 S - k_8 C_2 S + k_9 c_3], \\
s^{\alpha}\mathcal{L}[I] - s^{\alpha-1}I(0) &= \mathcal{L}[-k_4 EI + k_5 C_2 S - k_6 C_1 I + k_7 C_3], \\
s^{\alpha}\mathcal{L}[P] - s^{\alpha-1}P(0) &= \mathcal{L}[k_3 C_1], \\
s^{\alpha}\mathcal{L}[C_1] - s^{\alpha-1}C_1(0) &= \mathcal{L}[k_1 ES - k_2 C_1 - k_3 C_1 - k_6 C_1 I + k_7 C_3], \\
s^{\alpha}\mathcal{L}[C_2] - s^{\alpha-1}C_2(0) &= \mathcal{L}[k_4 EI - k_5 C_2 S - k_8 C_2 S + k_9 C_3], \\
s^{\alpha}\mathcal{L}[C_3] - s^{\alpha-1}C_3(0) &= \mathcal{L}[k_6 C_1 I - k_7 C_3 + k_8 C_2 S - k_9 C_3].
\end{aligned}
\tag{5.5}
$$

Taking the inverse Laplace and using the initial conditions

$$E(0) = 0.1, S(0) = 0.2, I(0) = 0.01, P(0) = C_1(0) = C_2(0) = C_3(0) = 0, \quad (5.6)$$

we obtain the following

$$\begin{aligned}
E(t) &= E_0 + \mathcal{L}^{-1}[\frac{1}{s}\mathcal{L}[-k_1ES + k_2C_1 + k_3C_1 - k_4EI + k_5C_2S]],\\
S(t) &= S_0 + \mathcal{L}^{-1}[\frac{1}{s}\mathcal{L}[-k_1ES + k_2C_1 + k_4EI - k_5C_2S - k_8C_2S + k_9c_3]],\\
I(t) &= I_0 + \mathcal{L}^{-1}[\frac{1}{s}\mathcal{L}[-k_4EI + k_5C_2S - k_6C_1I + k_7C_3]],\\
P(t) &= \mathcal{L}^{-1}[\frac{1}{s}\mathcal{L}[k_3C_1]], \qquad (5.7)\\
C_1(t) &= \mathcal{L}^{-1}[\frac{1}{s}\mathcal{L}[k_1ES - k_2C_1 - k_3C_1 - k_6C_1I + k_7C_3]],\\
C_2(t) &= \mathcal{L}^{-1}[\frac{1}{s}\mathcal{L}[k_4EI - k_5C_2S - k_8C_2S + k_9C_3]],\\
C_3(t) &= \mathcal{L}^{-1}[\frac{1}{s}\mathcal{L}[k_6C_1I - k_7C_3 + k_8C_2S - k_9C_3]].
\end{aligned}$$

Assuming the solutions $E(t), S(t), I(t), P(t), C_1(t), C_2(t)$ and $C_3(t)$ are infinite power series [59], we get:

$$\begin{aligned}
&E(t) = \textstyle\sum_{n=0}^{\infty} E_n(t), \quad S(t) = \sum_{n=0}^{\infty} S_n(t), \quad I(t) = \sum_{n=0}^{\infty} I_n(t), \quad E(t) = \sum_{n=0}^{\infty} E_n(t),\\
&C_1(t) = \textstyle\sum_{n=0}^{\infty} C_{1,n}(t), \quad C_2(t) = \sum_{n=0}^{\infty} C_{2,n}(t), \quad C_3(t) = \sum_{n=0}^{\infty} C_{3,n}(t),
\end{aligned} \qquad (5.8)$$

The nonlinear terms $ES(t), EI(t), C_1(t)\ and\ C_2(t)$ are obtained from Adomian polynomials and are given by

$$\begin{aligned}
ES(t) &= \textstyle\sum_{n=0}^{\infty} A_{1,n} = \frac{1}{n!}\left(\frac{d}{d\lambda}\right)^n \left(\sum_{k=0}^{n} \lambda^k E_k \sum_{k=0}^{n} \lambda^k S_k\right)\big|_{\lambda=0},\\
EI(t) &= \textstyle\sum_{n=0}^{\infty} A_{2,n} = \frac{1}{n!}\left(\frac{d}{d\lambda}\right)^n \left(\sum_{k=0}^{n} \lambda^k E_k \sum_{k=0}^{n} \lambda^k I_k\right)\big|_{\lambda=0},\\
C_1(t) &= \textstyle\sum_{n=0}^{\infty} A_{3,n} = \frac{1}{n!}\left(\frac{d}{d\lambda}\right)^n \left(\sum_{k=0}^{n} \lambda^k C_{1,k} \sum_{k=0}^{n} \lambda^k S_k\right)\big|_{\lambda=0},\\
C_2(t) &= \textstyle\sum_{n=0}^{\infty} A_{4,n} = \frac{1}{n!}\left(\frac{d}{d\lambda}\right)^n \left(\sum_{k=0}^{n} \lambda^k C_{2,k} \sum_{k=0}^{n} \lambda^k I_k\right)\big|_{\lambda=0}.
\end{aligned} \qquad (5.9)$$

5.2 Numerical Results

In this section, we solve the system using Laplace Decomposition Method and The Differential Transformation Method. We discuss two examples where the nonlinear reaction system (5.2) is solved for two different sets of parameters and initial conditions.

