ITERATIVE METHODS FOR THE NUMERICAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS

by

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Abstract

The aim of this thesis is twofold. First of all, in Chapters 1 and 2, we review the well-known Adomian Decomposition Method (ADM) and Variational Iteration Method (VIM) for obtaining exact and numerical solutions for ordinary differential equations, partial differential equations, integral equations, integro-differential equations, delay differential equations, and algebraic equations in addition to calculus of variations problems. These schemes yield highly accurate solutions. However, local convergence is a main setback of such approaches. It means that the accuracy deteriorates as the specified domain becomes larger, that is as we move away from the initial conditions. Secondly, we present an alternative uniformly convergent iterative scheme that applies to an extended class of linear and nonlinear third order boundary value problems that arise in physical applications. The method is based on embedding Green's functions into well-established fixed point iterations, including Picard's and Krasnoselskii-Mann's schemes. The effectiveness of the proposed scheme is established by implementing it on several numerical examples, including linear and nonlinear third order boundary value problems. The resulting numerical solutions are compared with both the analytical and the numerical solutions that exist in the literature. From the results, it is observed that the present method approximates the exact solution very well and yields more accurate results than the ADM and the VIM. Finally, the numerical results confirm the applicability and superiority of the introduced method for tackling various nonlinear equations.

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CHAPTER 0: INTRODUCTION

The aim of this thesis is twofold. First, we survey two major iterative methods that appear in the literature, which have been explored extensively for attaining analytical and/or numerical solutions for various linear and nonlinear problems, particularly those that model applications in the physical sciences. Secondly, we introduce a novel method based on manipulating Green's functions and some popular fixed point iterations schemes, such as Picard's and Mann's.

In the first Chapter, we will discuss and give a thorough review of the Adomian Decomposition Method (ADM). The decomposition method was first introduced and developed by George Adomian in [3, 5]. It has been receiving much attention from researchers in recent years in the field of applied mathematics, in general, and in the area of initial and boundary value problems in particular. The method efficiently handles a wide class of linear/nonlinear ordinary and partial differential equations, linear and nonlinear integral equations, and integro-differential equations. The ADM provides several significant advantages; it demonstrates fast convergence of the solution. It also handles the problem in a direct way without using linearization, perturbation, or any other restrictive assumptions that may change the physical behavior of the model under study. Furthermore, it provides an efficient numerical solution in the form of an infinite series that is obtained iteratively and usually converges to the exact solution using Adomian polynomials. The method is well addressed and used by many researchers in the literature [1-2, 4, 6-19].

In this Chapter, we apply the decomposition method on some problems of the linear/nonlinear ordinary and partial differential equations, algebraic equations, delay differential equations, integral equations, and integro-differential equations. The main advantage of the method is that it is capable of greatly reducing the size of computational work without affecting the accuracy of the numerical solutions. Also, it yields highly accurate solutions close to the initial conditions. However, the limitation of the ADM is that it converges locally, which means that as we move away from the left endpoints, the approximations deteriorate.

In the second Chapter, we will comprehensively review another well-known method, namely, the Variational Iteration Method (VIM). The VIM was first established by Ji-Huan He [29-32], and later used by several authors to solve various problems [20-28]. The technique is of great interest in the applied sciences. It was and still is utilized by mathematicians to handle a wide variety of applications that arise in engineering and sciences, such as homogeneous and inhomogeneous linear problems, as well as those that are nonlinear. Essentially, the VIM accurately computes the solutions in a series form that rapidly converges to the exact solution in an iterative fashion with specific features for each scheme. It yields several successive approximations by using the iteration of the correction functional. The VIM is a powerful and efficient method that results in approximations that are highly accurate, and also gives closed form solutions if they exist. This powerful technique handles both linear and nonlinear problems in a unified manner. Therefore, the VIM reduces the volume of calculations without requiring the use of Adomian polynomials, and hence the computations are direct and straightforward.

In this Chapter, we apply the variational iteration method on some problems, including linear/nonlinear ordinary and partial differential equations, calculus of variations problems, integral differential equations, and integro-differential equations. The advantages of the method are that it gives highly accurate numerical solutions and reduces the size of computational work. It is important to mention that a major shortcoming of the variational iteration method is that the error slowly deteriorates as we increase the values of x over the entire domain; hence, the convergence is local and not uniform.

In the third Chapter, the core part of the thesis begins. In this chapter, a new approach is introduced for the solution of a wide class of third order linear and nonlinear boundary value problems. The underlying strategy of the approach is based on manipulating Green's functions and fixed point iterations, such as Picard's and Krasnoselskii-Mann's schemes using a tailored and appropriately selected integral operator. The reliability and accuracy of the strategy are verified by implementing it on a number of test examples. The resulting numerical solutions are compared with both the analytical and the numerical solutions that are available in the literature. The proposed method provides an efficient computational tool for treating the third order linear and nonlinear boundary value problems.

