Maximum principles and overdetermined problems for Hessian equations

Principes du maximum et problèmes surdéterminés des équations Hessiennes

Cristian Enache[1], Monica Marras[2] and Giovanni Porru[2]

[1]Department of Mathematics and Statistics, American University of Sharjah, Sharjah, 26666, UAE
cenache@aus.edu
[2]Dipartimento di Matematica e Informatica, University of Cagliari, Cagliari, Italy
mmarras@unica.it, porru@unica.it

ABSTRACT. In this article we investigate some Hessian type equations. Our main aim is to derive new maximum principles for some suitable P-functions, in the sense of L.E. Payne, that is for some appropriate functional combinations of $u(x)$ and its derivatives, where $u(x)$ is a solution of the given Hessian type equations. To find the most suitable P-functions, we first investigate the special case of a ball, where the solution of our Hessian equations is radial, since this case gives good hints on the best functional to be considered later, for general domains. Next, we construct some elliptic inequalities for the well-chosen P-functions and make use of the classical maximum principles to get our new maximum principles. Finally, we consider some overdetermined problems and show that they have solutions when the underlying domain has a certain shape (spherical or ellipsoidal).

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1. Introduction

One of the most important and useful tools used in the study of partial differential equations is the maximum principle, which is nothing else that a natural extension to higher dimensions of the following elementary fact from calculus courses: any real-valued function f, which satisfies the inequality $f'' > 0$ on an open interval (a, b), cannot attain its maximum at a point of (a, b) (for more details, we refer the reader to the fundamental book of M.H. Protter and H.F. Weinberger [14]). For instance, if we consider a bounded domain $\Omega \subset \mathbb{R}^n, n \geq 2$, and a function $u(x)$ satisfying the equation

$$\Delta u = 1 \text{ in } \Omega,$$

then the maximum principle guarantees that $u(x)$ cannot attain its maximum in Ω.

In this paper we do not plan to investigate this kind of classical maximum principles, but rather maximum principles for P-functions, in the sense of L.E. Payne. More precisely, we will develop new maximum principles for some appropriate functional combinations of $u(x)$ and its derivatives (see the book of R. Sperb [18]), where $u(x)$ will be the solution of some Hessian equations. Moreover, we will find the best possible candidates for such maximum principles, in order to get the so-called best maximum principles. These candidates are the functional combinations which are identically constant on some particular type of domains, such as, for instance, domains with spherical or ellipsoidal shape. For instance, if Ω is the unit ball centered at the origin, then the function

$$u(x) = \frac{|x|^2 - 1}{2n},$$

satisfies $\Delta u = 1$ in Ω. Moreover, one can easily notice that the auxiliary functional defined as

$$\phi(x) := |\nabla u|^2 - \frac{2}{n} u,$$

is identically constant on Ω. Therefore, a question of interest is to investigate what is happening with the P-function ϕ when Ω is not a ball, but any other general domain. In such a case, for a general bounded domain Ω, C. Miranda proved that $\phi(x)$ satisfies a maximum principles, in the sense that $\phi(x)$ attains its maximum value on $\partial\Omega$, the boundary of Ω (see R. Sperb [18]).

Such maximum principles for appropriate P-functions have several important applications. First of all, they are very useful in getting isoperimetric estimates. For instance, when $u(x)$ also satisfies a Dirichlet boundary condition, that is $u = 0$ on $\partial\Omega$, and Ω is a strictly convex domain, then one can use the above mentioned Miranda's maximum principle and some computations in normal coordinates with respect to $\partial\Omega$, to obtain the following estimates (see R. Sperb, Ch. 6 [18]):

$$|\nabla u| \leq \frac{1}{nH_{\min}}, \qquad u(x) \geq -\frac{1}{2nH_{\min}^2},$$

where $H_{\min}$ is the minimum value of the mean curvature of $\partial\Omega$. These estimates are isoperimetric, since the equality holds if and only if Ω is a ball. The sufficient condition can be proved by using a second interesting application of the maximum principle for $\phi(x)$. Obviously, if we have equality above, then $u(x)$ satisfies the following overdetermined problem

$$\begin{cases} \Delta u = 1 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ |\nabla u| = c = const. \text{ on } \partial\Omega. \end{cases}$$

It was proved by J. Serrin, in a seminal paper [17], that this problem has a solution if and only if Ω is a ball. His method of proof is based on the classical maximum principle and the moving plane method. In a subsequent paper [19], H.F. Weinberger proved the same result making use of the maximum principle for $\phi(x)$, mentioned above. More precisely, using the boundary conditions on $u(x)$ and $|\nabla u(x)|$, the maximum principle implies that

either (I) $\phi(x) < c$ in Ω,   or (II) $\phi(x) \equiv c$ on Ω.

Next, using a Pohozaev type identity, (I) can be ruled out, so from (II) we get that

$$\Delta\phi(x) = 0 \text{ in } \Omega,$$

which implies that $D^2 u = \frac{1}{n} I_n$, so u is radial and Ω is a ball.

Finally, let us mention that the above maximum principle for ϕ and its applications have been extended by many authors to more general classes of nonlinear equations in divergence form (see [13], [18] and the references therein). The aim of this paper is to bring some new contributions, by extending such results to some fully nonlinear equations of Hessian type.

The paper is organized as follows. In Section 2, we first analyze the case of the ball, which gives inspiring hints about the P-functions to be considered. Thereafter, we develop some maximum principles for such P-functions. In Section 3, we consider some overdetermined problems and find the shapes of the underlying domain Ω for which we have solutions.