Example 5.1. *We solve the system* (5.2) *for the integer derivative* $\alpha = 1$ *subject to the following initial conditions*

$$E(0) = 0.1, S(0) = 0.2, I(0) = 0.01, P(0) = C_1(0) = C_2(0) = C_3(0) = 0, \quad (5.10)$$

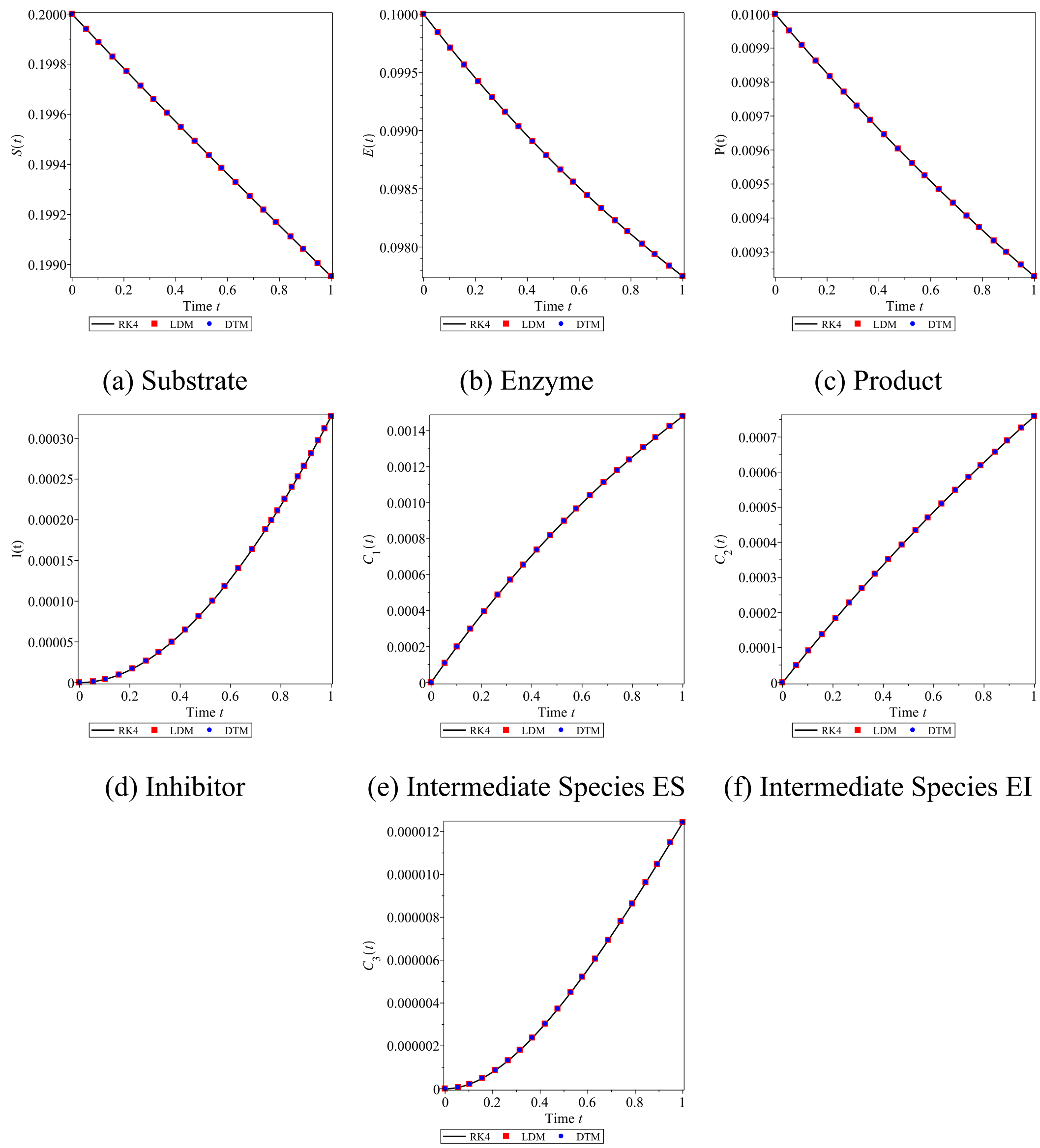
and the parameters k_i for $i = 1, 2, 3...9$ are

$$k_1 = 0.1, k_2 = 2, k_3 = 0.4, k_4 = 0.9, k_5 = 1, k_6 = 0.4, k_7 = 0.9, k_8 = 0.2, k_9 = 0.5 \tag{5.11}$$

The Laplace Decomposition Method and the Differential Transformation Method give identical solutions when $\alpha = 1$. The analytical expressions of the nonlinear biochemical reaction system are as follow

$$\begin{aligned}
E(t) &= 0.1 - 0.0029t + 0.00077805t^2 - 0.000152t^3 + 0.000024t^4 - 0.000003t^5,\\
S(t) &= 0.2 - 0.0011t + 0.000073t^2 - 0.000024t^3 + 0.000004t^4 - 0.0000003t^5,\\
I(t) &= 0.01 - 0.0009t + 0.00014t^2 - 0.000011t^3 - 0.000001t^4 + 0.0000007t^5,\\
P(t) &= 0.0004t^2 - 0.000085t^3 + 0.000014t^4 - 0.000002t^5 + 0.0000003t^6,\\
C_1(t) &= 0.002t - 0.000639t^2 + 0.000141t^3 - 0.000025t^4 + 0.000004t^5,\\
C_2(t) &= 0.0009t - 0.00016155t^2 + 0.000024t^3 - 0.000004t^4 + 0.000001t^5,\\
C_3(t) &= 0.000022t^2 - 0.000014t^3 + 0.000005t^4 - 0.000002t^5 + 0.000004t^6.
\end{aligned} \tag{5.12}$$

Also, the solutions strongly agree with the results obtained from RK4. We notice that the analytical solution is an exact fit to the numerical solution. Moreover, when solving the nonlinear fractional system (5.2) with respect to the initial conditions (5.11) using LDM and DTM where $\alpha = 1$, we noticed that the solutions obtained from these methods were identical for all variables. The analytical solution curves are presented in Figure 5.2 (a-g).