We start the Chapter with a concise survey of the properties of the Green's functions essential to implement the method. Also, we apply two well-known fixed point iterations, namely, Picard's and Krasnoselskii-Mann's schemes, on a carefully selected integral operator. Finally, we introduce the method, provide related proofs, and apply it on several problems. The numerical results are illustrated and depicted through a number of tables and graphs. The comparisons with other numerical methods and with the available exact solutions are included.

In the final Chapter, we will summarize this dissertation, as well as discuss directions for future research.

CHAPTER 1: ADMIAN DECOMPOSITION METHOD

1.1 Method Description

The Adomian Decomposition Method (ADM) is applied for solving a wide class of linear and nonlinear ordinary differential equations, partial differential equations, algebraic equations, difference equations, integral equations and integro-differential equations as well.

Consider the following equation:

$$Lu + Nu + Ru = g, \tag{1.1}$$

where L is a linear operator, N represents a nonlinear operator and R is the remaining linear part. By defining the inverse operator of L as L^{-1}, assuming that it exists, we get

$$u = L^{-1}g - L^{-1}Nu - L^{-1}Ru. \tag{1.2}$$

The Adomian Decomposition Method assumes that the unknown function u can be expressed by an infinite series of the form

$$u = \sum_{n=0}^{\infty} u_n, \tag{1.3}$$

or equivalently

$$u = u_0 + u_1 + u_2 + \cdots, \tag{1.4}$$

where the components u_n will be determined recursively. Moreover, the method defines the nonlinear term by the Adomian polynomials.

More precisely, the ADM assumes that the nonlinear operator $N(u)$ can be decomposed by an infinite series of polynomials given by

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{1.5}$$

where A_n are the Adomian's polynomials defined as $A_n = A_n(u_0, u_1, u_2, \dots, u_n)$. Substituting (1.3) and (1.4) into equation (1.2) and using the fact that R is a linear operator we obtain

$$\sum_{n=0}^{\infty} u_n = L^{-1}g - L^{-1}\left(\sum_{n=0}^{\infty} R(u_n)\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \dots, u_n)\right), \tag{1.6}$$

or equivalently

$$u_0 + u_1 + u_2 + \cdots = L^{-1}g - L^{-1}\left(\sum_{n=0}^{\infty} R(u_n)\right) - L^{-1}(A_0 + A_1 + \cdots) \tag{1.7}$$

Therefore the formal recurrence algorithm could be defined by

$$\begin{aligned} &u_0 = L^{-1}g, \\ &\ldots, \\ &u_{n+1} = -L^{-1}\big(R(u_n)\big) - L^{-1}(A_n(u_0, u_1, \ldots, u_n)), \end{aligned} \tag{1.8}$$

or equivalently,

$$\begin{aligned} &u_0 = L^{-1}g, \\ &u_1 = -L^{-1}\big(R(u_0)\big) - L^{-1}\big(A_0(u_0)\big), \\ &u_2 = -L^{-1}\big(R(u_1)\big) - L^{-1}\big(A_1(u_0, u_1)\big), \\ &\ldots. \end{aligned} \tag{1.9}$$

Consider the nonlinear function $f(u)$. Then, the infinite series generated by applying the Taylor's series expansion of f about the initial function u_0 is given by

$$f(u) = f(u_0) + f'(u_0)(u - u_0) + \frac{1}{2!}f''(u_0)(u - u_0)^2 + \cdots. \tag{1.10}$$

By substituting (1.4) into equation (1.10), we have:

$$f(u) = f(u_0) + f'(u_0)(u_1 + u_2 + \ldots) + \frac{1}{2!}f''(u_0)(u_1 + u_2 + \ldots)^2 + \cdots. \tag{1.11}$$

Now, we expand equation (1.11). To obtain the Adomian polynomials, we need first to reorder and rearrange the terms. Indeed, one needs to determine the order of each term in (1.11) which actually depends on both the subscripts and the exponents of the $u_n's$. For instance, u_1 is of order 1; ${u_1}^2$ is of order 2; ${u_2}^3$ is of order 6; and so on. In general, ${u_n}^k$ is of order kn. In case a particular term involves the multiplication of $u_n's$, its order is determined by the sum of the terms of the $u_n's$ in each term. For example, ${u_2}^3{u_1}^2$ is of order 8 since (3)(2) + (2)(1) = 8.