Throughout this paper the summation convention over repeated indices is employed and the following notations are adopted: $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/(\partial x_i \partial x_j)$. Using these notations we have, for instance,

$$u_{ij} u_i u_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}.$$

2. Maximum principles

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and let $\kappa_1, \cdots, \kappa_{n-1}$ be the principal curvatures of the boundary $\partial\Omega$ at a point $x \in \partial\Omega$. The κ-th curvature of $\partial\Omega$ at the point x is defined as

$$H_{(\kappa)} := H_{(\kappa)}(\kappa_1, \cdots, \kappa_{n-1}),$$

where $H_{(\kappa)}(\kappa_1, \cdots, \kappa_{n-1})$ denotes the κ-th elementary symmetric function of the curvatures $\kappa_1, \cdots, \kappa_{n-1}$. We say that Ω is κ-convex if $H_{(i)}(\kappa_1, \cdots, \kappa_{n-1}) \geq 0$ for $i = 1, \cdots, \kappa$ and for every $x \in \partial\Omega$. In this paper we assume Ω to be κ-convex.

Let us recall the definition of κ-convex functions in Ω. If $u = u(x)$ is a C^2 function, let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of the Hessian matrix $D^2 u$ and $\sigma_{(\kappa)}$ be the κ-th elementary symmetric function of the eigenvalues $\lambda_1, \cdots, \lambda_n$. Then we say that u is κ-convex if $\sigma_{(i)} \geq 0$ for $i = 1, \cdots, \kappa$ and for every $x \in \Omega$.

Let us now consider the following Hessian problem:

$$\sigma_{(\kappa)}(D^2 u) = f(u) \ \text{ in } \ \Omega, \ \ u = 0 \ \text{ on } \ \partial\Omega, \tag{1}$$

where f is a positive smooth function. We have $\sigma_{(1)} = \Delta u$ and $\sigma_{(n)} = \det(D^2 u)$. We say that a solution u of (1) is admissible if it is κ-convex. We refer to [16] for a discussion on existence and uniqueness of Problem (1).

Next, let us recall the definition of the ν-th Newton tensor associated with the Hessian matrix $D^2 u$:

$$T_{(\nu)} := \sigma_{(\nu)} I - T_{(\nu-1)} D^2 u, \quad \nu = 1, \cdots, n, \quad T_{(0)} := I,$$

where I is the identity matrix. An useful equation is the following (see [15])

$$\frac{\partial \sigma_{(\nu)}}{\partial u_{ij}} = T_{(\nu-1)}^{ij}, \quad i, j = 1, \cdots, n.$$

Since $D^2 u$ is symmetric, also the matrices $T_{(\nu)}$ are symmetric.

For $1 \leq \nu \leq \kappa$, we consider the P-function

$$\phi(x) := T_{(\nu-1)}^{kl} u_k u_l - 2 \binom{n-1}{\nu-1} \binom{n}{\kappa}^{-\frac{\nu}{\kappa}} \int_0^u f^{\frac{\nu}{\kappa}}(t) dt, \tag{2}$$

as well as the P-function

$$\psi(x) := T_{(\nu-1)}^{kl} u_k u_l - 2 \binom{n-1}{\nu-1} \binom{n}{\kappa}^{-\frac{\nu}{\kappa}} c_0^{\frac{\nu}{\kappa}} u, \tag{3}$$

where $c_0 = f(m_0)$, $m_0 = \min_\Omega u(x)$. We note that when f is a constant, $\phi(x) = \psi(x)$.

In this section we investigate the following problem: Do these P-functions, ϕ and ψ, attain their maximum or minimum values at the boundary $\partial\Omega$?

2.1. The radial case

In this subsection we'll investigate the case of a ball and obtain the following result:

Theorem 2.1. *Let Ω be a ball, u be a (radial) solution of (1), ϕ be the P-functions defined in (2) and ψ be the P-function defined in (3) We have:*
(i) if $f(t)$ is non-decreasing, then $\phi'(r) \geq 0$;
(ii) if $f(t)$ is non-increasing, then $\phi'(r) \leq 0$;
(iii) if $f(t)$ is non-decreasing, then $\psi'(r) \geq 0$.

Proof. We note that our equation (1) as well as our P-functions (2) and (3) are invariant under a rigid rotation. Therefore, we can assume $x = (r, 0, \cdots, 0)$ and we thus have

$$u_1 = u', \quad u_i = 0 \quad i = 2, \cdots, n,$$

respectively

$$D^2 u = \text{diag}\left\{ u'', \frac{u'}{r}, \cdots, \frac{u'}{r} \right\}. \tag{4}$$

Let us consider first the P-function $\phi(x)$, defined in (2), which can be written now as

$$\phi(r) = T^{11}_{(\nu-1)}(u')^2 - \alpha \int_0^u f(t)^{\frac{\nu}{\kappa}} dt,$$

where

$$\alpha := 2 \binom{n-1}{\nu-1} \binom{n}{\kappa}^{-\frac{\nu}{\kappa}}$$

. We claim that

$$T^{11}_{(\nu)} = \binom{n-1}{\nu} \left(\frac{u'}{r}\right)^{\nu}, \quad \nu = 0, \cdots, n-1. \tag{5}$$

Indeed, the relation is true (trivially) for $\nu = 0$. Next, we proceed by induction, assuming that (5) holds for $\nu - 1$, that is,

$$T^{11}_{(\nu-1)} = \binom{n-1}{\nu-1} \left(\frac{u'}{r}\right)^{\nu-1}.$$

Since

$$T_{(\nu)} = \sigma_{(\nu)} I - T_{(\nu-1)} D^2 u,$$

by (4) we find

$$T^{11}_{(\nu)} = \binom{n-1}{\nu-1} u'' \left(\frac{u'}{r}\right)^{\nu-1} + \binom{n-1}{\nu} \left(\frac{u'}{r}\right)^{\nu} - \binom{n-1}{\nu-1} u'' \left(\frac{u'}{r}\right)^{\nu-1}.$$

After simplification we find (5), and the claim is proved.

