Accepted manuscript of: Badawi, A. (2017). n-Absorbing Ideals of Commutative Rings and Recent Progress on Three Conjectures: A Survey. In *Rings, Polynomials, and Modules* (pp. 33–52). Springer International Publishing. https://doi.org/10.1007/978-3-319-65874-2_3

n-absorbing ideals of commutative rings and recent progress on three conjectures: A survey

Ayman Badawi

Abstract Let R be a commutative ring with $1 \neq 0$. Recall that a proper ideal I of R is called a *2-absorbing ideal* of R if $a,b,c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. A more general concept than 2-absorbing ideals is the concept of n-absorbing ideals. Let $n \geq 1$ be a positive integer. A proper ideal I of R is called an *n-absorbing ideal* of R if $a_1, a_2, ..., a_{n+1} \in R$ and $a_1 a_2 \cdots a_{n+1} \in I$, then there are n of the a_i's whose product is in I. The concept of n-absorbing ideals is a generalization of the concept of prime ideals (note that a prime ideal of R is a 1-absorbing ideal of R). In this survey article, we collect some old and recent results on n-absorbing ideals of commutative rings.

key words prime, primary, weakly prime, weakly primary, 2-absorbing, n-absorbing, weakly 2-absorbing, weakly n-absorbing, 2-absorbing primary, weakly 2-absorbing primary

Mathematics Subject Classification (2010): 13A15

Dedicated to Professors David F. Anderson and Luigi Salce on their retirement

1 introduction

We assume throughout that all rings are commutative with $1 \neq 0$. Over the past several years, there has been considerable attention in the literature to n-absorbing ideals of commutative rings and their generalizations, for example see ([1]–[62]). We recall from [8] that a proper ideal I of R is called a *2-absorbing ideal* of R if

Ayman Badawi
Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates, e-mail: abadawi@aus.edu

$a,b,c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. A more general concept than 2-absorbing ideals is the concept of n-absorbing ideals. Let $n \geq 1$ be a positive integer. A proper ideal I of R is called an *n-absorbing ideal* of R as in [5] if $a_1, a_2, ..., a_{n+1} \in R$ and $a_1 a_2 \cdots a_{n+1} \in I$, then there are n of the a_i's whose product is in I. The concept of n-absorbing ideals is a generalization of the concept of prime ideals (note that a prime ideal of R is a 1-absorbing ideal of R).

Let R be a (commutative) ring. Then $\dim(R)$ denotes the Krull dimension of R, $Spec(R)$ denotes the set of prime ideals of R, $Max(R)$ denotes the set of maximal ideals of R, $T(R)$ denotes the total quotient ring of R, $qf(R)$ denotes the quotient field of R when R is an integral domain, and $Nil(R)$ denotes the ideal of nilpotent elements of R. If I is a proper ideal of R, then $Rad(I)$ and $Min_R(I)$ denote the radical ideal of I and the set of prime ideals of R minimal over I, respectively. We will often let 0 denote the zero ideal.

The purpose of this survey article is to collect some properties of n-absorbing ideals in commutative rings. In particular, we state some recent progresses on three outstanding conjectures (see section 5). Our aim is to give the flavor of the subject, but not be exhaustive.

We recall some background material. A prime ideal P of a ring R is said to be a *divided prime ideal* if $P \subset xR$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. An integral domain R is said to be a *divided domain* if every prime ideal of R is a divided prime ideal.

An integral domain R is said to be a *valuation domain* if either $x|y$ or $y|x$ (in R) for all $0 \neq x, y \in R$ (a valuation domain is a divided domain). If I is a nonzero fractional ideal of a ring R, then $I^{-1} = \{ x \in T(R) \mid xI \subseteq R \}$. An integral domain R is called a *Dedekind* (resp., *Prüfer*) *domain* if $II^{-1} = R$ for every nonzero fractional ideal (resp., finitely generated fractional ideal) I of R. Moreover, an integral domain R is a Prüfer domain if and only if R_M is a valuation domain for every maximal ideal M of R.

Some of our examples use the $R(+)M$ construction. Let R be a ring and M an R-module. Then $R(+)M = R \times M$ is a ring with identity $(1,0)$ under addition defined by $(r,m)+(s,n) = (r+s, m+n)$ and multiplication defined by $(r,m)(s,n) = (rs, rn+sm)$.

2 Basic properties of n-absorbing ideals

Let I be a proper ideal of R. If I be an n-absorbing ideal of R for some positive integer n, then recall from [5] that $\omega_R(I) = \min\{ n \mid I \text{ is an } n\text{-absorbing ideal of } R \}$; otherwise, set $\omega_R(I) = \infty$. It is convenient to define $\omega_R(R) = 0$. We start by recalling some basic properties of n-absorbing ideals.

Theorem 2.1 *1. ([8, Theorem 2.3]). Let I be a 2-absorbing ideal of a ring R. Then there are at most two prime ideals of R that are minimal over I (i.e. $|Min_R(I)| = 1$ or 2).*

2. *([8, Theorem 2.1 and Theorem 2.4]). Let I be a 2-absorbing ideal of a ring R. Then $Rad(I)$ is a 2-absorbing ideal of R and $(Rad(I))^2 \subseteq I$.*
3. *([5, Theorem 2.5]). Let I be an n-absorbing ideal of a ring R. Then there are at most n prime ideals of R minimal over I. Moreover, $|Min_R(I)| \leq \omega_R(I)$.*
4. *([5, Theorem 2.9]). Let $M_1, \ldots, M_n$ be maximal ideals of a ring R (not necessarily distinct). Then $I = M_1 \cdots M_n$ is an n-absorbing ideal of R. Moreover, $\omega_R(I) \leq n$.*
5. *([8, Theorem 2.4]). Let I be an 2-absorbing ideal of a ring R with exactly two minimal prime ideals P_1, P_2 over I. Then $P_1P_2 \subseteq I$.*
6. *([5, Theorem 2.14]). Let I be an n-absorbing ideal of a ring R such that I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$. Then $P_1 \cdots P_n \subseteq I$. Moreover, $\omega_R(I) = n$.*
7. *([8, Theorem 2.5]). Let I be a 2-absorbing ideal of R such that $Rad(I) = P$ is a prime ideal of R and suppose that $I \neq P$. For each $x \in P \setminus I$ let $B_x = (I :_R x) = (I :_R x) = \{y \in R \mid yx \in I\}$. Then B_x is a prime ideal of R containing P. Furthermore, either $B_y \subseteq B_x$ or $B_x \subseteq B_y$ for every $x, y \in P \setminus I$.*
8. *([8, Theorem 2.6]). Let I be a 2-absorbing ideal of R such that $I \neq Rad(I) = P_1 \cap P_2$, where P_1 and P_2 are the only nonzero distinct prime ideals of R that are minimal over I. Then for each $x \in Rad(I) \setminus I$, $B_x = (I :_R x) = \{y \in R \mid xy \in I\}$ is a prime ideal of R containing P_1 and P_2. Furthermore, either $B_y \subseteq B_x$ or $B_x \subseteq B_y$ for every $x, y \in Rad(I) \setminus I$.*
9. *([5, Theorem 3.4]). Let I be an n-absorbing ideal of a ring R. Then $(I :_R x) = \{y \in R \mid yx \in I\}$ is an n-absorbing ideal of R containing I for all $x \in R \setminus I$. Moreover, $\omega_R(I_x) \leq \omega_R(I)$ for all $x \in R$.*
10. *([5, Theorem 3.5]). Let $n \geq 2$ and $I \subset Rad(I)$ be an n-absorbing ideal of a ring R. Suppose that $x \in Rad(I) \setminus I$, and let $m(\geq 2)$ be the least positive integer such that $x^m \in I$. Then $(I :_R x^{m-1}) = \{y \in R \mid yx^{m-1} \in I\}$ is an $(n-m+1)$-absorbing ideal of R containing I.*
11. *([5, Corollary 2.6]). Let $n \geq 2$ and $I \subset Rad(I)$ be an n-absorbing ideal of a ring R. Suppose that $x \in Rad(I) \setminus I$ and $x^n \in I$, but $x^{n-1} \notin I$. Then $(I :_R x^{n-1}) = \{y \in R \mid yx^{n-1} \in I\}$ is a prime ideal of R containing $Rad(I)$.*
12. *([5, Corollary 2.7]). Let $n \geq 2$ and I be an n-absorbing P-primary ideal of a ring R for some prime ideal P of R. If $x \in Rad(I) \setminus I$ and n is the least positive integer such that $x^n \in I$, then $(I :_R x^{n-1}) = \{y \in R \mid yx^{n-1} \in I\} = P$.*
13. *([5, Theorem 3.8]). Let $n \geq 2$ and $I \subset Rad(I)$ be an n-absorbing ideal of a ring R such that I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$. Suppose that $x \in Rad(I) \setminus I$, and let $m(\geq 2)$ be the least positive integer such that $x^m \in I$. Then every product of $n-m+1$ of the P_i's is contained in $(I :_R x^{m-1}) = \{y \in R \mid yx^{m-1} \in I\}$.*
14. *([5, Theorem 3.9]). Let I be a P-primary ideal of a ring R such that $P^n \subseteq I$ for some positive integer n (for example, if R is a Noetherian ring), and let $x \in P \setminus I$. If $x^m \notin I$ for some positive integer m, then $(I :_R x^m) = \{y \in R \mid yx^m \in I\}$ is an $(n-m)$-absorbing ideal of R.*

Assume that I is a proper ideal of R such that $I \neq Rad(I)$. The following two results give a characterization of 2-absorbing ideals in terms of $(I :_R x) = \{y \in R \mid yx \in I\}$, where $x \in Rad(I) \setminus I$.