As a result, rearranging the terms in the expansion (1.11) according to the order, we have

$$f(u) = f(u_0) + f'(u_0)u_1 + f'(u_0)u_2 + \frac{1}{2!}f''(u_0){u_1}^2 + f'(u_0)u_3$$
$$+ \frac{2}{2!}f''(u_0)u_1u_2 + \frac{1}{3!}f'''(u_0){u_1}^3 + f'(u_0)u_4 + \frac{1}{2!}f''(u_0){u_2}^2$$
$$+ \frac{2}{2!}f''(u_0)u_1u_3 + \frac{3}{3!}f'''(u_0){u_1}^2u_2 + \frac{1}{4!}f''''(u_0){u_1}^4 + \cdots. \qquad (1.12)$$

The Adomian polynomials are constructed in a certain way so that the polynomial A_1 consists of all terms in the expansion (1.12) of order 1, A_2 consists of all terms of order 2, and so on. In general, A_n consists of all terms of order n. Therefore, the first nine terms of Adomian polynomials are listed as follows:

$$A_0 = f(u_0),$$

$$A_1 = f'(u_0)u_1,$$

$$A_2 = f'(u_0)u_2 + \frac{1}{2!}f''(u_0){u_1}^2,$$

$$A_3 = f'(u_0)u_3 + \frac{2}{2!}f''(u_0)u_1u_2 + \frac{1}{3!}f'''(u_0){u_1}^3,$$

$$A_4 = f'(u_0)u_4 + \frac{1}{2!}f''(u_0)(2u_1u_3 + {u_2}^2) + \frac{3}{3!}f'''(u_0){u_1}^2u_2 + \frac{1}{4!}f''''(u_0){u_1}^4,$$

$$A_5 = f'(u_0)u_5 + \frac{1}{2!}f''(u_0)(2u_1u_4 + 2u_2u_3) + \frac{1}{3!}f'''(u_0)\left(3{u_1}^2u_3 + 3u_1{u_2}^2\right)$$
$$+ \frac{4}{4!}f^{(4)}(u_0){u_1}^3u_2 + \frac{1}{5!}f^{(5)}(u_0){u_1}^5,$$

$$A_6 = f'(u_0)u_6 + \frac{1}{2!}f''(u_0)(2u_1u_5 + 2u_2u_4 + {u_3}^2) + \frac{1}{3!}f'''(u_0)(3{u_1}^2u_4 + {u_2}^3 + 6u_1u_2u_3)$$
$$+ \frac{1}{4!}f^{(4)}(u_0)(4{u_1}^3u_3 + 6{u_1}^2{u_2}^2) + \frac{5}{5!}f^{(5)}(u_0){u_1}^4u_2 + \frac{1}{6!}f^{(6)}(u_0){u_1}^6,$$

$$A_7 = f'(u_0)u_7 + \frac{1}{2!}f''(u_0)(2u_1u_6 + 2u_2u_5 + 2u_3u_4)$$
$$+ \frac{1}{3!}f'''(u_0)(3{u_1}^2u_5 + 3u_1{u_3}^2 + 3u_3{u_2}^2 + 6u_1u_2u_4)$$
$$+ \frac{1}{4!}f^{(4)}(u_0)(4{u_1}^3u_4 + 12{u_1}^2u_2u_3 + 4u_1{u_2}^3)$$
$$+ \frac{1}{5!}f^{(5)}(u_0)\left(5{u_1}^4u_3 + 10{u_1}^3{u_2}^2\right) + \frac{1}{6!}f^{(6)}(u_0){u_1}^5u_2 + \frac{1}{7!}f^{(7)}(u_0){u_1}^7,$$

$$A_8 = f'(u_0)u_8 + \frac{1}{2!}f''(u_0)(2u_1u_7 + 2u_2u_6 + 2u_3u_5 + {u_4}^2)$$

$$+\frac{1}{3!}f'''(u_0)(3{u_1}^2u_6+3{u_2}^2u_4+3u_2{u_3}^2+6u_1u_2u_5+6u_1u_3u_4)$$
$$+\frac{1}{4!}f^{(4)}(u_0)(4{u_1}^3u_5+12{u_1}^2u_2u_4+12u_1{u_2}^2u_3+6{u_1}^2{u_3}^2$$
$$+{u_2}^4)+\frac{1}{5!}f^{(5)}(u_0)\left(5{u_1}^4u_4+20{u_1}^3u_2u_3+10{u_1}^2{u_2}^3\right)$$
$$+\frac{1}{6!}f^{(6)}(u_0)\left({u_1}^5u_3+15{u_1}^4{u_2}^2\right)+\frac{7}{7!}f^{(7)}(u_0){u_1}^6u_2 \qquad (1.13)$$
$$+\frac{1}{8!}f^{(8)}(u_0){u_1}^8.$$