So, we can write

$$\phi(r) = \binom{n-1}{\nu-1} \frac{(u')^{\nu+1}}{r^{\nu-1}} - \alpha \int_0^u f^{\frac{\nu}{\kappa}}(t) dt,$$

and obtain

$$\phi'(r) = u'\left\{\binom{n-1}{\nu-1}\left[(\nu+1)\left(\frac{u'}{r}\right)^{\nu-1}u'' - (\nu-1)\left(\frac{u'}{r}\right)^{\nu}\right] - \alpha f^{\frac{\nu}{\kappa}}\right\}. \tag{6}$$

Equation (1) in the radial case reads as

$$\binom{n-1}{\kappa-1}\left(\frac{u'}{r}\right)^{\kappa-1}u'' + \binom{n-1}{\kappa}\left(\frac{u'}{r}\right)^{\kappa} = f, \tag{7}$$

from which we find

$$\left(\frac{u'}{r}\right)^{\kappa-1}u'' = \binom{n-1}{\kappa-1}^{-1}\left(f - \binom{n-1}{\kappa}\left(\frac{u'}{r}\right)^{\kappa}\right).$$

Insertion of the latter equation into (6) yields

$$\begin{aligned}
\phi'(r) = u'\bigg\{&\binom{n-1}{\nu-1}\bigg[(\nu+1)\binom{n-1}{\kappa-1}^{-1}\left(\left(\frac{u'}{r}\right)^{\nu-\kappa}f\right.\\
&\left.-\binom{n-1}{\kappa}\left(\frac{u'}{r}\right)^{\nu}\right) - (\nu-1)\left(\frac{u'}{r}\right)^{\nu}\bigg] - \alpha f^{\frac{\nu}{\kappa}}\bigg\}.
\end{aligned} \tag{8}$$

By (7) we have

$$k\left(\frac{u'}{r}\right)^{\kappa-1}u'' + (n-\kappa)\left(\frac{u'}{r}\right)^{\kappa} = k\binom{n-1}{\kappa-1}^{-1}f,$$

$$\left((u')^{\kappa}\right)' + \frac{n-\kappa}{r}(u')^{\kappa} = r^{\kappa-1}k\binom{n-1}{\kappa-1}^{-1}f,$$

and

$$\left(r^{n-\kappa}(u')^{\kappa}\right)' = r^{n-1}k\binom{n-1}{\kappa-1}^{-1}f. \tag{9}$$

Since $u'(0) = 0$, from (9) we find that $u'(r) > 0$. If $f(t)$ is non-decreasing then $f(u(r))$ if non-decreasing. Therefore, integrating (9) we find

$$r^{n-\kappa}(u')^{\kappa} \leq \frac{r^n}{n}k\binom{n-1}{\kappa-1}^{-1}f,$$

and

$$\left(\frac{u'}{r}\right)^{\kappa} \leq \binom{n}{\kappa}^{-1}f. \tag{10}$$

Insertion of the latter estimate into (8) yields

$$\begin{aligned}
\phi'(r) \geq u'\bigg\{&\binom{n-1}{\nu-1}\bigg[(\nu+1)\binom{n-1}{\kappa-1}^{-1}\bigg[\binom{n}{\kappa}\\
&-\binom{n-1}{\kappa}\bigg] - (\nu-1)\bigg]\binom{n}{\kappa}^{-\frac{\nu}{\kappa}} - \alpha\bigg\}f^{\frac{\nu}{\kappa}}.
\end{aligned}$$

Since

$$\binom{n-1}{\kappa-1}^{-1}\left[\binom{n}{\kappa} - \binom{n-1}{\kappa}\right] = 1,$$

recalling the value of α we find

$$\phi'(r) \geq u' \left\{ 2 \binom{n-1}{\nu-1} \binom{n}{\kappa}^{-\frac{\nu}{\kappa}} - \alpha \right\} f^{\frac{\nu}{\kappa}} = 0. \tag{11}$$

Part (i) of the theorem is proved.

If $f(t)$ is non-increasing then $f(u(r))$ if non-increasing. Integrating (9) now we find

$$\left(\frac{u'}{r} \right)^{\kappa} \geq \binom{n}{\kappa}^{-1} f.$$

Insertion of the latter estimate into (8) yields

$$\phi'(r) \leq u' \left\{ 2 \binom{n-1}{\nu-1} \binom{n}{\kappa}^{-\frac{\nu}{\kappa}} - \alpha \right\} f^{\frac{\nu}{\kappa}} = 0.$$

Part (ii) of the theorem is proved.

The proof of the result for ψ is similar. If $f(t)$ is non-decreasing, in place of (11) we find

$$\psi'(r) \geq u' 2 \binom{n-1}{\nu-1} \binom{n}{\kappa}^{-\frac{\nu}{\kappa}} \left\{ f^{\frac{\nu}{\kappa}} - c_0^{\frac{\nu}{\kappa}} \right\} \geq 0,$$

and the part (iii) of the theorem is also proved. $\square$

Remark. If $f(t)$ is non-increasing, then one can find $\psi'(r) \leq 0$, provided that the constant $c_0 = f(m_0)$ is replaced by $c_1 = f(0)$. In case $f(t)$ is a constant, Theorem 2.1 yields $\phi'(r) = 0$. Recall that in this case we have $\phi(r) = \psi(r)$.