Theorem 2.2 *([8, Theorem 2.8]) . Let I be an ideal of R such that $I \neq Rad(I)$ and $Rad(I)$ is a prime ideal of R. Then the following statements are equivalent:*

1. I is a 2-absorbing ideal of R;
2. $B_x = \{y \in R \mid yx \in I\}$ is a prime ideal of R for each $x \in Rad(I) \setminus I$.

Theorem 2.3 *([8, Theorem 2.9]). Let I be an ideal of R such that $I \neq Rad(I) = P_1 \cap P_2$, where P_1 and P_2 are nonzero distinct prime ideals of R that are minimal over I. Then the following statement are equivalent:*

1. I is a 2-absorbing ideal of R;
2. $P_1P_2 \subseteq I$ and $B_x = \{y \in R \mid yx \in I\}$ is a prime ideal of R for each $x \in Rad(I) \setminus I$.
3. $B_x = \{y \in R \mid yx \in I\}$ is a prime ideal of R for each $x \in (P_1 \cup P_2) \setminus I$.

In view of Theorem 2.2, the following is an example of a prime ideal P of an integral domain R such that P^2 is not a 2-absorbing ideal of R.

Example 2.4 *([8, Example 3.9]). Let $R = \mathbb{Z} + 6X\mathbb{Z}[X]$ and $P = 6X\mathbb{Z}[X]$. Then P is a prime ideal of R. Since $6X^2 \in P \setminus P^2$ and $B_{6X^2} = \{y \in R \mid 6X^2y \in P^2\} = 6\mathbb{Z} + 6X\mathbb{Z}[X]$ is not a prime ideal of R, P^2 is not a 2-absorbing ideal of R by Theorem 2.2.*

The following result characterizes all P-primary ideals that are 2-absorbing ideals.

Theorem 2.5 *([8, Theorem 3.1]). Let I be a P-primary ideal of a ring R for some prime ideal P of R. Then I is a 2-absorbing ideal of R if and only if $P^2 \subseteq I$. In particular, M^2 is a 2-absorbing ideal of R for each maximal ideal M of R.*

The following is an example of a prime ideal P of an integral domain R such that P^2 is a 2-absorbing ideal of R, but P^2 is not a P-primary ideal of R.

Example 2.6 *([8, Example 3.11]). Let $R = \mathbb{Z} + 3x\mathbb{Z}[X]$ and let $P = 3X\mathbb{Z}[X]$ be a prime ideal of R. Since $3(3X^2) \in P^2$, we conclude that P^2 is not a P-primary ideal of R. It is easy to verify that if $d \in P \setminus P^2$, then either $B_d = \{y \in R \mid yd \in I\} = P$ or $B_d = 3\mathbb{Z} + 3X\mathbb{Z}[X]$ is a prime ideal of R. Hence P^2 is a 2-absorbing ideal by Theorem 2.2.*

Let I be an ideal of R such that $Rad(I) = P$ is a nonzero divided prime ideal of R. The following result characterizes all such ideals that are 2-absorbing ideals.

Theorem 2.7 *([8, Theorem 3.6]). Suppose that P is a nonzero divided prime ideal of R and I is an ideal of R such that $Rad(I) = P$. Then the following statements are equivalent:*

1. I is a 2-absorbing ideal of R;
2. I is a P-primary ideal of R such that $P^2 \subseteq I$.

Theorem 2.8 *([8, Theorem 3.7] and [5, Theorem 3.3]). Let $n \geq 1$ be a positive integer. Suppose that $Nil(R)$ and P are divided prime ideals of a ring R such that $P \neq Nil(R)$. Then P^n is a P-primary ideal of R, and thus P^n is an n-absorbing ideal of R with $\omega_R(P^n) \leq n$. Moreover, $\omega_R(P^n) = n$ if $P^{n+1} \subset P^n$.*

In view of Theorems 2.5, 2.7 and 2.8, for $n \geq 3$, we have the following two results.

Theorem 2.9 *([5, Theorem 3.1]). Let P be a prime ideal of a ring R, and let I be a P-primary ideal of R such that $P^n \subseteq I$ for some positive integer n (for example, if R is a Noetherian ring). Then I is an n-absorbing ideal of R. Moreover, $\omega_R(I) \leq n$. In particular, if P^n is a P-primary ideal of R, then P^n is an n-absorbing ideal of R with $\omega_R(P^n) \leq n$, and $\omega_R(P^n) = n$ if $P^{n+1} \subset P^n$.*

Theorem 2.10 *([5, Theorem 3.2]). Let P be a divided prime ideal of a ring R, and let I be an n-absorbing ideal of R with $Rad(I) = P$. Then I is a P-primary ideal of R.*

Mostafanasab and Darani in [51] proved the following result.

Theorem 2.11 *([51, Theorem 2.15]). (n-absorbing avoidance theorem). Let $I_1, I_2, \ldots, I_m$ $(m \geq 2)$ be ideals of R such that I_i is an n_i-absorbing ideal of R for every $3 \leq i \leq m$. Suppose that $I_i \nsubseteq (I_j :_R x^{n_j - 1}) \subset R$ for every $x \in Rad(I_j) \setminus I_j$ with $i \neq j$. If I is an ideal of R such that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_m$, then $I \subseteq I_i$ for some $1 \leq i \leq m$.*

3 Extensions of n-absorbing ideals

The following results show the stability of n-absorbing ideals in various ring-theoretic constructions. These results generalize well-known results about prime ideals.

Theorem 3.1 *1. ([5, Theorem 4.1]). Let I be an n-absorbing ideal of a ring R, and let S be a multiplicatively closed subset of R with $I \cap S = \emptyset$. Then I_S is an n-absorbing ideal of R_S. Moreover, $\omega_{R_S}(I_S) \leq \omega_R(I)$.*

2. Let $f : R \longrightarrow T$ be a homomorphism of rings.

a. ([5, Theorem 4.1]). Let J be an n-absorbing ideal of T. Then $f^{-1}(J)$ is an n-absorbing ideal of R. Moreover, $\omega_R(f^{-1}(J)) \leq \omega_T(J)$.

b. Let f be surjective and I be an n-absorbing ideal of R containing $ker(f)$. Then $f(I)$ is an n-absorbing ideal of T if and only if I is an n-absorbing ideal of R. Moreover, $\omega_T(f(I)) = \omega_R(I)$. In particular, this holds if f is an isomorphism.

In the following result, we determine the n-absorbing ideals in the product of any two rings.

Theorem 3.2 *([5, Theorem 4.7]). Let I_1 be an m-absorbing ideal of a ring R_1 and I_2 an n-absorbing ideal of a ring R_2. Then $I_1 \times I_2$ is an $(m+n)$-absorbing ideal of the ring $R_1 \times R_2$. Moreover, $\omega_{R_1 \times R_2}(I_1 \times I_2) = \omega_{R_1}(I_1) + \omega_{R_2}(I_2)$.*

Let R be a ring, M be an R-module, and $T = R(+)M$. If I is an n-absorbing ideal of R, then it is easy to show that $I(+)M$ is an n-absorbing ideal of T. In fact, $\omega_T(I(+)M) = \omega_R(I)$. We have the following result for the special case $T = R(+)R$, where R is an integral domain.