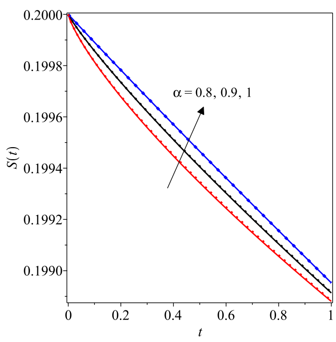
(a) Substrate (b) Enzyme (c) Product

(d) Inhibitor (e) Intermediate Species ES (f) Intermediate Species EI

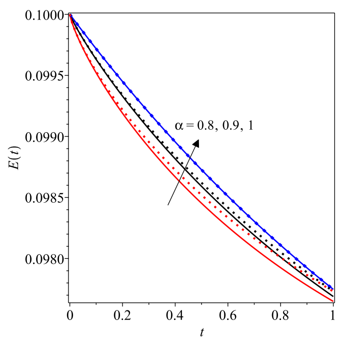
(g) Intermediate Species ESI

Figure 5.2: Analytical and numerical solution curves of the system (5.2) when $\alpha = 1$.

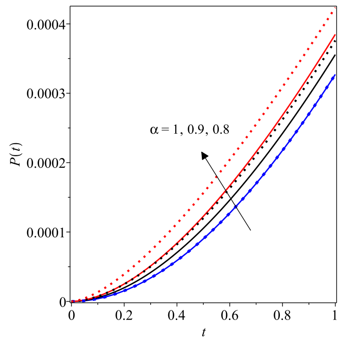
The considered system (5.2) is also solved for the fractional derivatives when $\alpha = 0.9$ and $\alpha = 0.8$. Figure 5.3 (a-g) present the analytical solution curves for the fractional values of α. One can clearly see strong agreements between LDM and DTM.



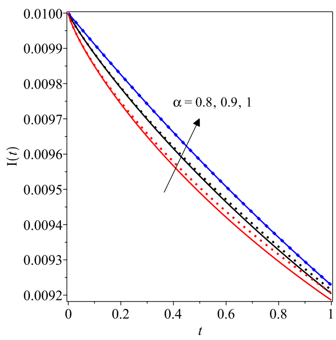
(a) Substrate



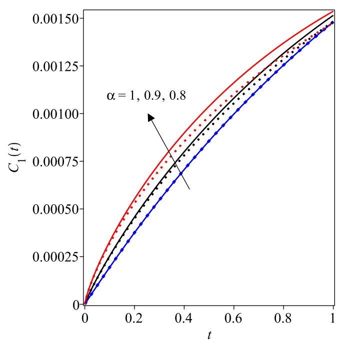
(b) Enzyme



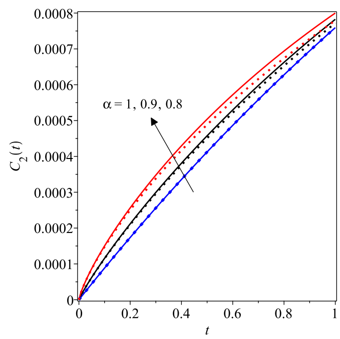
(c) Product



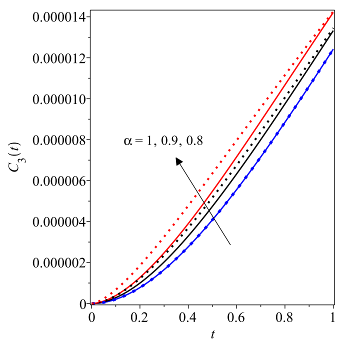
(d) Inhibitor



(e) Intermediate Species ES



(f) Intermediate Species EI



(g) Intermediate Species ESI

Figure 5.3: Analytical solution curves of the system (5.2) when $\alpha = 1$, $\alpha = 0.9$ and $\alpha = 0.8$. The dotted and solid curves correspond to the solutions of DTM and the LDM respectively.

The analytical expression solution of the nonlinear biochemical reaction system (5.2) when $\alpha = 0.9$ is

$$\begin{aligned}
E(t) &= 0.1 - 0.003015t^{\frac{9}{10}} + 0.000852t^{\frac{19}{10}} - 0.000172t^{\frac{29}{10}} + 0.000028t^{\frac{39}{10}} - 0.000004t^{\frac{49}{10}} \\
S(t) &= 0.2 - 0.001144t^{\frac{9}{10}} + 0.00008t^{\frac{19}{10}} - 0.000027t^{\frac{29}{10}} + 0.000004t^{\frac{39}{10}} - 0.0000003t^{\frac{49}{10}} \\
I(t) &= 0.01 - 0.000936t^{\frac{9}{10}} + 0.000153t^{\frac{19}{10}} - 0.000012t^{\frac{29}{10}} - 0.000001t^{\frac{39}{10}} + 0.0000009t^{\frac{49}{10}} \\
P(t) &= 0.000438t^{\frac{19}{10}} - 0.000096t^{\frac{29}{10}} + 0.000016t^{\frac{39}{10}} - 0.000002t^{\frac{49}{10}} + 0.0000003t^{\frac{59}{10}} \\
C_1(t) &= 0.00208t^{\frac{9}{10}} - 0.000699t^{\frac{19}{10}} + 0.00016t^{\frac{29}{10}} - 0.000029t^{\frac{39}{10}} + 0.000005t^{\frac{49}{10}} \\
C_2(t) &= 0.000936t^{\frac{9}{10}} - 0.000177t^{\frac{19}{10}} + 0.000028t^{\frac{29}{10}} - 0.000005t^{\frac{39}{10}} + 0.000001t^{\frac{49}{10}} \\
C_3(t) &= 0.000024t^{\frac{19}{10}} - 0.000016t^{\frac{29}{10}} + 0.000006t^{\frac{39}{10}} - 0.000002t^{\frac{49}{10}} + 0.000005t^{\frac{59}{10}}
\end{aligned} \tag{5.13}$$