2.2. *General* Ω, $\nu = 1$

A maximum principle for ϕ in this situation has been already proved by the first author:

Theorem 2.2. *Let Ω be a bounded κ-convex domain, and let f be non-decreasing. If u is an admissible solution to Problem (1), with convex level sets, the function ϕ defined in (2) with $\nu = 1$ attains its maximum value at the boundary $\partial\Omega$.*

Proof. We refer the reader to Theorem 2.1 of [6]. $\square$

We note that similar maximum (or minimum) principles for more general fully nonlinear equations have been obtained in [2], [7], [8] and [9]. In what follows, we will show that we can prove a similar maximum principle (with weaker conditions) for the function ψ. More precisely, we have:

Theorem 2.3. *Let Ω be a κ-convex domain, and let f be non-decreasing. If u is a κ-convex solution to Problem (1), the function ψ defined in (3) with $\nu = 1$ attains its maximum value at the boundary $\partial\Omega$.*

Proof. With $\nu = 1$ the function ψ reads as

$$\psi(x) = u_k u_k - 2 \binom{n}{\kappa}^{-\frac{1}{\kappa}} c_0^{\frac{1}{\kappa}} u.$$

We find

$$\psi_{ii} = 2u_{iik}u_k + 2u_{ki}u_{ki} - 2\binom{n}{\kappa}^{-\frac{1}{\kappa}}c_0^{\frac{1}{\kappa}}u_{ii}.$$

Let us make a suitable rotation around a point $\bar{x}$ such that D^2u is diagonal at $\bar{x}$. Then,

$$\psi_{ii} = 2u_{iik}u_k + 2u_{ii}u_{ii} - 2\binom{n}{\kappa}^{-\frac{1}{\kappa}}c_0^{\frac{1}{\kappa}}u_{ii}.$$

Since

$$T_{(\kappa-1)}^{ii}u_{ii} = \kappa\sigma_{(\kappa)} = \kappa f,$$

we get

$$T_{(\kappa-1)}^{ii}\psi_{ii} = 2\left[T_{(\kappa-1)}^{ii}u_{iik}u_k + T_{(\kappa-1)}^{ii}u_{ii}u_{ii} - \binom{n}{\kappa}^{-\frac{1}{\kappa}}c_0^{\frac{1}{\kappa}}\kappa f\right]. \tag{12}$$

From Equation (1) we find

$$T_{(\kappa-1)}^{ij}u_{ijk} = f'u_k.$$

Since D^2u is supposed to be diagonal at $\bar{x}$, then $T_{(\kappa-1)}$ is also diagonal, so

$$T_{(\kappa-1)}^{ii}u_{iik}u_k = f'u_ku_k. \tag{13}$$

We claim that

$$T_{(\kappa-1)}^{ii}u_{ii}u_{ii} \geq \binom{n}{\kappa}^{-\frac{1}{\kappa}}f^{\frac{1}{\kappa}}\kappa f. \tag{14}$$

Indeed, we know that

$$T_{(\kappa)} = \sigma_{(\kappa)}I - T_{(\kappa-1)}D^2u,$$

from which we find

$$T_{(\kappa-1)}(D^2u)^2 = \sigma_{(\kappa)}D^2u - T_{(\kappa)}D^2u.$$

Therefore,

$$T_{(\kappa-1)}^{ii}u_{ii}u_{ii} = \text{trace}(T_{(\kappa-1)}(D^2u)^2) = \sigma_{(\kappa)}\text{trace}(D^2u) - T_{(\kappa)}^{ii}u_{ii},$$

and

$$T_{(\kappa-1)}^{ii}u_{ii}u_{ii} = \sigma_{(\kappa)}\sigma_{(1)} - (\kappa+1)\sigma_{(\kappa+1)}.$$

In view of the latter equation, (14) holds if

$$\sigma_{(\kappa)}\sigma_{(1)} \geq (\kappa+1)\sigma_{(\kappa+1)} + \binom{n}{\kappa}^{-\frac{1}{\kappa}}\sigma_{(\kappa)}^{\frac{1}{\kappa}}\kappa\sigma_{(\kappa)}. \tag{15}$$

We know that

$$\sigma_{(1)} \geq n\binom{n}{\kappa}^{-\frac{1}{\kappa}}\sigma_{(\kappa)}^{\frac{1}{\kappa}}, \tag{16}$$

$$\sigma_{(1)} \geq n \binom{n}{\kappa+1}^{-\frac{1}{\kappa+1}} \sigma_{(\kappa+1)}^{\frac{1}{\kappa+1}}, \tag{17}$$

and

$$\sigma_{(\kappa)} \geq \binom{n}{\kappa}\binom{n}{\kappa+1}^{-\frac{\kappa}{\kappa+1}} \sigma_{(\kappa+1)}^{\frac{\kappa}{\kappa+1}}. \tag{18}$$

Now, by (16) we find

$$\frac{\kappa}{n}\sigma_{(\kappa)}\sigma_{(1)} \geq \binom{n}{\kappa}^{-\frac{1}{\kappa}} \sigma_{(\kappa)}^{\frac{1}{\kappa}} \kappa \sigma_{(\kappa)}. \tag{19}$$

By (17) and (18) we find

$$\left(1 - \frac{\kappa}{n}\right)\sigma_{(\kappa)}\sigma_{(1)} \geq (\kappa+1)\sigma_{(\kappa+1)}. \tag{20}$$

Adding (19) and (20) we find (15), and (14) is proved.