Theorem 3.3 *([5, Theorem 4.10]). Let D be an integral domain, $R = D(+)D$, and I be an n-absorbing ideal of D that is not an $(n-1)$-absorbing ideal of D. Then $0(+)I$ is an $(n+1)$-absorbing ideal of R that is not an n-absorbing ideal of R; so $\omega_R(0(+)I) = \omega_D(I)+1$. In particular, if P is a prime ideal of D, then $0(+)P$ is a 2-absorbing ideal of R.*

Let T be a ring extension of an integral domain D and P a prime ideal of D. Then $0(+)P$ need not be a 2-absorbing ideal of the ring $R = D(+)T$; so Theorem 3.3 does not extend to general R. We have the following example.

Example 3.4 *([5, Example 4.12]). Let $R = \mathbb{Z}(+)\mathbb{Q}$. Then $I = 0(+)2\mathbb{Z}$ is an ideal of R with $Rad(I) = 0(+)\mathbb{Q}$. Let $x = (0, \frac{1}{2}) \in Rad(I) \setminus I$. Then $B_x = (I :_R x) = (4\mathbb{Z})(+)\mathbb{Q}$ is not a prime ideal of R ($\omega_R(B_x) = 2$), and hence I is not a 2-absorbing ideal of R by Theorem 2.2. In fact, one can easily show that I is not an n-absorbing ideal of R for any positive integer n. For each positive integer n, let $x_i = (2,0)$ for $1 \leq i \leq n$ and $x_{n+1} = (0, \frac{1}{2^{n-1}})$. Then $x_1 \cdots x_{n+1} = (0,2) \in I$, but no proper subproduct of the x_i's is in I. Thus $\omega_R(I) = \infty$.*

We next consider extensions of n-absorbing ideals of R in the polynomial ring $R[X]$ and the power series ring $R[[X]]$.

Theorem 3.5 *Let I be a proper ideal of a ring R. Then*

1. *([5, Theorem 4.13]). (I,X) is an n-absorbing ideal of $R[X]$ if and only if I is an n-absorbing ideal of R. Moreover, $\omega_{R[X]}((I,X)) = \omega_R(I)$.*
2. *([5, Theorem 4.15]). $I[X]$ is a 2-absorbing ideal of $R[X]$ if and only if I is a 2-absorbing ideal of R. (If $n \geq 3$ and I is an n-absorbing ideal of R, does it follow that $I[X]$ is an n-absorbing ideal of $R[X]$? (See section 5.)*
3. *([44, Proposition 2.13]) $I[[X]]$ is a 2-absorbing ideal of $R[[X]]$ if and only if I is a 2-absorbing ideal of R (and therefore $I[X]$ is a 2-absorbing ideal of $R[X]$ if and only if I is a 2-absorbing ideal of R).*

Let K be a field. For rings of the form $D + XK[[X]]$, where D is a subring of K, we have the following result.

Theorem 3.6 *([5, Theorem 4.17]). Let D be a subring of a field K and $R = D + XK[[X]]$.*

(a) If D is a field, then every proper ideal of R is an n-absorbing ideal of R for some positive integer n.

(b) If D is a proper subring of K with $qf(D) = K$, then the nonzero n-absorbing ideals of R have the form $I + XK[[X]]$, where I is an n-absorbing ideal of D, or $X^m K[[X]]$ for m an integer with $1 \leq m \leq n$. Moreover, $\omega_R(I + XK[[X]]) = \omega_D(I)$ and $\omega_R(X^m K[[X]]) = m$.

4 n-absorbing ideals in specific rings

If R is a Noetherian ring, then we have the following result.

Theorem 4.1 *([5, Theorem 5.3]). Let R be a Noetherian ring. Then every proper ideal of R is an n-absorbing ideal of R for some positive integer n.*

A characterization of Dedekind domains in terms of 2-absorbing ideals is first given in [8, Theorem 3.15] and a similar characterization of Dedekind domains in terms of n-absorbing ideals ($n \geq 2$) is given in [5, Theorem 5.1].

Theorem 4.2 *([8, Theorem 3.15] and [5, Theorem 5.1]). If R be a Noetherian integral domain. Then the following statements are equivalent.*

1. *R is a Dedekind domain.*
2. *If I is an n-absorbing ideal of R, then $I = M_1 \cdots M_m$ for maximal ideals $M_1, \ldots, M_m$ of R with $1 \leq m \leq n$.*
   Moreover, if $I = M_1 \cdots M_n$ for maximal ideals $M_1, \ldots, M_n$ of a Dedekind domain R which is not a field, then $\omega_R(I) = n$.

All 2-absorbing ideals of a valuation domain are determined in [8, Theorem 3.10]. If $n \geq 3$, then a similar result [5, Theorem 5.5] determines all n-absorbing ideals of a valuation domain.

Theorem 4.3 *([8, Theorem 3.10] and [5, Theorem 5.5]). Let R be a valuation domain and n a positive integer. Then the following statements are equivalent for an ideal I of R.*
(1) I is an n-absorbing ideal of R.
(2) I is a P-primary ideal of R for some prime ideal P of R and $P^n \subseteq I$.
(3) $I = P^m$ for some prime ideal $P(= Rad(I))$ of R and integer m with $1 \leq m \leq n$.
Moreover, $\omega_R(P^n) = n$ for P a nonidempotent prime ideal of R.

Theorem 4.4 *([51, Proposition 2.10]). Let V be a valuation domain with quotient field K, and let I be a proper ideal of V. Then I is an n-absorbing ideal of V if and only if whenever $x_1x_2 \cdots x_{n+1} \in I$ with $x_1, x_2, ..., x_{n+1} \in K$, then there are n of $x_1, x_2, ..., x_{n+1}$ whose product is in I.*

All 2-absorbing ideals of a Prüfer domain are determined in [8, Theorem 3.14].

Theorem 4.5 *([5, Theorem 3.14]). Let R be a Prüfer domain and I be a nonzero ideal of R. Then the following statements are equivalent:*

1. *I is a 2-absorbing ideal of R;*
2. *I is a prime ideal of R or $I = P^2$ is a P-primary ideal of R or $I = P_1 \cap P_2$, where P_1 and P_2 are nonzero prime ideals of R.*

If n is a positive integer and R is a Prüfer domain, then we have the following result.

Theorem 4.6 *([8, Theorem 5.7]). Let R be a Prüfer domain. Then an ideal I of R is an n-absorbing ideal of R for some positive integer n if and only if I is a product of prime ideals of R. Moreover, if $P_1, \ldots, P_k$ are incomparable prime ideals of R and $n_1, \ldots, n_k$ are positive integers with $n_i = 1$ if P_i is idempotent, then $\omega_R(P_1^{n_1} \cdots P_k^{n_k}) = n_1 + \cdots + n_k$.*

5 Strongly *n*-absorbing ideals and recent progresses on three Conjectures

It is well known that a proper ideal I of a ring R is a prime ideal of R if and only if whenever $I_1I_2 \subseteq I$ for ideals I_1, I_2 of R, then either $I_1 \subseteq I$ or $I_2 \subseteq I$. Let n be a positive integer. We recall from [5] that a proper ideal I of a ring R is a *strongly n-absorbing ideal* if whenever $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \dots, I_{n+1}$ of R, then the product of some n of the I_j's is in I. Thus a strongly 1-absorbing ideal is just a prime ideal, and the intersection of n prime ideals is a strongly n-absorbing ideal. It is clear that a strongly n-absorbing ideal of R is also an n-absorbing ideal of R, and in [8, Theorem 2.13], it was shown that these two concepts agree when $n = 2$.

Theorem 5.1 *([8, Theorem 2.13]). Let I be a proper ideal of R. Then I is a 2-strongly absorbing ideal of R if and only if I is a 2-absorbing ideal of R.*

If R is a Prüfer domain and I is a proper ideal of of R, it was shown in ([5, Corollary 6.9]) that I is an n-strongly absorbing ideal of R if and only if I is an n-absorbing ideal of R..

Theorem 5.2 *([5, Corollary 6.9]). Let R be a Prüfer domain and n a positive integer. Then an ideal I of R is a strongly n-absorbing ideal of R if and only if I is an n-absorbing ideal of R.*

In view of Theorem 5.1, the following result is a generalization of Theorem 2.5 ([8, Theorem 3.1]).