The Adomian polynomial A_n was first introduced by Adomian himself; it was defined via the general formula

$$A_n(u_0,u_1,\dots,u_n)=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{\infty}u_k\lambda^k\right)\right]_{\lambda=0}, \quad n=0,1,2,\dots. \qquad (1.14)$$

To find the A_n's by Adomian general formula, these polynomials will be computed as follows:

$$A_0=N(u_0),$$

$$A_1=\frac{d}{d\lambda}N(u_0+u_1\lambda)|_{\lambda=0}=N(u_0)u_1,$$

$$A_2=\frac{1}{2!}\frac{d}{d\lambda}\left((u_1+2u_2\lambda)N''(u_0+u_1\lambda)\right)|_{\lambda=0}=N'(u_0)u_2+\frac{1}{2!}N''(u_0){u_1}^2,$$

$$\cdots. \qquad (1.15)$$

In the subsequent sections, the ADM method and a modified version of it will be used for solving several interesting linear and nonlinear equations which are of physical importance.

1.2 Convergence Analysis

It is clear from (1.14) that the $A_n's$ are indeed polynomials and hence the u_{n+1} term is obtained from (1.8). Cherruault [1] has given the first proof of convergence of the Adomian Decomposition Method and he used fixed point theorems for abstract functional equations. The order of convergence of the ADM was discussed by Babolian and Biazar [4].

Consider the general functional equation

$$u-N(u)=f, \qquad \text{for } u\in \mathbb{H}. \qquad (1.16)$$

where $\mathbb{H}$ is a Hilbert space and N is a nonlinear operator where $N: \mathbb{H} \longrightarrow \mathbb{H}$ and f is a given function in $\mathbb{H}$. The decomposition method assumes a series solution for u given by

$$u = \sum_{n=0}^{\infty} u_n, \tag{1.17}$$

while the nonlinear term $N(u)$ as the sum of the series

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{1.18}$$

where the A_n's are the Adomian polynomials in $u_0, \dots, u_n$ obtained by

$$A_n(u_0, u_1, \dots, u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{\infty} u_k \lambda^k\right)\right]_{\lambda=0}, \qquad n = 0,1,2, \dots. \tag{1.19}$$

Substituting equations (1.17) and (1.18) into the functional equation (1.16) gives

$$\sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n = f, \tag{1.20}$$

The method consists of the following scheme:

$$\begin{cases} u_0 = f, \\ u_{n+1} = A_n(u_0, u_1, \dots, u_n). \end{cases} \tag{1.21}$$

The Adomian technique is equivalent to determining the sequence $S_n = u_1 + u_2 + \dots + u_n$ by using the iterative scheme defined by

$$\begin{aligned} S_0 &= 0, \\ S_{n+1} &= N_n(u_0 + S_n), \end{aligned} \tag{1.22}$$

where $N_n(u_0 + S_n) = \sum_{n=0}^{i} A_i$.

If there exist limits

$$S = \lim_{n \to \infty} S_n , \qquad N = \lim_{n \to \infty} N_n, \tag{1.23}$$

in a Hilbert space $\mathbb{H}$, then S solves the functional equation $S = N(u_0 + S)$ in $\mathbb{H}$. The convergence of the Adomian decomposition method has been proved in [1-2], under the following two conditions:

$$||N|| < 1, \qquad ||N_n - N|| = \varepsilon_n \to 0 \;\; as \;\; n \to \infty. \tag{1.24}$$

In the first condition, the nonlinear function $N(u)$ has to be a contraction, while the second condition implies the convergence of the series $\sum_{n=0}^{\infty} A_n$.

Theorem 1.1 Let N be a nonlinear operator from a Hilbert space $\mathbb{H}$ where: $\mathbb{H} \longrightarrow \mathbb{H}$ and u be the exact solution of (1.16). The decomposition series $\sum_{n=0}^{\infty} u_n$ of u converges to u when $\exists\ \alpha < 1, \|u_{n+1}\| \leq \alpha \|u_n\|, \forall\ n \in N \cup \{0\}$.

Proof:
We have the sequence

$$S_n = u_1 + u_2 + \cdots + u_n \tag{1.24}$$

We need to show that this sequence is a Cauchy sequence in the Hilbert space $\mathbb{H}$. To do that let

$$\|S_{n+1} - S_n\| = \|u_{n+1}\| \leq \alpha \|u_n\| \leq \alpha^2 \|u_{n-1}\| \leq \cdots \leq \alpha^{n+1} \|u_0\|. \tag{1.25}$$