Finally, insertion of (13) and (14) into (12) yields

$$T_{(\kappa-1)}^{ii}\psi_{ii} \geq 2\left[f'u_k u_k + \binom{n}{\kappa}^{-\frac{1}{\kappa}}\left(f^{\frac{1}{\kappa}} - c_0^{\frac{1}{\kappa}}\right)\kappa f\right] \geq 0.$$

Since u is κ-convex, the Newton matrix $T_{(\kappa-1)}$ is positive definite. Therefore, the theorem follows from the classical maximum principle applied to the latter inequality. □

2.3. General Ω, $\nu = n$

A maximum principle for ψ (defined in (3)), when $\kappa = \nu = n$, has been proved by Feng and Shi (see [11], Theorem 1.1). When $f = 1$, the last result was proved in [4]. We discuss here the case of the P-function ϕ (defined in (2)). When $\kappa = \nu = n = 2$, a maximum principle has been obtained by the first and the third authors in [10]. In what follows we obtain an extension of their result to any higher dimension:

Theorem 2.4. *Let Ω be a convex domain, and let f be non-decreasing. If u is a convex solution to Problem (1) with $\kappa = n$, the function ϕ defined in (2) with $\nu = n$ attains its maximum value at the boundary $\partial\Omega$.*

Proof. Our method is very similar to that used in [4, 11]. Since $T_{(n)} = O$ (the null matrix) and $\sigma_{(n)} = f$, from the equation

$$T_{(n)} = \sigma_{(n)}I - T_{(n-1)}D^2 u,$$

we find

$$T_{(n-1)} = (D^2 u)^{-1}f.$$

If $[u^{kl}] = (D^2 u)^{-1}$, the P-function (2) with $\nu = n$ reads as

$$\phi(x) := u^{kl}u_k u_l f - 2\int_0^u f(s)\, ds. \tag{21}$$

We compute

$$\phi_i = u_i^{kl} u_k u_l f + 2u^{kl} u_{ki} u_l f + u^{kl} u_k u_l u_i f' - 2f u_i.$$

Since

$$u^{kl} u_{ki} = \delta_i^l \quad \text{(the Kroneckel delta function)}$$

we find

$$\phi_i = u_i^{kl} u_k u_l f + u^{kl} u_k u_l u_i f', \quad i = 1, \cdots, n.$$

Further differentiation yields

$$\phi_{ii} = u_{ii}^{kl} u_k u_l f + 2u_i^{kl} u_{ki} u_l f + 2u_i^{kl} u_k u_l u_i f'$$
$$+ 2u^{kl} u_{ki} u_l u_i f' + u^{kl} u_k u_l u_{ii} f' + u^{kl} u_k u_l u_i^2 f''.$$

Next, we make a suitable rotation around a point $\bar{x}$ such that $D^2 u$ is diagonal at $\bar{x}$. In particular, for i fixed we have

$$u^{ii} u_{ii} = 1, \quad i = 1, \cdots, n.$$

Then we find

$$\phi_{ii} = u_{ii}^{kl} u_k u_l f + 2u_i^{il} u_{ii} u_l f + 2u_i^{kl} u_k u_l u_i f'$$
$$+ 2u_i^2 f' + u^{ll} u_l^2 u_{ii} f' + u^{ll} u_l^2 u_i^2 f'', \quad i = 1, \cdots, n.$$

Multiplying by u^{ii} (and adding from $i = 1$ up to $i = n$) we get

$$u^{ii} \phi_{ii} = u^{ii} u_{ii}^{kl} u_k u_l f + 2u_i^{il} u_l f + 2u^{ii} u_i^{kl} u_k u_l u_i f'$$
$$+ 2u^{ii} u_i^2 f' + n u^{ll} u_l^2 f' + u^{ii} u^{ll} u_l^2 u_i^2 f''.$$

Since $(D^2 u)^{-1} = T_{n-1}$ is divergence free (see [15]) we have

$$u_i^{il} = 0, \quad l = 1, \cdots, n.$$

Therefore,

$$u^{ii} \phi_{ii} = u^{ii} u_{ii}^{kl} u_k u_l f + 2u^{ii} u_i^{kl} u_k u_l u_i f'$$
$$+ 2u^{ii} u_i^2 f' + n u^{ll} u_l^2 f' + u^{ii} u^{ll} u_l^2 u_i^2 f''. \tag{22}$$

Differentiating equation (1) (with $\kappa = n$) with respect to x^k we find

$$\frac{\partial \sigma_{(n)}}{\partial u_{ij}} u_{ijk} = f' u_k, \quad k = 1, \cdots, n.$$

Recalling that

$$\frac{\partial \sigma_{(n)}}{\partial u_{ij}} = T_{(n-1)}^{ij} = f u^{ij}$$

we have

$$f u^{ij} u_{ijk} = f' u_k,$$

and

$$u^{ij}u_{ijk} = \frac{f'u_k}{f}. \tag{23}$$

Further differentiation yields,

$$u^{ij}u_{ijkl} + u_l^{ij}u_{ijk} = \frac{f(f''u_ku_l + f'u_{kl}) - (f')^2u_ku_l}{f^2}. \tag{24}$$

Since

$$u^{il}u_{lj} = \delta_j^i, \quad i,j = 1,\cdots,n,$$

we have

$$u_l^{ij} = -u^{ip}u^{jq}u_{pql}. \tag{25}$$

At the point $\bar{x}$ we have

$$u_l^{ij} = -u^{ii}u^{jj}u_{ijl}.$$

By (24) and the latter equation, at $\bar{x}$ we find

$$u^{ii}u_{iikl} = u^{ii}u^{jj}u_{ijl}u_{ijk} + \frac{f(f''u_ku_l + f'u_{kl}) - (f')^2u_ku_l}{f^2}. \tag{26}$$