Theorem 5.3 *([5, Theorem 6.6]). Let I be a P-primary ideal of a ring R and n a positive integer. Then the following statements are equivalent.*
(1) I is an n-absorbing ideal of R and $P^n \subseteq I$.
(2) I is a strongly n-absorbing ideal of R.
In particular, if P^n is P-primary, then P^n is a strongly n-absorbing ideal of R.

For a Notherian ring R, we have the following result.

Theorem 5.4 *([5, Corollary 6.8]). Let R be a Noetherian ring. Then every proper ideal of R is a strongly n-absorbing ideal of R for some positive integer n.*

Theorem 5.5 *([5, Corollary 6.7]). Let $M_1, \dots, M_n$ be maximal ideals of a ring R. Then $I = M_1 \cdots M_n$ is a strongly n-absorbing ideal of R.*

In view of Theorem 5.1, the following result is a generalization of ([8, Theorem 2.4]).

Theorem 5.6 *([5, Theorem 6.2]). Let n be a positive integer and I a strongly n-absorbing ideal of a ring R such that I has exactly $m (\leq n)$ minimal prime ideals $P_1, \dots, P_m$. Then $P_1^{n_1} \cdots P_m^{n_m} \subseteq I$ for positive integers $n_1, \dots, n_m$ with $n = n_1 + \cdots + n_m$. In particular, if $Rad(I) = P$ is a prime ideal of R, then $P^n \subseteq I$.*

Theorem 5.7 *([51, Corollary 2.14]). Let I_i be a strongly n_i-absorbing ideal of a ring R for every $1 \leq i \leq m$ $(m \geq 2)$. If I is an ideal of R such that $I \subseteq I_1 \cup I_2 \cdots \cup I_m$, then $I^{n_i} \subseteq I_i$ for some $1 \leq i \leq m$.*

Three outstanding conjectures on n-absorbing ideals are the following (see Anderson and Badawi [5] and also Cahen et al. [14, Problem 30]) :

1. **Conjecture one**. If an ideal of R is n-absorbing, then it is strongly n-absorbing.
2. **Conjecture two**. If an ideal I of R is n-absorbing, then $I[X]$ is an n-absorbing ideal of $R[X]$.
3. **Conjecture three**. If an ideal I of R is n-absorbing, then $(Rad(I))^n \subseteq I$.

Laradji in [44] gave an affirmative answer for Conjecture three when $n = 3$. Note that an affirmative answer for Conjecture three was given in Theorem 2.1(2) when $n = 2$.

Theorem 5.8 *([44, Proposition 2.7]. Let I be a 3-absorbing ideal of R. Then $(Rad(I))^3 \subseteq I$.*

Recently, Choi and Walker in [21, Theorem 1] gave an an affirmative answer for Conjecture three for any positive integer n.

Theorem 5.9 *([21, Theorem 1]). Let n be a positive integer and I be an n-absorbing ideal of R. Then $(Rad(I))^n \subseteq I$.*

It was shown [5, Theorem 6.1] that Conjecture one implies Conjecture three.

Theorem 5.10 *([5, Theorem 6.1]). Let n be a positive integer and I be a strongly n-absorbing ideal of R. Then $(Rad(I))^n \subseteq I$.*

Laradji in [44] showed that Conjecture two implies Conjecture one.

Theorem 5.11 *([44, Proposition 2.9(i)]). Let I be a proper ideal of R and n be a positive integer. If $I[X]$ is an n-absorbing ideal of $R[X]$, then I is a strongly n-absorbing ideal of R.*

Let $f(X) = a_m x^m + \cdots + a_0 \in R[X]$, for some positive integer m and for some $a_m, ..., a_0 \in R$. Then $c(f) = (a_m, ..., a_0)R$ is an ideal of R and it is called the the *content* of $f(X)$. We recall that a ring R is called *Armandariz* if whenever $f(X)g(X) = 0 \in R[X]$ for some $f(X), g(X) \in R[X]$, then $c(f)c(g) = 0 \in R$.

Let I be a strongly n-absorbing ideal of R. The author in [44] showed that if R/I is Armandariz, then Conjecture one implies Conjecture two.

Theorem 5.12 *([44, Proposition 2.9(ii)]). Let I be a strongly n-absorbing ideal of R for some positive integer n. If R/I is Armandariz, then $I[X]$ is an n-absorbing ideal of $R[X]$.*

Note that Theorem 5.2 gives an affirmative answer for Conjecture one when R is a Prüfer domain.

Let I be an n-absorbing ideal of R. Darani and Puczylowski in [29] proved that Conjecture one holds if the additive group of R/I is torsion-free.

Theorem 5.13 *([29, Theorem 4.2]). Let I be a proper ideal of R and n be a positive integer. If I is an n-absorbing ideal of R such that the additive group of R/I is torsion-free, then I is a strongly n-absorbing ideal of R.*

Donadze in [31] proved that Conjecture one holds in the following case.

Theorem 5.14 *([31, Proposition 2.2]). Let R be a ring and $n \geq 2$ be an integer. Suppose that R contains $n-1$ distinct invertible elements $u_1, ..., u_{n-1}$ such that $u_i - u_j$ is also invertible for all $i \neq j$, $1 \leq i, j \leq n-1$. Then every n-absorbing ideal of R is strongly n-absorbing.*

Laradji in [44] proved that Conjecture two holds in the following cases.

Theorem 5.15 *([44, Proposition 2.10]). Let n be a positive integer, I be an n-absorbing ideal of R, and let $S = R/I$. Then $I[X]$ is an n-absorbing ideal of $R[X]$ in each of the following cases.*

1. *S is Armendariz and $|R/M| \geq n$ for each maximal ideal M of R containing I.*
2. *S is Armendariz and is $(n-1)!$-torsion free as an additive group.*
3. *S is torsion-free as an additive group.*

Donadze in [31] proved the following result.

Theorem 5.16 *([31, Corollary 2.10]). If Conjecture two holds for $\mathbb{Z}[X_1, ..., X_m]$ for all $m \geq 1$, then Conjecture one holds for any commutative ring R.*

Recall that *R* is called *arithmetical* ring if the set of ideals of every localization of *R* by a prime ideal of *R* is totally ordered by inclusion.

Laradji in [44] proved that Conjecture two holds if *R* is arithmetical.

Theorem 5.17 *(([44, Corollary 2.11]). Let n be a positive integer and I be an n-absorbing ideal of an arithmetical ring R. Then $I[X]$ is an n-absorbing ideal of $R[X]$.*

In light of Theorem 5.17, Theorem 5.11, and Theorem 5.10, we conclude that all three Conjectures hold if *R* is arithmetical.

Theorem 5.18 *Let R be an arithmatical ring (for example, if R is a Prüfer domain). If I is an n-absorbing ideal of R for some positive integer n, then the following statements are true:*

1. *I is a strongly n-absorbing ideal of R;*
2. *$I[X]$ is an n-absorbing ideal of $R[X]$;*
3. *$(Rad(I))^n \subseteq I$.*

Laradji in [44] showed that when attempting to prove either Conjecture one, Conjecture two, or Conjecture three, it is enough to restrict our attention to the zero ideal of some total quotient rings.

Theorem 5.19 *([44, Proposition 2.15]). Let I be a proper ideal of R and $T(R/I)$ be the total quotient ring of R/I. If Conjecture one, Conjecture two, or Conjecture three holds for the zero ideal of $T(R/I)$, then it holds for I.*

Let I be a proper ideal of R. Badawi and Anderson in [5] conjectured that $\omega_{R[X]}(I[X]) = \omega_R(I)$.

In view of Theorem 5.18, we have the following result.

Theorem 5.20 *Let R be an arithmatical ring (for example, if R is a Prüfer domain). Then $\omega_{R[X]}(I[X]) = \omega_R(I)$ for every proper ideal I of R.*

Nesehpour in [52, Corollary 10], independently, proved that $\omega_{R[X]}(I[X]) = \omega_R(I)$ for every proper ideal I of a Prüfer domain R.

6 n-Krull dimension of commutative rings

From [46], we recall the following definitions.

Definition ([46]).