and for $\alpha = 0.8$, the solution becomes

$$\begin{aligned}
E(t) &= 0.1 - 0.003114t^{\frac{4}{5}} + 0.000929t^{\frac{9}{5}} - 0.000194t^{\frac{14}{5}} + 0.000032t^{\frac{19}{5}} - 0.000005t^{\frac{24}{5}} \\
S(t) &= 0.2 - 0.001181t^{\frac{4}{5}} + 0.000087t^{\frac{9}{5}} - 0.00003t^{\frac{14}{5}} + 0.000005t^{\frac{19}{5}} - 0.0000004t^{\frac{24}{5}} \\
I(t) &= 0.01 - 0.000966t^{\frac{4}{5}} + 0.000166t^{9}5 - 0.000014t^{\frac{14}{5}} - 0.0000016t^{\frac{19}{5}} + 0.000001t^{\frac{24}{5}} \\
P(t) &= 0.000477t^{\frac{9}{5}} - 0.000109t^{\frac{14}{5}} + 0.000019t^{\frac{19}{5}} - 0.000003t^{\frac{24}{5}} \\
C_1(t) &= 0.002147t^{\frac{4}{5}} - 0.000762t^{\frac{9}{5}} + 0.00018t^{\frac{14}{5}} - 0.000034t^{\frac{19}{5}} + 0.000006t^{\frac{24}{5}} \\
C_2(t) &= 0.000966t^{\frac{4}{5}} - 0.000193t^{\frac{9}{5}} + 0.000031t^{\frac{14}{5}} - 0.000001t^{\frac{19}{5}} + 0.000001t^{\frac{24}{5}} \\
C_3(t) &= 0.000026t^{\frac{9}{5}} - 0.000017t^{\frac{14}{5}} + 0.000007t^{\frac{19}{5}} - 0.000002t^{\frac{24}{5}} + 0.000001t^{\frac{29}{5}}
\end{aligned} \tag{5.14}$$

The actual variations for the fractional cases between the Laplace Decomposition Method and Differential Transformation Method when $\alpha = 0.9$ and $\alpha = 0.8$ are shown in Table 5.2 and 5.1.

Table 5.1: Maximum variation between LDM and DTM computed concentrations when α = 0.8

Concentration	Max difference	Occurred at x
Enzyme	0.0000843	0.850
Substrate	0.0000049	0.600
Inhibition	0.0000206	1.000
Production	0.0000408	0.825
Complex ES	0.0000643	0.825
Complex EI	0.0000208	1.950
Complex ESI	0.0000009	0.450

Table 5.2: Maximum variation between LDM and DTM computed concentrations when α = 0.9

Concentration	Max difference	Occurred at x
Enzyme	0.0000424	1.000
Substrate	0.0000024	0.007
Inhibition	0.0000103	1.000
Production	0.0000205	0.925
Complex ES	0.0000323	0.925
Complex EI	0.0000105	1.000
Complex ESI	0.0000004	0.525

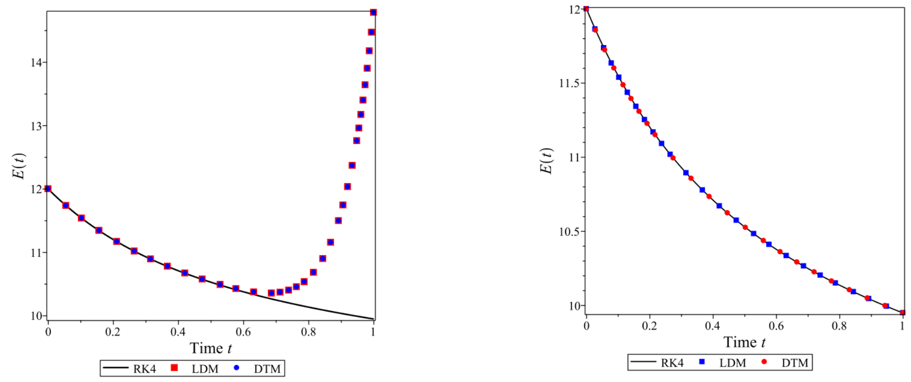
Example 5.2. *We solve the system* (5.2) *subject to the following initial conditions*

$$E(0) = 12, S(0) = 5, I(0) = 2, P(0) = C_1(0) = C_2(0) = C_3(0) = 0, \tag{5.15}$$

and where the $k_i's$ are given by

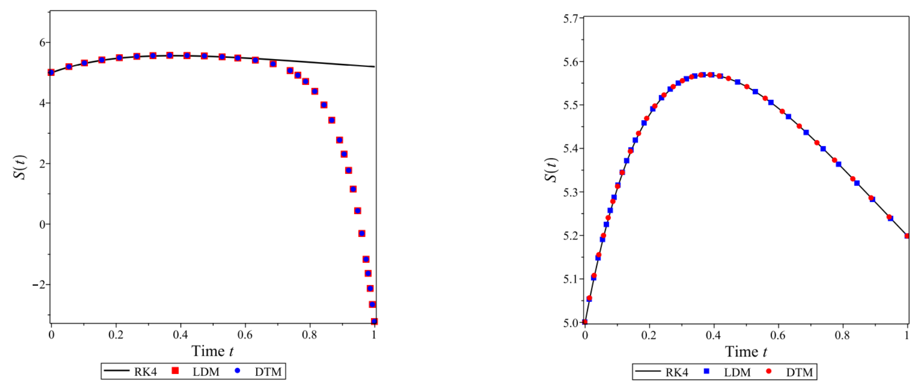
$$k_1 = 0.1, k_2 = 0.2, k_3 = 0.04, k_4 = 0.19,\ k_5 = 0.1, k_6 = 0.4, k_7 = 0.09, k_8 = 0.22, k_9 = 0.05, \tag{5.16}$$

The nonlinear system (5.2) is solved with respect to parameters (5.16) and subject to initial conditions (5.15) using the two methods. When $\alpha = 1$, the analytical series solutions for LDM and DTM are identical. However, the series solution diverges over a small time domain. We can control this divergence using Padé approximation. The figures 5.4 and 5.5 show how Padé approximation helps overcoming the divergence obstacle.



(a) Divergence Solution (b) Convergence Solution

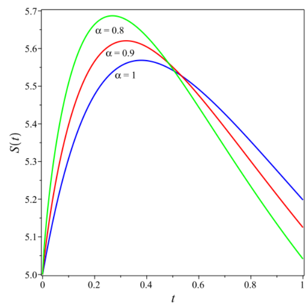
Figure 5.4: Analytical concentration curve of Enzyme E(t) for the system (5.2) with initial conditions (5.15) and parameters (5.16)



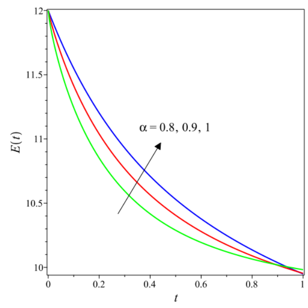
(a) Divergence Solution (b) Convergence Solution

Figure 5.5: Analytical concentration curve of Substrate S(t) for the system (5.2) with initial conditions (5.15) and parameters (5.16).