Let us now write (25) as

$$u_i^{kl} = -u^{kp}u^{lq}u_{pqi}.$$

Further differentiation yields, for $i = 1,\cdots,n$,

$$\begin{aligned}
u_{ii}^{kl} &= (u^{ks}u^{rq}u^{pl} + u^{kq}u^{ps}u^{rl})u_{pqi}u_{rsi} - u^{kp}u^{lq}u_{pqii} \\
&= 2u^{kk}u^{ll}u^{jj}u_{jki}u_{jli} - u^{kk}u^{ll}u_{klii}.
\end{aligned}$$

Multiplying by u^{ii} we find

$$u^{ii}u_{ii}^{kl} = 2u^{ii}u^{jj}u^{kk}u^{ll}u_{jki}u_{jli} - u^{kk}u^{ll}u^{ii}u_{iikl}. \tag{27}$$

Insertion of (26) into (27) leads to

$$\begin{aligned}
u^{ii}u_{ii}^{kl} &= 2u^{ii}u^{jj}u^{kk}u^{ll}u_{jki}u_{jli} \\
&\quad - u^{kk}u^{ll}\left(u^{ii}u^{jj}u_{ijl}u_{ijk} + \frac{f(f''u_ku_l + f'u_{kl}) - (f')^2u_ku_l}{f^2}\right).
\end{aligned}$$

Simplifying we get

$$u^{ii}u_{ii}^{kl} = u^{ii}u^{jj}u^{kk}u^{ll}u_{jki}u_{jli} - u^{kk}u^{ll}\frac{f(f''u_ku_l + f'u_{kl}) - (f')^2u_ku_l}{f^2}. \tag{28}$$

Therefore, since (for l fixed) $u^{ll}u_{ll} = 1$, we have

$$\begin{aligned}
u^{ii}u_{ii}^{kl}u_ku_lf &= u^{ii}u^{jj}\left(u^{kk}u_{jki}u_k\right)^2f - \left(u^{kk}u_k^2\right)^2f'' \\
&\quad - u^{kk}u_k^2f' + u^{kk}u^{ll}\frac{(f')^2}{f}u_k^2u_l^2.
\end{aligned} \tag{29}$$

Inserting (25) and (29) into (22) we find

$$u^{ii}\phi_{ii} = u^{ii}u^{jj}\left(u^{kk}u_{jki}u_k\right)^2 f - \left(u^{kk}u_k^2\right)^2 f'' + u^{kk}u^{ll}\frac{(f')^2}{f}u_k^2 u_l^2$$
$$+ (n+1)u^{kk}u_k^2 f' - 2u^{ii}u^{kk}u^{ll}u_{kli}u_k u_l u_i f' + \left(u^{ii}u_i^2\right)^2 f''.$$

Hence,

$$u^{ii}\phi_{ii} = u^{ii}u^{jj}\left(u^{kk}u_{jki}u_k - \frac{f'}{f}u_i u_j\right)^2 f + (n+1)u^{kk}u_k^2 f' \geq 0. \tag{30}$$

Since u is convex, the matrix $(D^2 u)^{-1}$ is positive definite. Therefore, the theorem follows by (30) and the classical maximum principle. $\qquad\square$

Remarks. (i) Consider a $n \times n$ symmetric positive definite matrix A such that

$$\det A = 1.$$

If $v(x) = u(Ax)$ then

$$Dv = ADu,$$
$$D^2 v = AD^2 uA,$$
$$(D^2 v)^{-1} = A^{-1}(D^2 u)^{-1}A^{-1}.$$

Hence,

$$\det D^2 v = \det D^2 u,$$

and

$$v^{ij}v_i v_j = (Dv)^t (D^2 v)^{-1} Dv = (Du)^t AA^{-1}(D^2 u)^{-1}A^{-1}ADu$$
$$= (Du)^t (D^2 u)^{-1} Du = u^{ij}u_i u_j.$$

It follows that equation (1) and the P-function (2) with $\nu = n$ are invariant under the transformation $x \to Ax$. Clearly this transformation maps the domain Ω into $A(\Omega)$, but it preserves the volume, since $\det A = 1$.

(ii) As far as we know, there is no maximum principle known for ϕ or ψ, when $2 < \nu < n$, except the one for the case of a ball, obtained in this paper. Therefore, the problem of finding a similar maximum principle for general domains remain open.

3. Overdetermined problems

Let us first recall a result proved by the first author:

Theorem 3.1. *Let Ω be strictly convex, and let u be an admissible solution of the problem (1) with $f = \binom{n}{\kappa}$. If u satisfies the following further boundary condition*

$$H_{(\kappa-1)}|Du|^{\kappa-1} = \binom{n-1}{\kappa-1} \quad on \ \partial\Omega,$$

then Ω must be a ball.

Proof. We refer the reader to Theorem 3.2 (i) of [6]. □

We prove here the following generalization.

Theorem 3.2. *Let Ω be strictly convex, and let u be an admissible solution of the problem (1), with $f = \binom{n}{\kappa}$. Fix an integer r such that $1 \leq r < \kappa$. If u satisfies the following further boundary condition*

$$\frac{H_{(\kappa-1)}}{H_{(r-1)}}|Du|^{\kappa-r} = \binom{n-r}{\kappa-r}\binom{\kappa-1}{r-1}^{-1} \quad on \ \partial\Omega, \tag{31}$$

then Ω must be a ball.