1. Let R be a ring and n a positive integer. A chain of ideals: $I_0 \subset I_1 \cdots \subset I_m$, where $I_0, I_1, \ldots, I_m$ are distinct n-absorbing ideals of R, is called a chain of n-absorbing ideals of length m. The *n-Krull dimension* of R, denoted by $dim_n(R)$, is defined to be the supremum of the lengths of these chains. Thus $dim_1(R)$ is just the usual Krull dimension, dim(R), of R.
2. An n-absorbing ideal I of R is called a *minimal n-absorbing* ideal of the ideal J if $J \subseteq I$ and there is no n-absorbing ideal L such that $J \subseteq L \subset I$. An n-absorbing ideal I of R is called *a minimal n-absorbing ideal of R* if I is a minimal n-absorbing ideal of 0.
3. If I is an n-absorbing ideal of R, the n-height of I, denoted by $ht_n(I)$, is defined to be the supremum of lengths of chains $I_0 \subset I_1 \cdots \subset I_m$ of n-absorbing ideals of R for which $I_m = I$ if this supremum exists, and ∞ otherwise.
4. If I is a proper ideal of R (not necessarily an n-absorbing ideal) and n a positive integer, the n-height of I, denoted by $ht_n(I)$, is defined to be $min\{ht_n(J) \mid J$ is an n-absorbing ideal and $I \subseteq J\}$.

Remark 6.1 *Although every prime ideal of R is an n-absorbing ideal for each $n \geq 1$, there exists a minimal prime ideal which is not a minimal n-absorbing ideal for each $n \geq 2$. For example, if $R = K[X]$ is the polynomial ring in one variable X over a field K, the minimal prime ideal $P = RX$ of (0) is not a minimal* 2*-absorbing ideal of 0, since by [5, Lemma 2.8], RX^2 is a* 2*-absorbing ideal of R.*

Let $l(R)$ denotes the length of a composition series for a ring R which is of finite length. We recall the following results.

Theorem 6.2 *Let R be a ring. Then*

1. *([46, Theorem 2.1]). For each positive integer n, there is an n-absorbing ideal of R which is minimal among all n-absorbing ideals of R.*
2. *([46, Theorem 2.1]. If I a proper ideal of R, then for each positive integer n, there is an n-absorbing ideal of R which is minimal among all n-absorbing ideals of R containing I.*
3. *([46, Theorem 2.7]). Let n a positive integer. If* $dim_n(R)$ *is finite, then* $dim_n(R) = sup\{ht_n(M) \mid M \text{ is a maximal ideal of } R\}$.
4. *([46, Theorem 2.8]). If R is an Artinian ring, then* $dim_n(R)$ *is finite for each positive integer n.*
5. *([46, Theorem 2.9]). If* (R,M) *is a quasilocal Noetherian domain with maximal ideal M such that* $dim_1(R) = 1$*, then* $dim_2(R)$ *is finite.*
6. *([46, Theorem 2.12]). If* (R,M) *is a quasilocal Artinian ring and n is the smallest positive integer such that* $M^n = 0$*, then* $dim_k R = l(R) - 1$ *for each* $k \geq n$.
7. *([46, Theorem 2.13]). If R is an Artinian ring with k maximal ideals, then there exists a positive integer n such that* $dim_n(R) = l(R) - k$.

It was shown [46, Theorem 2.10] that if Conjecture three holds (see Section 5), then Theorem 6.2(5) can be extended to any positive integer *n*. Hence in view of Theorem 5.9, we have the following result.

Theorem 6.3 *([21, Theorem 1] and [46, Theorem 2.10]). If* (R,M) *is a quasilocal Noetherian domain with maximal ideal M such that* $dim_1(R) = 1$*, then* $dim_n(R)$ *is finite for every positive integer n.*

In light of Theorem 4.2, the following result provides a characterization of Dedekind domains in terms of n-Krull dimension.

Theorem 6.4 *([46, Theorem 2.13]). Let R be a Noetherian integral domain which is not a field. Then the following statements are equivalent.*

1. *R is a Dedekind domain.*
2. $dim_n(R) = n$ *for every positive integer n.*
3. $dim_2(R) = 2$.

Theorem 6.5 *([46, Theorem 2.21]). Let* (R,M) *be a discrete valuation ring and I an ideal of R. Then*

1. *I is an n-absorbing ideal for some positive integer n and* $\omega_R(I) = l_R(R/I)$.
2. *For every positive integer n,* $dim_n(R) = l_R(R/M^n) = n$.

7 (m,n)-closed ideals and quasi-n-absorbing ideals

We start by recalling some definitions.

Definition. Let *I* be a proper ideal *I* of *R*. Then

1. ([4]). I is called a *semi-n-absorbing* ideal of R if $x^{n+1} \in I$ for $x \in R$ implies $x^n \in I$. More generally, for positive integers m and n, I is said to be an (m,n)*-closed* ideal of R if $x^m \in I$ for $x \in R$ implies $x^n \in I$ (observe that I is a semi-n-absorbing ideal of R if and only if I is a $(n+1,n)$-closed ideal of R).
2. ([4]). For positive integers m and n, I is said to be an *strongly* (m,n)*-closed* ideal of R if $J^m \subseteq I$ for some ideal J of R implies $J^n \subseteq I$.
3. ([51]). I is called a *quasi-n-absorbing* ideal if whenever $a^n b \in I$ for some $a, b \in R$, then $a^n \in I$ or $a^{(n-1)} b \in I$.
4. [51]. I is called a *strongly quasi-n-absorbing* ideal if whenever $I_1^n I_2 \subseteq I$ for some ideals I_1, I_2 of R, then $I_1^n \subseteq I$ or $I_1^{(n-1)} I_2 \subseteq I$.

Remark 7.1 *Note that Mostafanasab and Darani in [51] called a proper ideal I of R to be a semi-*(m,n)*-absorbing ideal if I is an* (m,n)*-closed ideal.*

The following examples show that for every integer $n \geq 2$, there is a semi-n-absorbing ideal (i.e., $(n+1,n)$-closed ideal) that is neither a radical ideal nor an n-absorbing ideal, and that there is an ideal that is not a semi-n-absorbing ideal (i.e., $(n+1,n)$-closed ideal) for any positive integer n.

Example 7.2 *([4, Example 2.2]).*

1. *Let* $R = \mathbb{Z}$, $n \geq 2$ *an integer, and* $I = 2 \cdot 3^n \mathbb{Z}$. *Then I is a semi-n-absorbing ideal (i.e.,* $(n+1,n)$*-closed ideal) of R. In fact, I is a semi-m-absorbing ideal for every integer* $m \geq n$. *However,* $(2 \cdot 3^{n-1})^2 \in I$ *and* $2 \cdot 3^{n-1} \notin I$*; so I is not a radical ideal of R. Moreover,* $2 \cdot 3^n \in I$, $3^n \notin I$, *and* $2 \cdot 3^{n-1} \notin I$*; so I is not an n-absorbing ideal of R (but, I is an* $(n+1)$*-absorbing ideal of R). Note that for* $n = 1$, $I = 6\mathbb{Z}$ *is a semi-*1*-absorbing ideal (i.e., radical ideal) of R, but not a* 1*-absorbing ideal (i.e., prime ideal) of R.*
2. *Let* $R = \mathbb{Q}[\{X_n\}_{n \in \mathbb{N}}]$ *and* $I = (\{X_n^n\}_{n \in \mathbb{N}})$. *Then* $X_{n+1}^{n+1} \in I$ *and* $X_{n+1}^n \notin I$ *for every positive integer n; so I is not a semi-n-absorbing ideal (i.e.,* $(n+1,n)$*-closed ideal) for any positive integer n. Thus I is* (m,n)*-closed if and only if* $1 \leq m \leq n$.
3. *Let R be a commutative Noetherian ring. Then every proper ideal of R is an n-absorbing ideal of R, and hence a semi-n-absorbing ideal of R, for some positive integer n (Theorem 4.1). Thus, for every proper ideal I of R, there is a positive integer n such that I is* (m,n)*-closed for every positive integer m. Note that the ring in (2) is not Noetherian.*
4. *Clearly, an n-absorbing ideal of R is also an* $(n+1)$*-absorbing ideal of R. However, this need not be true for semi-n-absorbing ideals. For example, it is easily seen that* $I = 16\mathbb{Z}$ *is a semi-*2*-absorbing ideal (i.e.,* $(3,2)$*-closed ideal) of* $\mathbb{Z}$*, but not a semi-*3*-absorbing ideal (i.e.,* $(4,3)$*-closed ideal) of* $\mathbb{Z}$.
5. *Let R be a valuation domain. Then it is known that a radical ideal of R is also a prime ideal of R, i.e., a semi-*1*-absorbing ideal of R is a* 1*-absorbing ideal of R. However, a semi-n-absorbing ideal of R need not be an n-absorbing ideal of R for* $n \geq 2$. *For example, let* $R = \mathbb{Z}_{(2)}$ *and* $I = 16\mathbb{Z}_{(2)}$. *Then R is a DVR, and it is easily verified that I is a semi-*2*-absorbing ideal (i.e.,* $(3,2)$*-closed ideal) of R, but not a* 2*-absorbing ideal of R.*

It was conjectured (see Conjecture one in section 5) that a proper ideal I of R is an n-absorbing ideal of R if and only if I is a strongly n-absorbing ideal of R. However, an (m,n)-closed ideal of R need not be a strongly (m,n)-closed ideal of R; we have the following example.