Since

$$\begin{aligned}
\|S_m - S_n\| &= \|(S_m - S_{m-1}) + (S_{m-1} - S_{m-2}) + \cdots + (S_{n+1} - S_n)\| \\
&\leq \|S_m - S_{m-1}\| + \|S_{m-1} - S_{m-2}\| + \cdots + \|S_{n+1} - S_n\| \\
&\leq \alpha^m \|u_0\| + \alpha^{m-1} \|u_0\| + \cdots + \alpha^{n+1} \|u_0\| \\
&\leq (\alpha^{n+1} + \alpha^{n+2} + \cdots) \|u_0\| = \frac{\alpha^{n+1}}{1-\alpha} \|u_0\|, \quad \text{for } n, m \in N, m \geq n.
\end{aligned} \tag{1.26}$$

Thus, S_m converges to S_n and

$$\lim_{n,m \to \infty} \|S_m - S_n\| = 0. \tag{1.27}$$

From (1.27), the sequence $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space $\mathbb{H}$.
Hence,

$$\lim_{n \to \infty} S_n = S, \qquad \text{for } S \in \mathbb{H}, \tag{1.28}$$

where $S = \sum_{n=0}^{\infty} u_n$.

Solving Eq. (1.16) is the same as solving the functional $N(u_0 + S)$; by assuming that N is a continuous operator we get

$$N(u_0 + S) = N\left(\lim_{n \to \infty} (u_0 + S_n)\right) = \lim_{n \to \infty} N(u_0 + S_n) = \lim_{n \to \infty} S_{n+1} = S. \tag{1.29}$$

Therefore, the solution of Equation (1.16) is S.

1.3 Algebraic Equations

In this section, we will apply the ADM for obtaining solution of algebraic equations. First, we will show an alternate proof of the quadratic formula using an iterative decomposition approach.

Consider the quadratic equation

$$ax^2 + bx + c = 0. \tag{1.30}$$

Upon completing the square on (1.30) leads to the widely known quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{1.31}$$

where $a, b,$ and c are real numbers with $a \neq 0$. Next we give another proof.

Proof:
First, we rewrite the quadratic equation as

$$x = -\frac{c}{b} - \frac{a}{b}x^2. \tag{1.32}$$

The Adomian decomposition method suggests the solution of (1.30) to be decomposed as an infinite series of the form

$$x = x_0 + x_1 + x_2 + \cdots = \sum_{n=0}^{\infty} x_n. \tag{1.33}$$

More specifically, we can write the quadratic equation in the operator form

$$Lu + Nu = g, \tag{1.34}$$

where $Nx = ax^2$, $Lx = bx$ and $g = -c$.

Then we have

$$bx = -c - ax^2, \tag{1.35}$$

or equivalently

$$Lu = -g - Nu. \tag{1.36}$$

Now, to solve the quadratic equation (1.31), we first rewrite it by dividing both sides of the equation by b, so we have the equation (1.32). Let $\alpha = -\frac{c}{b}$, $\beta = -\frac{a}{b}$, then (1.37) becomes

$$x = \alpha + \beta x^2. \tag{1.37}$$

The solution x of (1.32) is decomposed by the above infinite series of the components, while the nonlinear term x^2 is expressed in terms of an infinite series of polynomials

$$x^2 = A_0 + A_1 + A_2 + \cdots = \sum_{n=0}^{\infty} A_n, \tag{1.38}$$

where the $A_n{}'s$ are the Adomian polynomials. Substituting (1.33) and (1.38) into (1.37) gives

$$\sum_{n=0}^{\infty} x_n = \alpha + \beta \sum_{n=0}^{\infty} A_n. \tag{1.39}$$

The various components x_n of the solution x can be easily determined by using the recursive relation

$$x_0 = \alpha,$$

$$x_{n+1} = \beta A_n, \quad n \geq 0. \tag{1.40}$$

Since $Nx = x^2,$ therefore by using (1.13), the first few Adomian polynomials $A_n$are given by

$$\begin{aligned}
A_0 &= x_0{}^2,\\
A_1 &= 2x_0x_1,\\
A_2 &= x_1{}^2 + 2x_0x_2,\\
A_3 &= 2x_1x_2 + 2x_0x_3,\\
A_4 &= x_2{}^2 + 2x_1x_3 + 2x_0x_4,\\
A_5 &= 2x_0x_5 + 2x_1x_4 + 2x_2x_3,\\
A_6 &= x_3{}^2 + 2x_0x_6 + 2x_1x_5 + 2x_2x_4,\\
A_7 &= 2x_0x_7 + 2x_1x_6 + 2x_2x_5 + 2x_3x_4,\\
A_8 &= x_4{}^2 + 2x_0x_8 + 2x_1x_7 + 2x_2x_6 + 2x_3x_5,\\
&\vdots
\end{aligned} \tag{1.41}$$