Proof. Recall the following Newton inequality: if $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of D^2u, then we have

$$\left(\frac{\sigma_{(r)}}{\binom{n}{r}}\right) \geq \left(\frac{\sigma_{(\kappa)}}{\binom{n}{\kappa}}\right)^{\frac{r}{\kappa}}, \tag{32}$$

with equality if and only if $\lambda_1 = \cdots = \lambda_n$ (see [12]).

From equation (1), with $f = \binom{n}{\kappa}$, we have

$$\frac{\sigma_{(\kappa)}}{\binom{n}{\kappa}} = 1.$$

Therefore, by (32) we have

$$\frac{\sigma_{(r)}}{\binom{n}{r}} \geq \frac{\sigma_{(\kappa)}}{\binom{n}{\kappa}}. \tag{33}$$

Recalling that $T_{(r-1)}$ is divergence free and using Green's formula we find

$$
\begin{aligned}
\int_\Omega \frac{\sigma_{(r)}}{\binom{n}{r}}dx &= \binom{n}{r}^{-1}\frac{1}{r}\int_\Omega T_{(r-1)}^{ij}u_{ij}dx = \binom{n}{r}^{-1}\frac{1}{r}\int_\Omega \left(T_{(r-1)}^{ij}u_i\right)_j dx \\
&= \binom{n}{r}^{-1}\frac{1}{r}\int_{\partial\Omega} T_{(r-1)}^{ij}u_iu_j\frac{1}{|Du|}ds = \binom{n}{r}^{-1}\frac{1}{r}\int_{\partial\Omega} H_{(r-1)}|Du|^r ds,
\end{aligned} \tag{34}
$$

where the equation

$$T_{(r-1)}^{ij}u_iu_j = H_{(r-1)}|Du|^{r+1} \quad on \ \partial\Omega$$

has been used (we refer the reader to [15] for a proof). Similarly, we find

$$
\begin{aligned}
\int_\Omega \frac{\sigma_{(\kappa)}}{\binom{n}{\kappa}}dx &= \binom{n}{\kappa}^{-1}\frac{1}{\kappa}\int_\Omega T_{(\kappa-1)}^{ij}u_{ij}dx = \binom{n}{\kappa}^{-1}\frac{1}{\kappa}\int_\Omega \left(T_{(\kappa-1)}^{ij}u_i\right)_j dx \\
&= \binom{n}{\kappa}^{-1}\frac{1}{\kappa}\int_{\partial\Omega} T_{(\kappa-1)}^{ij}u_iu_j\frac{1}{|Du|}ds = \binom{n}{\kappa}^{-1}\frac{1}{\kappa}\int_{\partial\Omega} H_{(\kappa-1)}|Du|^\kappa ds.
\end{aligned} \tag{35}
$$

Using the boundary condition (31) we have

$$\binom{n}{r}^{-1}\frac{1}{r}\int_{\partial\Omega} H_{(r-1)}|Du|^r ds = \binom{n}{\kappa}^{-1}\frac{1}{\kappa}\int_{\partial\Omega} H_{(\kappa-1)}|Du|^\kappa ds.$$

Therefore, from (34) and (35) we find

$$\int_{\Omega} \left(\frac{\sigma_{(r)}}{\binom{n}{r}} - \frac{\sigma_{(\kappa)}}{\binom{n}{\kappa}} \right) dx = 0.$$

The latter result and inequality (33) imply

$$\frac{\sigma_{(r)}}{\binom{n}{r}} = \frac{\sigma_{(\kappa)}}{\binom{n}{\kappa}}, \quad x \in \Omega.$$

Since

$$\frac{\sigma_{(\kappa)}}{\binom{n}{\kappa}} = 1$$

we can also write our previous equation as

$$\frac{\sigma_{(r)}}{\binom{n}{r}} = \left(\frac{\sigma_{(\kappa)}}{\binom{n}{\kappa}} \right)^{\frac{r}{\kappa}}.$$

Then, equality must hold in (32), and the eigenvalues $\lambda_1, \cdots, \lambda_n$ of $D^2 u$ must be equal. By Equation (1) with $f = \binom{n}{\kappa}$, $D^2 u$ is the identity matrix, and the result follows. $\qquad \square$

We consider now an overdetermined problem already solved in [3] by using a different method. We need the following Pohozaev type identity.

Lemma 3.3. *If Ω is κ-convex, if $e = (e^1, \cdots, e^n)$ is the exterior normal to $\partial\Omega$ and if u is an admissible solution to problem (1) with $f = 1$, then*

$$\int_{\partial\Omega} x^\ell e^\ell T^{ij}_{(\kappa-1)} u_i u_j ds = \kappa(n+2) \int_{\Omega} (-u) dx. \tag{36}$$

Proof. Since $u = 0$ on $\partial\Omega$ we have $u_j = |Du| e^j$ and $e^\ell = \frac{u_\ell}{|Du|}$. Therefore,

$$\begin{aligned}
\int_{\partial\Omega} x^\ell e^\ell T^{ij}_{(\kappa-1)} u_i u_j ds &= \int_{\partial\Omega} x^\ell u_\ell T^{ij}_{(\kappa-1)} u_i e^j ds = \int_{\Omega} \left(x^\ell u_\ell T^{ij}_{(\kappa-1)} u_i \right)_{,j} dx \\
&= \int_{\Omega} T^{ij}_{(\kappa-1)} u_i u_j dx + \int_{\Omega} x^\ell u_{\ell j} T^{ij}_{(\kappa-1)} u_i dx + \int_{\Omega} x^\ell u_\ell T^{ij}_{(\kappa-1)} u_{ij} dx.
\end{aligned} \tag{37}$$