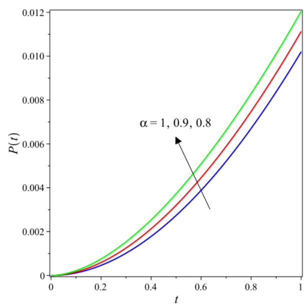
The nonlinear system (5.2) is also solved when $\alpha = 0.8$ and $\alpha = 0.9$ with respect to parameters (5.16) and subject to initial conditions (5.15) using LDM and DTM. Figure (5.6) (a-g) shows how these methods are in strong agreement for the fractional cases of α for all seven variables.



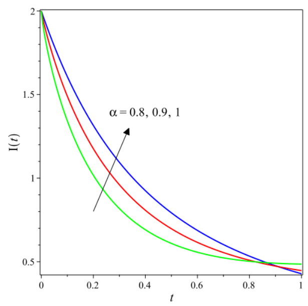
(a) Substrate



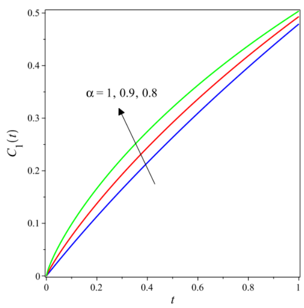
(b) Enzyme



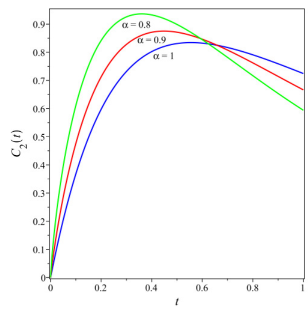
(c) Product



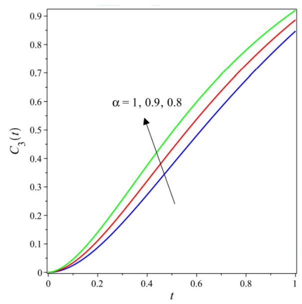
(d) Inhibitor



(e) Intermediate Species ES



(f) Intermediate Species EI



(g) Intermediate Species ESI

Figure 5.6: Analytical and numerical solution curves of the system (5.2) when $\alpha = 1$, $\alpha = 0.9$ and $\alpha = 0.8$.

5.3 COVID-19 Model

COVID-19 is a worldwide pandemic that was first discovered in December 2019 in Wuhan province of China. It started emerging worldwide in the beginning of 2020 caused by the SARS-CoV-2 virus and has been spreading globally since then. Scientists are learning more about this virus every day since it is so new. It can cause serious illness and even death, despite the fact that most people who have it have moderate symptoms. Mathematical models are important for understanding how an infection behaves as it reaches a population and determining if it can be eradicated or not. Based on some research done, we conclude that human to human contact could be the most important cause of the outbreak of COVID-19 [60]. Hence, isolating infected people was taken into account to potentially reduce the spread of the disease. Based on this, A model is constructed based on five compartments: susceptible, exposed, infected, isolated and recovered.

Table 5.3: Parameters and description

Parameters	Value[61]	Description
S	1.0	Susceptible population
E	1.0	Exposed population
I	0.00002	Infected population
Q	0.000095	Isolated population
R	0.000095	Recovered population
β	0.000002	Rate at which susceptible population moves to infected and exposed class
π	0.00567	Rate at which exposed population moves to infected one
γ	0.000095	Presents the rate at which exposed people take onside as isolated
σ	0.0028404	Rate at which infected people were added to isolated individual
θ	0.000095	Rate at which isolated persons recovered
μ	0.000001	Natural death rate plus disease-related death rate

The parameters used in our model are presented in Table (5.3) that contributed in controlling the infection. In this section, we study a nonlinear fractional model of COVID-19. We will use The Laplace Decomposition Method and the Differential Transformation Method. It is a powerful technique used to get an approximate solution of the considered system. Furthermore, The two methods presented can be coupled with Padé approximation to avoid divergent solution over large intervals and to obtain convergent series solutions [62]. In addition, a comparison will be made with fourth-order Runge-Kutta method.

The epidemic mathematical model with respect to the parameters and variables in Table (5.3) is presented as follows[61]:

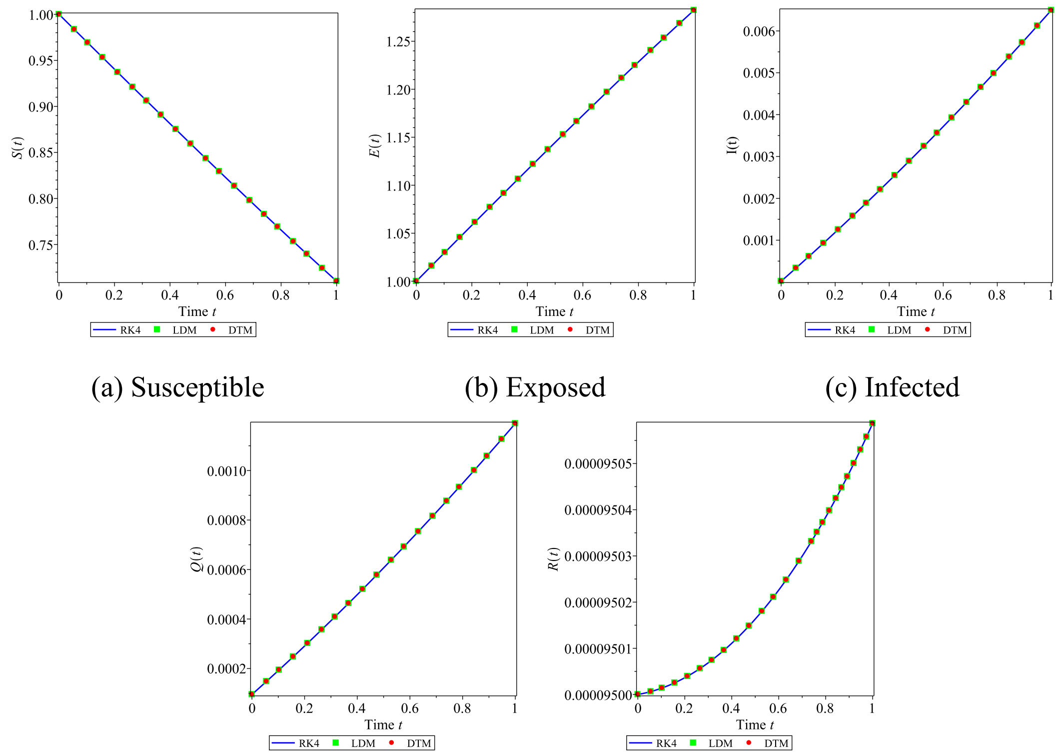
$$\begin{cases} \frac{dS(t)}{dt} = A - \mu S(t) - \beta(N)S(t)(E(t) + I(t)) \\ \frac{dE(t)}{dt} = \beta(N)S(t)(E(t) + I(t)) - \pi E(t) - (\mu + \gamma)E(t) \\ \frac{dI(t)}{dt} = \pi E(t) - \sigma I(t) - \mu I(t), \\ \frac{dQ(t)}{dt} = \gamma E(t) + \sigma I(t) - \theta Q(t) - \mu Q(t) \\ \frac{dR(t)}{dt} = \theta Q(t) - \mu R(t) \end{cases} \tag{5.17}$$

We apply fractional derivatives to the model to get:

$$\begin{cases} D_t^\alpha S(t) = \mu - \mu S(t) - \beta(N)S(t)(E(t) + I(t)) \\ D_t^\alpha E(t) = \beta(N)S(t)(E(t) + I(t)) - \pi E(t) - (\mu + \gamma)E(t) \\ D_t^\alpha I(t) = \pi E(t) - \sigma I(t) - \mu I(t), \\ D_t^\alpha Q(t) = \gamma E(t) + \sigma I(t) - \theta Q(t) - \mu Q(t) \\ D_t^\alpha R(t) = \theta Q(t) - \mu R(t) \end{cases} \tag{5.18}$$

With respect to the initial conditions and parameter presented in Table (5.3).

We solved the nonlinear fractional system (5.18) with respect to the initial conditions presented in Table (5.3) using LDM and DTM where $\alpha = 1$. We noticed that the solutions obtained from these methods were identical for all variables. Furthermore, they strongly agree with the fourth-order Runge-Kutta method. The analytical solution curves are presented in Figure (5.7) (a-e).



(a) Susceptible (b) Exposed (c) Infected

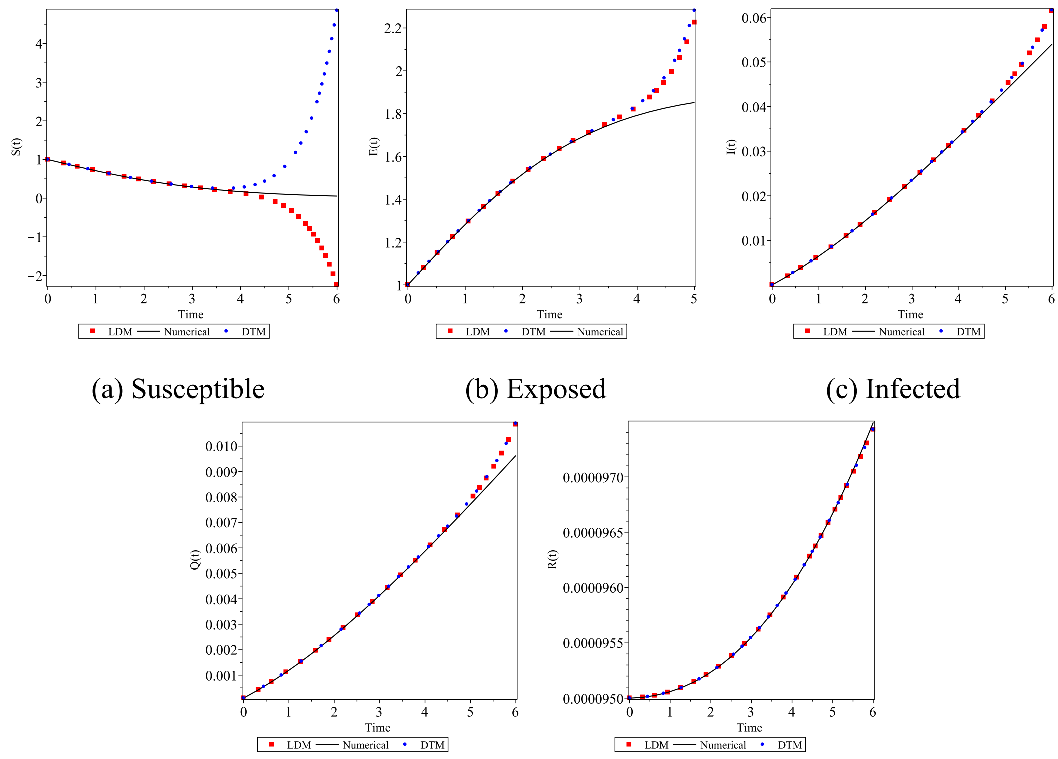
(d) Isolated (e) Recovered

Figure 5.7: Analytical solution curves of the system (5.18) when $\alpha = 1$, $\alpha = 0.9$ and $\alpha = 0.8$ where the dotted and solid curves correspond to the solutions of DTM and the LDM respectively.