Example 7.3 *([4, Example 2.5]). Let $R = \mathbb{Z}[X,Y]$, $I = (X^2, 2XY, Y^2)$, and $J = \sqrt{I} = (X,Y)$. Suppose that $a^m \in I$ for $a \in R$ and m a positive integer. Then $a \in \sqrt{I}$, and thus $a = bX + cY$ for some $b,c \in R$. Hence $a^2 = (bX+cY)^2 = b^2X^2 + 2bcXY + c^2Y^2 \in I$, and thus I is an $(m,2)$-closed ideal of R for every positive integer m. It is easily checked that $J^m \subseteq I$ for every integer $m \geq 3$. However, $J^2 \not\subseteq I$ since $XY \notin I$; so I is not a strongly $(m,2)$-closed ideal of R for any integer $m \geq 3$.*

In view of Example 7.3, we have the following result.

Theorem 7.4 *([4, Theorem 2.6]). Let R be a commutative ring, m a positive integer, I an $(m,2)$-closed ideal of R, and J an ideal of R.*

1. *If $J^m \subseteq I$, then $2J^2 \subseteq I$.*
2. *Suppose that $2 \in U(R)$. Then I is a strongly $(m,2)$-closed ideal of R.*

In view of Theorem 7.4(2), we have the following result.

Theorem 7.5 *([51, Corollary 4.11]). Let R be a ring and n be a positive integer such that $n!$ is a unit in R. Then every semi-n-absorbing ideal of R is strongly semi-n-absorbing.*

We have the following result.

Theorem 7.6 *([51, Proposition 4.6]). Let I be an ideal of a ring R and n be a positive integer. If for every ideal J of R, we have $J^{n+1} \subseteq I \subseteq J$ implies $J^n \subseteq I$, then I is a strongly semi-n-absorbing ideal of R.*

The following result is a characterization of zero-dimensional rings in terms of (m,n)-closed ideals.

Theorem 7.7 *([4, Theorem 2.15]). Let R be a commutative ring and n a positive integer. Then the following statements are equivalent.*

1. *Every proper ideal of R is (m,n)-closed for every positive integer m.*
2. *There is an integer $m > n$ such that every proper ideal of R is (m,n)-closed.*
3. *For every proper ideal I of R, there is an integer $m_I > n$ such that I is (m_I,n)-closed.*
4. *Every proper ideal of R is a semi-n-absorbing ideal (i.e., $(n+1,n)$-closed ideal) of R.*
5. *$dim(R) = 0$ and $w^n = 0$ for every $w \in nil(R)$.*

Let R be an integral domain and m,k be fixed positive integers. The next result determines the smallest positive integer n such that $I = p^kR$ is (m,n)-closed. As usual, $\lfloor x \rfloor$ is the greatest integer, or floor function.

Theorem 7.8 *([4, Theorem 3.10]). Let R be an integral domain and $I = p^kR$, where p is a prime element of R and k is a positive integer. Let m be a positive integer and n be the smallest postive integer such that I is (m,n)-closed.*

1. *If $m \geq k$, then $n = k$.*
2. *Let $m < k$ and write $k = ma + r$, where a is a positive integer and $0 \leq r < m$.*

   a. *If $r = 0$, then $n = m$.*
   b. *If $r \neq 0$ and $a \geq m$, then $n = m$.*
   c. *If $r \neq 0$, $a < m$, and $(a+1)|k$, then $n = k/(a+1)$.*
   d. *If $r \neq 0$, $a < m$, and $(a+1) \nmid k$, then $n = \lfloor k/(a+1) \rfloor + 1$.*

Let R be an integral domain and n,k be fixed positive integers. The next result determines the largest positive integer m such that $I = p^kR$ is (m,n)-closed.

Theorem 7.9 *([4, Theorem 3.11]). Let R be an integral domain, n a positive integer, and $I = p^kR$, where p is a prime element of R and k is a positive integer.*

1. *If $n \geq k$, then I is (m,n)-closed for every postive integer m.*
2. *Let $n < k$ and write $k = na + r$, where a is a positive integer and $0 \leq r < n$. Let m be the largest positive integer such that I is (m,n)-closed.*

   a. *If $a > n$, then $m = n$.*
   b. *If $a = n$ and $r = 0$, then $m = n + 1$.*
   c. *If $a = n$ and $r \neq 0$, then $m = n$.*
   d. *If $a < n$, $r = 0$, and $(a-1)|k$, then $m = k/(a-1) - 1$.*
   e. *If $a < n$, $r = 0$, and $(a-1) \nmid k$, then $m = \lfloor k/(a-1) \rfloor$.*
   f. *If $a < n$, $r \neq 0$, and $a|k$, then $m = k/a - 1$.*
   g. *If $a < n$, $r \neq 0$, and $a \nmid k$, then $m = \lfloor k/a \rfloor$.*

In view of Theorem 7.8 and Theorem 7.9, let I be a proper ideal of a commutative ring R and m and n positive integers. Anderson and Badawi in [4] defined $f_I(m) = \min\{n \mid I \text{ is } (m,n)-closed\} \in \{1,\ldots,m\}$ and $g_I(n) = sup\{m \mid I \text{ is } (m,n)-closed\} \in \{n, n+1, \ldots\} \cup \{\infty\}$. We have the following example.

Example 7.10 *Let R be an integral domain and $I = p^{30}R$ for p a prime element of R. By Theorem 7.8, one may easily calculate that $f_I(m) = m$ for $1 \leq m \leq 6$, $f_I(7) = 6, f_I(8) = f_I(9) = 8, f_I(m) = 10$ for $10 \leq m \leq 14, f_I(m) = 15$ for $15 \leq m \leq 29$, and $f_I(m) = 30$ for $m \geq 30$. Using Theorem 7.9, one may easily calculate that $g_I(n) = n$ for $1 \leq n \leq 5, g_I(6) = g_I(7) = 7, g_I(8) = g_I(9) = 9, g_I(n) = 14$ for $10 \leq n \leq 14, g_I(n) = 29$ for $15 \leq n \leq 29$, and $g_I(n) = \infty$ for $n \geq 30$.*

If R is a Prüfer domain, we have the following result.

Theorem 7.11 *([51, Corollary 3.26]). Let R be a Prüfer domain, n be a positive integer, and I be an ideal of R.*

1. *If I is a strongly quasi-n-absorbing (resp. strongly semi-n-absorbing) ideal of R, then $I[X]$ is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of $R[X]$.*

2. *If $I[X]$ is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of $R[X]$, then I is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of R.*

The following result determines the quasi-n-absorbing ideals in the product of any two rings.

Theorem 7.12 *([51, Proposition 4.20]). Let $n \geq 2$ be an integer, R_1, R_2 be rings, $R = R_1 \times R_2$, and L be a quasi-n- absorbing ideal of R. Then either $L = I_1 \times R_2$, where I_1 is a quasi-n-absorbing ideal of R_1 or $L = R_1 \times I_2$, where I_2 is a quasi-n-absorbing ideal of R_2 or $L = I_1 \times I_2$, where I_1 is a semi-$(n-1)$-absorbing ideal of R_1 and I_2 is a semi-$(n-1)$-absorbing ideal of R_2.*

8 2-absorbing primary ideals of commutative rings

We recall the following definition from [12] which is a generalization of primary ideal. A proper ideal I of R is said to be a 2-*absorbing primary ideal* of R if whenever $a,b,c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in Rad(I)$ or $bc \in Rad(I)$.

In the following result, we collect some basic properties of 2-absorbing primary ideals of commutative rings.