Integrating by parts and using the fact that $T_{(\kappa-1)}$ is divergence free, we find

$$\int_{\Omega} T^{ij}_{(\kappa-1)} u_i u_j dx = \int_{\Omega} T^{ij}_{(\kappa-1)} u_{ij} (-u) dx = \kappa \int_{\Omega} (-u) dx. \tag{38}$$

Next, since $\sigma_{(\kappa)} = 1$ we have

$$T_{(\kappa)} = I - T_{(\kappa-1)} D^2 u,$$

and

$$T^{ij}_{(\kappa-1)} u_{\ell j} = \delta^i_\ell - T^{i\ell}_{(\kappa)}.$$

On the other hand, since $I - T_{(\kappa)}$ is divergence free, it follows that

$$\left(u_{\ell j}T^{ij}_{(\kappa-1)}\right)_i = 0.$$

Therefore,

$$\int_\Omega x^\ell u_{\ell j}T^{ij}_{(\kappa-1)}u_i dx = \int_\Omega \left(x^\ell u_{\ell j}T^{ij}_{(\kappa-1)}\right)_i (-u)dx$$

$$= \int_\Omega \delta^\ell_i u_{\ell j}T^{ij}_{(\kappa-1)}(-u)dx = \kappa\int_\Omega (-u)dx. \tag{39}$$

Finally, we also have

$$\int_\Omega x^\ell u_\ell T^{ij}_{(\kappa-1)}u_{ij}dx = \kappa\int_\Omega x^\ell u_\ell dx = \kappa n\int_\Omega (-u)dx. \tag{40}$$

Inserting now (38), (39) and (40) into (37), we are lead to (36). The proof of the lemma is thus achieved. □

Now let us consider problem (1), with $\kappa = n$ and $f = 1$. Our new further boundary condition is the following

$$T^{ij}_{(n-1)}u_i u_j = c^2 \quad \text{on } \partial\Omega. \tag{41}$$

Condition (41) can be written as

$$H_{(n-1)}|Du|^{n+1} = c^2 \quad on \ \partial\Omega.$$

We note that the exponent of $|Du|$ is now $n+1$, whereas in Theorem 3.1, with $\kappa = n$, the exponent of $|Du|$ is $n-1$. Condition (41) does not imply that Ω is a ball. Indeed, condition (41) and equation (1), with $\kappa = n$, are invariant under the linear transformation $y = Ax$, where A is a symmetric matrix with $\det A = 1$. By Theorem 4.2 of [5], condition (41) appears naturally while optimizing the domain dependent functional

$$E(\Omega) = \int_\Omega \left(T^{ij}_{(n-1)}u_i u_j u + (n+n^2)u\right)dx$$

(where u is the solution to Problem (1) with $\kappa = n$ and $f = 1$), among all domains Ω of given measure. Note that we can express the previous functional as

$$E(\Omega) = n^2\int_\Omega u\,dx.$$

By Theorem 1.1 of [4] (or our Theorem 2.4 with $f = 1$), we have

$$T^{ij}_{(n-1)}u_i u_j - 2u \leq \max_{\partial\Omega} T^{ij}_{(n-1)}u_i u_j \quad \forall x \in \Omega. \tag{42}$$

Theorem 3.4. *Let u be a convex solution to Problem (1), with $\kappa = n$ and $f = 1$. If we have*

$$T^{ij}_{(n-1)}u_i u_j = c^2 \quad on \ \partial\Omega \tag{43}$$

then equality holds in (42).

Proof. By Lemma 3.3, with $\kappa = n$, we have

$$\int_{\partial\Omega} x^\ell e^\ell T^{ij}_{(n-1)} u_i u_j ds = n(n+2) \int_\Omega (-u) dx.$$

Using condition (43), from the latter equation we find

$$c^2 \int_{\partial\Omega} x^\ell e^\ell d\sigma = n(n+2) \int_\Omega (-u) dx,$$

so

$$c^2 |\Omega| = (n+2) \int_\Omega (-u) dx. \tag{44}$$

On the other hand, integration of (42) over Ω yields

$$\int_\Omega T^{ij}_{(n-1)} u_i u_j dx + 2 \int_\Omega (-u)\, dx \leq c^2 |\Omega|. \tag{45}$$

Since

$$\int_\Omega T^{ij}_{(n-1)} u_i u_j dx = \int_\Omega T^{ij}_{(n-1)} u_{ij} (-u)\, dx = n \int_\Omega (-u)\, dx,$$

from (45) we find

$$(n+2) \int_\Omega (-u)\, dx \leq c^2 |\Omega|. \tag{46}$$

By (44), equality holds in (46). Hence, equality must hold in (45), and the assertion of the theorem follows. The proof is thus achieved. $\square$

Remarks. (i) Under the conditions of Theorem 3.4, we find $c^2 = -2u_{min}$. Hence,

$$T^{ij}_{(n-1)} u_i u_j = 2(u - u_{min}), \quad x \in \Omega.$$

As a consequence, if $u_{min} < t < 0$ and if $\Omega_t = \{x \in \Omega : u(x) < t\}$, we have

$$T^{ij}_{(n-1)} u_i u_j = 2(t - u_{min}) \quad \text{on} \quad \partial\Omega_t,$$

so

$$H_{(n-1)}(\partial\Omega_t) |Du|^{n+1} = 2(t - u_{min}) \quad \text{on} \quad \partial\Omega_t.$$

(ii) A solution of the overdetermined problem of Theorem 3.4 is an ellipse if $n = 2$ (see [1]) and an ellipsoid if $n > 2$ (see [3]).

Acknowledgement

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