Therefore the $x_k{}'s$ are given by

$$\begin{aligned}
x_0 &= \alpha,\\
x_1 &= \beta A_0 = \beta(x_0{}^2),\\
x_2 &= \beta A_1 = \beta(2x_0x_1),\\
x_3 &= \beta A_2 = \beta(x_1{}^2 + 2x_0x_2),
\end{aligned}$$

$$x_4 = \beta A_3 = \beta(2x_0x_4 + 2x_1x_2),$$
$$\vdots \tag{1.42}$$

Solving the equations (1.42) iteratively we get,

$$x_0 = \alpha,$$
$$x_1 = \beta x_0^2 = \beta\alpha^2,$$
$$x_2 = \beta(2x_0x_1) = \beta[2\alpha(\alpha^2\beta)] = 2\beta^2\alpha^3,$$
$$x_3 = \beta({x_1}^2 + 2x_0x_2) = \beta[\beta^2\alpha^4 + 2\alpha(2\beta^2\alpha^3)] = 5\beta^2\alpha^4,$$
$$x_4 = \beta(2x_0x_4 + 2x_1x_2) = \beta[2\alpha(5\beta^3\alpha^4) + 2\alpha^2\beta(2\beta^2\alpha^3)] = 14\beta^4\alpha^5,$$
$$\vdots \tag{1.43}$$

Hence, the infinite series solution of the quadratic equation is given by:

$$x = x_0 + x_1 + x_2 + \cdots,$$
$$= \alpha + \beta\alpha^2 + 2\beta^2\alpha^3 + 5\beta^2\alpha^4 + 14\beta^4\alpha^5 + \cdots,$$
$$= \frac{1}{2\beta}[2\alpha\beta + 2(\alpha\beta)^2 + 4(\alpha\beta)^3 + 10(\alpha\beta)^4 + 28(\alpha\beta)^5 + \cdots],$$
$$= \frac{1}{2\beta}[1 - 1 + 2\alpha\beta + 2(\alpha\beta)^2 + 4(\alpha\beta)^3 + 10(\alpha\beta)^4 + 28(\alpha\beta)^5 + \cdots],$$
$$= \frac{1}{2\beta} - \frac{1}{2\beta}[1 - 2\alpha\beta - 2(\alpha\beta)^2 - 4(\alpha\beta)^3 - 10(\alpha\beta)^4 - 28(\alpha\beta)^5 - \cdots]. \tag{1.44}$$

Notice that the last expansion is almost identical to the Maclaurin series expansion of the following root function:

$$\sqrt{1-4x} = 1 - 2x - \sum_{n=2}^{\infty} 1.3.5 \ldots (2n-3)\frac{2^n}{n!}x^n$$
$$= 1 - 2x - 2x^2 - 4x^3 - 10x^4 - 28x^5 - \cdots. \tag{1.45}$$

Thus the latter solution can be rewritten as

$$x = \frac{1}{2\beta} - \frac{1}{2\beta}\sqrt{1 - 4(\alpha\beta)}. \tag{1.46}$$

The Maclaurin expansion in (1.46) converges for

$$|4x| = |4\alpha\beta| = 4\left|\frac{-c}{b}\frac{-a}{b}\right| = \frac{4|ac|}{b^2} < 1, \tag{1.47}$$

which implies that the scheme converges if

$$b^2 - 4|ac| > 0. \tag{1.48}$$

Substituting the values of $\alpha = -\frac{c}{b}$, $\beta = -\frac{a}{b}$, given in equation (1.32), then the solution in (1.46) becomes:

$$x = \frac{1}{2\beta} - \frac{1}{2\beta}\sqrt{1 - 4(\alpha\beta)} = \frac{-b}{2a} + \frac{b}{2a}\sqrt{1 - 4\frac{ac}{b^2}}$$

$$= \frac{-b}{2a} + \frac{b}{2a|b|}\sqrt{1 - 4ac}. \tag{1.49}$$

Two cases follow from the solution given in equation (1.49).

Case 1: If $b > 0$ then the solution in (1.49) becomes

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}. \tag{1.50}$$

Therefore, the technique converges to the only one root, and the second solution can be approximated by factoring the quadratic equation. The solution in (1.50) implies that:

- If $b > 0$ & $a > 0$ then the method converges to the smaller solution.
- If $b > 0$ & $a < 0$ then the method converges to the larger solution.

Case 2: If $b < 0$ then the solution in (1.49) becomes

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{1.51}$$

As for the second solution, again it can be approximated by factoring. The solution in (1.51) implies that:

- If $b < 0$ & $a > 0$ then the method converges to the larger solution.
- If $b < 0$ & $a < 0$ then the method converges to the smaller solution.

Next, we use the ADM to solve a number of nonlinear algebraic equations.