Theorem 8.1 *1. ([12, Theorem 2.2]). If I is a 2-absorbing primary ideal of R, then $Rad(I)$ is a 2-absorbing ideal of R.*

2. *([12, Theorem 2.3]). Suppose that I is a 2-absorbing primary ideal of R. Then one of the following statements must hold.*
   a. *$Rad(I) = P$ is a prime ideal,*
   b. *$Rad(I) = P_1 \cap P_2$, where P_1 and P_2 are the only distinct prime ideals of R that are minimal over I.*
3. *([12, Corollary 2.5]). Let R be a commutative ring with $1 \neq 0$, and let P_1, P_2 be prime ideals of R. If P_1^n is a P_1-primary ideal of R for some positive integer $n \geq 1$ and P_2^m is a P_2-primary ideal of R for some positive integer $m \geq 1$, then $P_1^n P_2^m$ and $P_1^n \cap P_2^m$ are 2-absorbing primary ideals of R. In particular, $P_1 P_2$ is a 2-absorbing primary ideal of R.*
4. *([12, Theorem 2.8]). Let I be an ideal of R. If $Rad(I)$ is a prime ideal of R, then I is a 2-absorbing primary ideal of R. In particular, if P is a prime ideal of R, then P^n is a 2-absorbing primary ideal of R for every positive integer $n \geq 1$.*
5. *([12, Theorem 2.10]). Let R be a commutative divided ring with $1 \neq 0$ (for example, if R is a valuation domain). Then every proper ideal of R is a 2-absorbing primary ideal of R.*
6. *([12, Theorem 2.20]). Let $f : R \to R'$ be a homomorphism of commutative rings. Then the following statements hold.*
   a. *If I' is a 2-absorbing primary ideal of R', then $f^{-1}(I')$ is a 2-absorbing primary ideal of R.*

b. If f is an epimorphism and I is a 2-absorbing primary ideal of R containing $Ker(f)$, then $f(I)$ is a 2-absorbing primary ideal of R'.

7. *([12, Theorem 2.22]). Let R be a commutative ring with $1 \neq 0$, S be a multiplicatively closed subset of R, and I be a proper ideal of R. Then the following statements hold.*

   a. If I is a 2-absorbing primary ideal of R such that $I \cap S = \varnothing$, then $S^{-1}I$ is a 2-absorbing primary ideal of $S^{-1}R$.
   b. If $S^{-1}I$ is a 2-absorbing primary ideal of $S^{-1}R$ and $S \cap Z_I(R) = \varnothing$, then I is a 2-absorbing primary ideal of R.

The following result is a characterization of Dedekind domains in terms of 2-absorbing primary ideals.

Theorem 8.2 *([12, Theorem 2.11]). Let R be a Noetherian integral domain with $1 \neq 0$ that is not a field. Then the following statements are equivalent.*

1. *R is a Dedekind domain.*
2. *A nonzero proper ideal I of R is a 2-absorbing primary ideal of R if and only if either $I = M^n$ for some maximal ideal M of R and some positive integer $n \geq 1$ or $I = M_1^n M_2^m$ for some maximal ideals M_1, M_2 of R and some positive integers $n, m \geq 1$.*
3. *If I is a nonzero proper 2-absorbing primary ideal of R, then either $I = M^n$ for some maximal ideal M of R and some positive integer $n \geq 1$ or $I = M_1^n M_2^m$ for some maximal ideals M_1, M_2 of R and some positive integers $n, m \geq 1$.*
4. *A nonzero proper ideal I of R is a 2-absorbing primary ideal of R if and only if either $I = P^n$ for some prime ideal P of R and some positive integer $n \geq 1$ or $I = P_1^n P_2^m$ for some prime ideals P_1, P_2 of R and some positive integers $n, m \geq 1$.*
5. *If I is a nonzero proper 2-absorbing primary ideal of R, then either $I = P^n$ for some prime ideal P of R and some positive integer $n \geq 1$ or $I = P_1^n P_2^m$ for some prime ideals P_1, P_2 of R and some positive integers $n, m \geq 1$.*

The following result determines the 2-absorbing primary ideals in the product of any finite number of rings.

Theorem 8.3 *([12, Theorem 2.24]). Let $R = R_1 \times R_2 \times \cdots \times R_n$, where $2 \leq n < \infty$, and R_1, R_2, ..., R_n are commutative rings with $1 \neq 0$. Let J be a proper ideal of R. Then the following statements are equivalent.*

1. *J is a 2-absorbing primary ideal of R.*
2. *Either $J = \times_{t=1}^n I_t$ such that for some $k \in \{1, 2, ..., n\}$, I_k is a 2-absorbing primary ideal of R_k, and $I_t = R_t$ for every $t \in \{1, 2, ..., n\} \setminus \{k\}$ or $J = \times_{t=1}^n I_t$ such that for some $k, m \in \{1, 2, ..., n\}$, I_k is a primary ideal of R_k, I_m is a primary ideal of R_m, and $I_t = R_t$ for every $t \in \{1, 2, ..., n\} \setminus \{k, m\}$.*

A proper ideal I of R is said to be a *strongly 2-absorbing primary ideal* of R if whenever I_1, I_2, I_3 are ideals of R with $I_1 I_2 I_3 \subseteq I$, then $I_1 I_2 \subseteq I$ or $I_1 I_3 \subseteq I$ or $I_2 I_3 \subseteq I$. We have the following result.

Theorem 8.4 *[12, Theorem 2.19]). Let I be a proper ideal of R. Then I is a 2-absorbing primary ideal of R if and only if I is a strongly 2-absorbing primary ideal of R.*

Remark 8.5 *Many topics related to the concept of n-absorbing ideals have been left untouched; the interested reader may consult the many articles mentioned in the references and MathSciNet. In the following, we will outline some of the related topics.*

1. *For topics on 2-absorbing preradicals, see ([23]-[25])*
2. *For topics related to 2-absorbing commutative semigroups, see [29].*
3. *For topics related to (weakly) n-absorbing ideals of commutative rings, see [2], [3], [6], [7], [9], [10], [12], ([15]-[17]), [20], [30], and ([36]-[38]).*
4. *For topics related to n-absorbing ideals in semirings, see [18], [22], [35], [42], [43], [57], [58], and [61].*
5. *For topics related to (weakly) n-absorbing submodules, see [19], ([25]-[27]), [32], [34], ([47]-[50]), [53], [55], [60], and [62].*

References

1. Filiz Kose Akca and Gulsen Ulucak, Uniformly primary elements of multiplicative lattices. International Mathematical Forum **10**, 157-163(2015)
2. D. D. Anderson and Malik Bataineh, Generalizations of prime ideals. Comm. Algebra **36**, 686-696(2008)
3. D. D. Anderson and E. Smith, Weakly prime ideals. Houston J. Math. **29**, 831-840(2003)
4. D. F. Anderson and A. Badawi, On (m,n)-closed ideals of commutative rings. Journal of Algebra and Its Applications **16**, (21 pages)(2017)
5. D. F. Anderson and A. Badawi, On n-absorbing ideals of commutative rings. Comm. Algebra **39**, 1646-1672(2011)
6. Sachin Ballal and Vilas Kharat, On generalization of prime, weakly prime and almost prime elements in multiplicative lattices. International Journal of Algebra **8**, 439-449(2014)
7. A. Badawi, On weakly semiprime ideals of commutative rings. Beitrge zur Algebra und Geometrie / Contributions to Algebra and Geometry **57**, 589-597(2016)
8. A. Badawi, On 2-absorbing ideals of commutative rings. Bull. Austral. Math. Soc. **75**, 417-429(2007)
9. A. Badawi and A.Yousef Darani, On weakly 2-absorbing ideals of commutative rings. Houston J. Math. **39**, 441-452 (2013)
10. A. Badawi, Unsal Tekir, and Ece Yeikin, Generalizations of 2-absorbing primary ideals of commutative rings. Turk. J. Math. **40**, 703-717(2016)
11. A. Badawi, Unsal Tekir, and Ece Yeikin, On weakly 2-absorbing primary ideals of commutative rings. J. Korean Math. Soc. **52**, 97-111(2015)
12. A. Badawi, Unsal Tekir, and Ece Yeikin, On 2-absorbing primary ideals of commutative rings. Bull. Korean Math. Soc. **51**, 1163-1173(2014)
13. Alison Elaine Becker, Results on n-Absorbing Ideals of Commutative Rings, M.S. thesis, University of Wisconsin-Milwaukee, Milwaukee, U. S. A., 2015
14. P. J. Cahen, M. Fontana, S. Frisch, and S. Glaz, Open problems in commutative ring theory, Commutative Algebra. Springer, 353-375(2014)
15. Fethi Callialp, Ece Yetkin and Unsal Tekir, On 2-absorbing primary and weakly 2-absorbing elements in multiplicative lattices. Italian Journal of Pure and Applied Mathematics **34**, 263-276(2015)