The analytical expressions for the system (5.18) solved via LDM and DTM when $\alpha = 1$ is as follows:

$$
\begin{aligned}
S(t) &= 1 + 0.000011t^7 - 0.000316t^5 + 0.008860t^3 + 0.000143t^2 - 0.298594t \\
E(t) &= 1 + 0.000316t^5 + 0.000024t^4 - 0.008858t^3 - 0.001109t^2 + 0.291972t \\
I(t) &= 0.00002 - 0.000013t^4 - 2.872898 \times 10^{-6}t^3 + 0.000820t^2 + 0.005670t \\
Q(t) &= 0.000095 - 2.105717 \times 10^{-6}t^4 - 2.105717 \times 10^{-6}t^4 + 0.000147t^2 + 0.000950t \\
R(t) &= 0.000095 + 4.645267 \times 10^{-9}t^3 + 4.512726 \times 10^{-8}t^2 + 8.93 \times 10^{-9}t
\end{aligned}
\tag{5.19}
$$

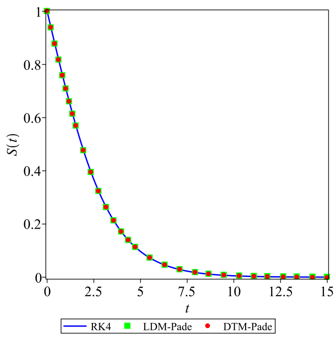
Over a larger time domain, we notice that the solution curves diverges over small time domain as shown in figure (5.8). However, the divergence is being controlled using $[5/5]$ Padé approximation for each analytical solution. Figure (5.8) shows the divergent solution curve of $S(t)$ obtained by RK4, LDM and DTM when $\alpha = 1$ while Figure (5.9) shows how using $[5/5]$ Padé approximation fixes that obstacle.



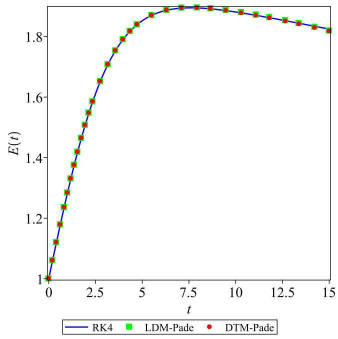
(a) Susceptible (b) Exposed (c) Infected

(d) Isolated (e) Recovered

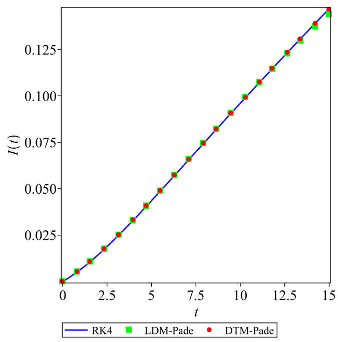
Figure 5.8: Divergent solution of the system (5.18) when $\alpha = 1$ obtained from RK4, LDM and DTM without using Padé approximation.



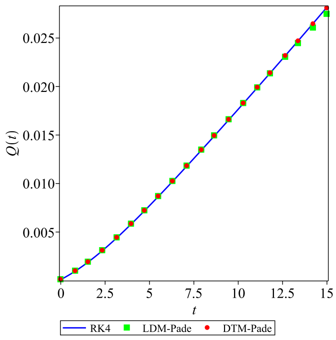
(a) Susceptible



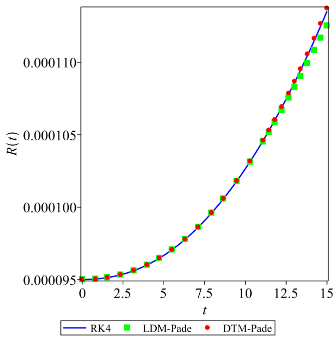
(b) Exposed



(c) Infected



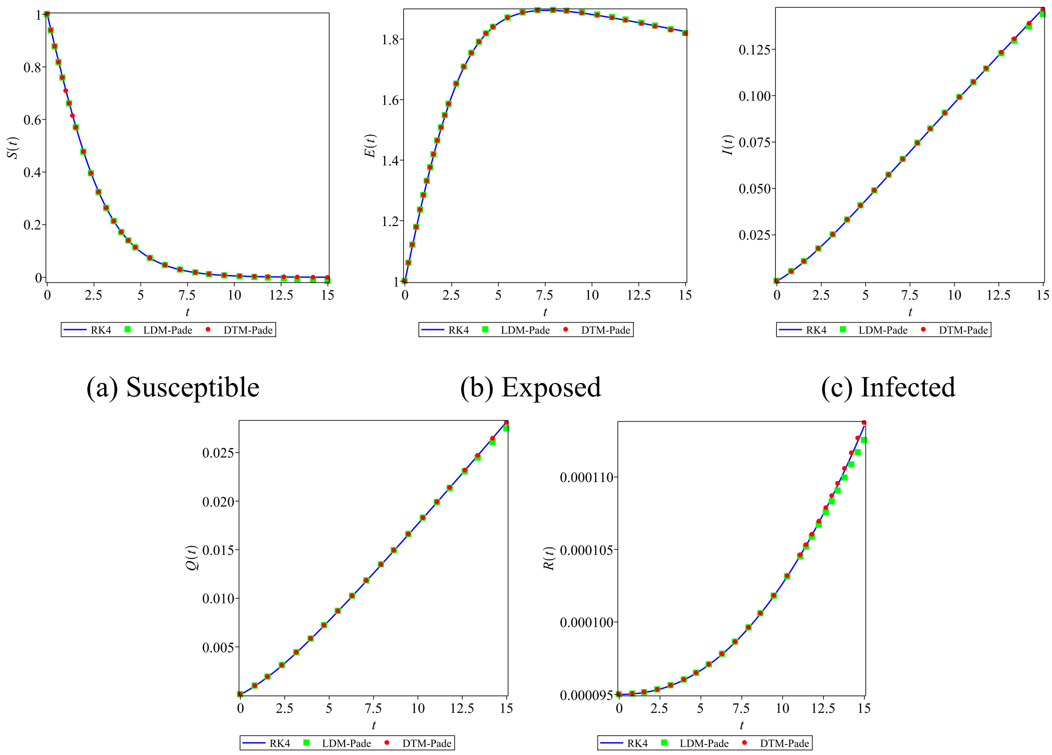
(d) Isolated



(e) Recovered

Figure 5.9: Convergence solution of (5.18) when $\alpha = 1$ after being coupled with Padé approximation

In order to obtain a convergence solution over a bigger time domain, a larger order of Padé approximation is needed. Figures (5.10) (a-e) show the solution curves when $0 \leq t \leq 15$ where we used [13/12] Padé approximation to get the convergence solution. Tables 5.4 and 5.5 show the maximum error in DTM and LDM. Its obvious that the error decreases as we couple Padé approximation to the two methods and it decreases further more as we increase the terms of the series solution and we couple the methods with Padé approximation.