Example 1.1 Consider the equation

$$x^2 + 4x + 3 = 0, \tag{1.52}$$

whose solutions are $x = -1$ and $x = -3$. Write it in the form

$$4x = -3 - x^2, \tag{1.53}$$

or

$$x = -\frac{3}{4} - \frac{1}{4}x^2. \tag{1.54}$$

Apply the ADM: decompose x as an infinite series while express the nonlinear term x^2 as an infinite series of Adomian polynomials. Hence we have

$$\begin{aligned} x_0 + x_1 + x_2 + \cdots &= -\frac{3}{4} - \frac{1}{4}\sum_{n=0}^{\infty} A_n \\ &= -\frac{3}{4} - \frac{1}{4}A_0 - \frac{1}{4}A_1 - \cdots. \end{aligned} \tag{1.55}$$

Therefore the $x_k{}'s$ are given by

$$\begin{aligned} x_0 &= -\frac{3}{4}, \\ x_1 &= -\frac{1}{4}A_0 = -\frac{1}{4}({x_0}^2), \\ x_2 &= -\frac{1}{4}A_1 = -\frac{1}{4}(2x_0x_1), \\ x_3 &= -\tfrac{1}{4}A_2 = -\tfrac{1}{4}({x_1}^2 + 2x_0x_2), \\ x_4 &= -\tfrac{1}{4}A_3 = -\tfrac{1}{4}(2x_0x_4 + 2x_1x_2), \\ &\vdots \end{aligned} \tag{1.56}$$

Thus, upon solving these last equations, the scheme (1.56) yields the following values:

$$\begin{aligned} x_0 &= -0.75, \\ x_1 &= -0.140625, \\ x_2 &= -0.052734, \\ x_3 &= -0.024719, \\ x_4 &= -0.012977, \\ x_5 &= -0.007300. \end{aligned} \tag{1.57}$$

n- Term Approximation $\sum_{k=0}^{n-1} x_k$	Numerical Solutions	Absolute Errors
$n = 1$	-0.750000	0.250
$n = 2$	-0.890625	0.109
$n = 3$	-0.943359	0.057
$n = 4$	-0.968078	0.032
$n = 5$	-0.981055	0.019
$n = 6$	-0.988355	0.012

Table 1.1 Comparison of the n-th term approximation of ADM to the exact solution x of Example 1.1.

In Table 1.1 we compute the absolute error and it is obvious that the ADM converges fast using only few terms of the iterative scheme. Note that the scheme is approaching the negative root which is -1.

Example 1.2 Consider the equation

$$-x^2 + 5x + 6 = 0, \tag{1.58}$$

whose solutions are $x = -1$ and $x = 6$. Write it in the form

$$5x = -6 + x^2,$$

$$x = -\frac{6}{5} + \frac{1}{5}x^2. \tag{1.59}$$

Applying the decomposition approach we have

$$x_0 + x_1 + x_2 + \cdots = -\frac{6}{5} + \frac{1}{5}\sum_{n=0}^{\infty} A_n$$

$$x_0 + x_1 + x_2 + \cdots = -\frac{6}{5} + \frac{1}{5}A_0 + \frac{1}{5}A_1 + \cdots. \tag{1.60}$$

Matching both sides, as was done in the previous example, the latter scheme yields the following first five values of the iterates

$$x_0 = -1.2000,$$

$$x_1 = 0.2880,$$

$$x_2 = -0.1382,$$

$$x_3 = 0.0829,$$

$$x_4 = -0.0557,$$

$$x_5 = 0.06790. \tag{1.61}$$

n- Term Approximation $\sum_{k=0}^{n-1} x_k$	Numerical Solutions	Absolute Errors
$n = 1$	-1.200	0.200
$n = 2$	-0.912	0.088
$n = 3$	-1.050	0.050
$n = 4$	-0.967	0.032
$n = 5$	-1.023	0.023
$n = 6$	-0.983	0.017

Table 1.2 Comparison of the n-th term approximation of ADM to the exact solution x of Example 1.2.

From the numerical results in Table 1.2, it is clear the scheme yields numerical values that converge pretty fast to the smaller root, which is $x = -1$.

Example 1.3 Consider the fifth order algebraic equation

$$x^5 - 3x^4 + 2x^3 + 5x^2 - 6x - 4 = 0, \tag{1.62}$$

whose solutions are $x = 1.76518195942719$, $x = -1.09890396313245$ and $x = -0.528896048966185$. Write it in the form

$$6x = x^5 - 3x^4 + 2x^3 + 5x^2 - 4,$$

$$x = -\frac{4}{6} + \frac{5}{6}x^2 + \frac{2}{6}x^3 - \frac{3}{6}x^4 + \frac{1}{6}x^5. \tag{1.63}$$