16. Ece Yetkin Celikel, Emel A. Ugurlu and Gulsen Ulucak, On ϕ-2-absorbing elements in multiplicative lattices. Palest. J. Math. **5**, 127-135(2016)
17. Ece Yetkin Celikel, Emel A. Ugurlu and Gulsen Ulucak, On ϕ-2-absorbing primary elements in multiplicative lattices. Palest. J. Math. **5**, 136-146(2016)
18. J. N. Chaudhari, 2-absorbing ideals in semirings. International Journal of Algebra **6**, 265-270(2012)
19. J. N. Chaudhari and H. P. Bendale, Weakly prime ternary subsemimodules of ternary semimodules. Journal of Algebra and Related Topics **2**, 63-72(2014)
20. Sirawich Chinwarakorn and Sajee Pianskool, On almost generalized 2-absorbing and weakly almost generalized 2-absorbing structures. ScienceAsia **41**, 64-72(2015)
21. Hyun Seung Choi and Andrew Walker, The radical of an *n*-absorbing ideal. arXiv:1610.10077 [math.AC] (2016)
22. A. Yousefian Darani, On 2-absorbing and weakly 2-absorbing ideals of commutative semirings. Kyungpook Math. J. **52**, 91-97(2012)
23. A. Yousefian Darani and Hojjat Mostafanasab, Generalizations of semicoprime preradicals. Algebra and Discrete Mathematics **21**, 214-238(2016)
24. A. Yousefian Darani and Hojjat Mostafanasab, On 2-absorbing preradicals. Journal of Algebra and Its Applications **14**, (22 pages)(2015)
25. A. Yousefian Darani and Hojjat Mostafanasab, Co-2-absorbing preradicals and submodules. Journal of Algebra and Its Applications **14**, (23 pages)(2015)
26. A. Yousefian Darani and Fatemeh Soheilnia, On n-absorbing submodules. Math. Commun. **17**, 547-557(2012)
27. A. Yousefian Darani and Fatemeh Soheilnia, 2-absorbing and weakly 2-absorbing submodules. Thai Journal of Mathematics **9**, 577-584(2011)
28. A. Yousefian Darani and Ghader Ghasemi, On L-fuzzy 2-absorbing ideals. Italian Journal of Pure and Applied Mathematics **36**, 147-154(2016)
29. A. Yousefian Darani and E.R. Puczyowski, On 2-absorbing commutative semigroups and their applications to rings. Semigroup Forum **86**, 83-91(2013)
30. Bijan Davvaz and Brahim Fahid, Rough fuzzy *n*-absorbing (*n*-absorbing primary) ideal in commutative rings. Italian Journal of Pure and Applied Mathematics **36**, 473-482(2016)
31. G. Donadze, The Anderson-Badawi conjecture for commutative algebras over infinite fields. Indian J Pure Appl Math, **47**, 691-696(2016)
32. Manish Kant Dubey and Pakhi Aggarwal, On n-absorbing submodules of modules over commutative rings. Beitrge zur Algebra und Geometrie / Contributions to Algebra and Geometry **57**, 679-690(2016)
33. Manish Kant Dubey and Pakhi Aggarwal, On 2-absorbing primary submodules of modules over commutative ring with unity. Asian-European J. Math. **8**, (12 pages)(2015)
34. Manish Kant Dubey and Poonam Sarohe, on 2-absorbing semimodules. Quasigroups and Related Systems **21**, 175-184 (2013)
35. Manish Kant Dubey and Poonam Sarohe, On $(n1,n)-\phi$-Prime ideals in semirings. Matematiqki Vesnik **67**, 222-232(2015)
36. M. Ebrahimpour, On generalization of almost prime and weakly prime ideals. Bull. Iranian Math. Soc. **40**, 531-540(2014)
37. M. Ebrahimpour, On generalizations of prime ideals (II). Comm. Algebra **42**, 3861-3875(2014)
38. M. Ebrahimpour and R. Nekooei, On generalization of prime ideals. Comm. Algebra **40**, 1268-1279(2012)
39. Farkhonde Farzalipour and Peyman Ghiasvand, On graded semiprime and graded weakly semiprime ideals. International Electronic Journal of Algebra **13**, 15-32 (2013)
40. Mohammad Hamoda and Arwa Eid Ashour, On graded *n*-absorbing submodules. Le Matematiche **70**, 243-254(2015)
41. C. Jayaram, Unsal Tekir and Ece Yetkin, 2-absorbing and weakly 2-absorbing elements in multiplicative lattices. Comm. Algebra **42**, 2338-2353(2014)
42. Pratibha Kumar, Manish Kant Dubey and Poonam Sarohe, On 2-absorbing primary ideals in commutative semirings. European Journal of Pure & Applied Mathematics **9**, 186-195(2016)

43. Pratibha Kumar, Manish Kant Dubey and Poonam Sarohe, Some results on 2-absorbing ideals in commutative semirings. Journal of Mathematics and Applications **38**, 5-13(2015)
44. A. Laradji, On n-absorbing rings and ideals. To appear in Colloq. Math.
45. C. S. Manjarekar and A. V. Bingi, Absorbing elements in lattice modules. International Electronic Journal of Algebra **19**, 58-76(2016)
46. H. Fazaeli Moghimi and S. Rahimi Naghani, On n-absorbing ideals and the n-Krull dimension of a commutative ring. J. Korean Math. Soc. **53**, 1225-1236(2016)
47. S. Moradi and A. Azizi, 2-absorbing and n-weakly prime submodules. Miskolc Mathematical Notes **13**, 75-86(2012)
48. S. Moradi and A. Azizi, Weakly 2-absorbing submodules of modules. Turk J Math **40**, 350-364(2016)
49. R. Moradi and M. Ebrahimpour, On ϕ-2-absorbing primary submodules. Acta Mathematica Vietnamica, (First Online) (2016). DOI: 10.1007/s40306-016-0181-0
50. Hojjat Mostafanasab, Ece Yetkin, Unsal Tekir and A. Yousefian Darani, On 2-absorbing primary submodules of modules over commutative rings. An. St. Univ. Ovidius Constanta**24**, 335-351(2016)
51. Hojjat Mostafanasab and A. Yousefian Darani, On n-absorbing ideals and two generalizations of semiprime ideals. (on line), to appear in Thai Journal of Mathematics.
52. Peyman Nasehpour, On the Anderson-Badawi $\omega_{R[X]}(I[X]) = \omega_R(I)$ conjecture. Archivum Mathematicum (BRNO) **52**, 71-78(2016)
53. Sh. Payrovi and S. Babaei, On 2-absorbing submodules over commutative rings. Lobachevskii Journal of Mathematics **36**, 58-64(2015)
54. Sh. Payrovi and S. Babaei, On the 2-absorbing ideals and zero divisor graph of equivalence classes of zero divisors. Journal of Hyperstructures **3**, 1-9(2014)
55. Sh. Payrovi and S. Babaei, On 2-absorbing submodules. Algebra Colloquium **19**, 913-920(2012)
56. Sh. Payrovi and S. Babaei, On the 2-absorbing ideals. International Mathematical Forum **7**, 265-271(2012)
57. Pattarawan Petchkaew, Amorn Wasanawichit and Sajee Pianskool, Generalizations of n-absorbing ideals of commutative semirings. Thai Journal of Mathematics **14**, 477-489(2016)
58. Fatemeh Soheilnia, On 2-absorbing and weakly 2-absorbing primary ideals of a commutative semiring. Kyungpook Mathematical Journal **56**, 107-120(2016)
59. Unsal Tekir, Suat Ko, Kursat Hakan Oral and Kar Ping Shum, On 2-absorbing quasi-primary ideals in commutative rings. Communications in Mathematics and Statistics **4**, 55-62(2016)
60. Unsal Tekir, Hojjat Mostafanasab and Krat Hakan Oral, Classical 2-absorbing submodules of modules over commutative rings. European Journal of Pure & Applied Mathematics **8**, 417-430(2015)
61. Yibeltal Yitayew Tessema, Berhanu Bekele Belayneh and K. Venkateswarlu, Certain generalized prime ideals in boolean like semirings. International Journal of Algebra **8**, 663-669(2014)
62. Mohammad Yasein and Rashid Abu-Dawwas, On almost 2-absorbing submodules. Italian Journal of Pure and Applied Mathematics **36**, 923-928(2016)

Acknowledgment: I would like to thank the referee for the great effort in proofreading the manuscript.