(a) Susceptible (b) Exposed (c) Infected

(d) Isolated (e) Recovered

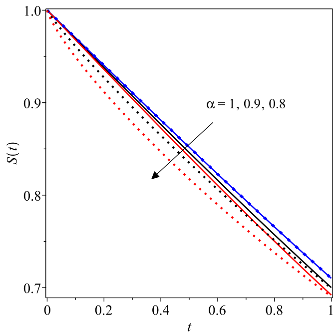
Figure 5.10: Analytical solution curve for the system (5.18) when $\alpha = 1$ and $0 \leq t \leq 15$ using [13/12] Padé approximation.

Table 5.4: Maximum error in DTM

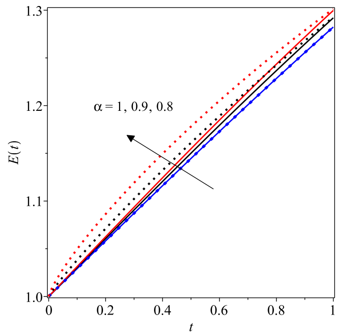
Population	Maximum error in DTM	DTM followed by Padé	More terms of DTM followed by Padé
Susceptible	46103.96	0.0085159	0.0021529
Exposed	16040.35	0.0030087	0.0068216
Infected	119.1669	0.0364303	0.0004470
Isolated	19.87663	0.0100607	0.0001102
Recovered	0.0004452	0.0000065	0.0000002

Table 5.5: Maximum error in LDM

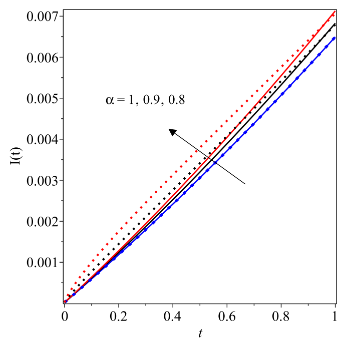
Population	Maximum error in LDM	LDM followed by Padé	More terms of LDM followed by Padé
Susceptible	13492.98	0.0700698	0.0152590
Exposed	13356.19	0.0076430	0.0058501
Infected	117.2429	0.0756698	0.0031094
Isolated	19.55336	0.0242172	0.0007176
Recovered	0.0044967	0.0000065	0.0000009



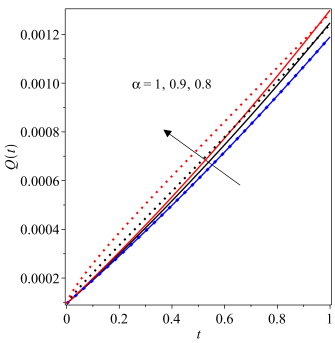
(a) Susceptible



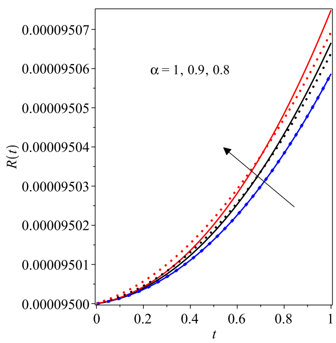
(b) Exposed



(c) Infected



(d) Isolated



(e) Recovered

Figure 5.11: Analytical solution curves of the system (5.18) when $\alpha = 1$, $\alpha = 0.9$ and $\alpha = 0.8$ where the dotted and solid curves correspond to the solutions of DTM and the LDM respectively.

The considered system (5.18) is also solved for the fractional derivatives when $\alpha = 0.9$ and $\alpha = 0.8$. Figure (5.11) presents the analytical solution curves for the fractional values of α where it shows strong agreements between LDM and DTM. Tables (5.6) and (5.7) show the actual variations between the LDM and DTM for the fractional case when $\alpha = 0.9$ and $\alpha = 0.8$.

Table 5.6: Maximum variation between LDM and DTM when α = 0.9

Population	Maximim Variation	Occurred at x
Susceptible	0.011896467	0.400
Exposed	0.011632328	0.400
Infected	0.000233417	0.300
Isolated	0.000039366	0.400
Recovered	0.000000003	1.000

Table 5.7: Maximum variation between LDM and DTM when α = 0.8

Population	Maximum Variation	Occurred at x
Susceptible	0.008916089	0.300
Exposed	0.025544899	0.300
Infected	0.000514733	0.300
Isolated	0.000086447	0.300
Recovered	0.000000006	1.000

Chapter 6: Conclusions and Future Work

In this thesis, we investigated the numerical solution of fractional differential equations with initial and boundary conditions. Laplace Decomposition Method and the Differential Transformation Method were implemented to obtain an analytical solution for a complex nonlinear fractional Enzyme Inhibitor Reaction Model and a complex nonlinear COVID-19 model.The two methods were in strong agreement with the fourth order Runge-Kutta method. Then the systems were solved for the fractional order when $\alpha = 0.9$ and $\alpha = 0.8$ where the two methods presented agreeable solutions. We also coupled the LDM and DTM with Padé approximation to obtain a convergent solution over a larger time interval. We can also acquire more accurate results by increasing the terms of the series solution. The proposed methods were employed to obtain highly accurate results for fractional differential equations. Furthermore, it helped us guarantee convergence, stability, and helped us reach a better understanding of the dynamical behavior of the systems.

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Vitae

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