Applying the decomposition approach we have

$$x_0 + x_1 + x_2 + \cdots = -\frac{4}{6} + \frac{5}{6}\sum_{n=0}^{\infty} A_n + \frac{2}{6}\sum_{n=0}^{\infty} B_n - \frac{3}{6}\sum_{n=0}^{\infty} C_n + \frac{1}{6}\sum_{n=0}^{\infty} D_n$$

$$x_0 + x_1 + x_2 + \cdots = -\frac{4}{6} + \frac{5}{6}A_0 + \frac{2}{6}B_0 - \frac{3}{6}C_0 + \frac{1}{6}D_0 + \cdots. \tag{1.64}$$

where A_n, B_n, $C_n$and D_n are Adomian polynomials. Matching both sides, as was done in the previous example, the latter scheme yields the following first four values of the iterates

$$x_0 = -\frac{4}{6} = -0.6666666667,$$

$$x_1 = \frac{5}{6}x_0^2 + \frac{2}{6}x_0^3 - \frac{3}{6}x_0^4 + \frac{1}{6}x_0^5 = 0.1508916324,$$

$$x_2 = \frac{5}{3}x_0x_1 + x_0^2x_1 - 2x_0^3x_1 + \frac{5}{6}x_0^4x_1 = 0.0136609708,$$

$$\mathrm{x}_3 = \frac{5}{3}\mathrm{x}_0\mathrm{x}_2 + \frac{5}{6}x_1^2 + \frac{5}{6}x_0^4\mathrm{x}_2 + \frac{5}{3}x_0^3x_1^2 - 2x_0^3\mathrm{x}_2 - 3x_0^2x_1^2 + x_0^2\mathrm{x}_2 + \mathrm{x}_0x_1^2 \qquad (1.65)$$
$$= -0.03656980055.$$

Consequently, the solution in a series form is given by

$$x = \mathrm{x}_0 + \mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3 + \cdots = -0.5386838640. \qquad (1.66)$$

Example 1.4 Consider the system of nonlinear algebraic equations

$$\begin{cases} x_1^2 - 10x_1 + x_2^2 + 8 = 0 \\ x_1x_2^2 + x_1 - 10x_2 + 8 = 0 \end{cases}, \qquad (1.67)$$

with exact solution

$$X^* = (x_1^*, x_2^*)^t = (1,1)^t. \qquad (1.68)$$

By rewriting the system (1.67)

$$\begin{cases} \sum_{n=0}^{\infty} x_{1,n} = 0.8 + 0.1\left(\sum_{n=0}^{\infty} A_{1,n}\right) + 0.1\left(\sum_{n=0}^{\infty} A_{2,n}\right) \\ \sum_{n=0}^{\infty} x_{2,n} = 0.8 + 0.1\sum_{n=0}^{\infty} x_{1,n}\left(\sum_{n=0}^{\infty} A_{2,n}\right) + 0.1\left(\sum_{n=0}^{\infty} x_{1,n}\right) \end{cases}. \qquad (1.69)$$

Applying the decomposition approach we have

$$\begin{cases} x_1 = 0.8 + 0.1x_1^2 + 0.1x_2^2 \\ x_2 = 0.8 + 0.1x_1x_2^2 + 0.1x_1 \end{cases}. \qquad (1.70)$$

The latter scheme yields the following first seven values of the iterates

$$x_1 = x_{1,0} + x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} + x_{1,5} + x_{1,6} = 0.997853,$$

$$x_2 = x_{2,0} + x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} + x_{2,5} + x_{2,6} = 0.997562. \qquad (1.71)$$

The absolute errors are

$$AE1 = |x_1^* - 0.997853\ | = 2.14 \times 10^{-3},$$

$$AE2 = |x_2^* - 0.997562\ | = 2.43 \times 10^{-3}. \qquad (1.72)$$

To our knowledge, no research papers exist in the literature that examines the approximation of complex roots rather than real ones using ADM. However, through a number of experiments, we found out that if the initial term x_0 is appropriately chosen as a complex number close to the root, then the ADM might converge to a complex root.

1.4 Ordinary Differential Equations

In this section, we will employ the Adomian Decomposition Method to linear and nonlinear ordinary differential equations (ODEs). These will include initial value problems (IVPs) as well as boundary value problems (BVPs). We will implement the method on both will first-order and higher order ODEs.

1.4.1 Linear ODEs

To apply the Adomian Decomposition Method for solving linear ordinary differential equations, we consider the following general equation written in operator form:

$$L(u) + Ru = g(x), \tag{1.73}$$

where the linear differential operator L may be considered as the highest order derivative in the equation, R is the remainder of the differential operator, and $g(x)$ is an inhomogeneous term. If L is a first order operator defined by

$$L = \frac{d}{dx}, \tag{1.74}$$

Then, assuming that L is invertible, then the inverse operator L^{-1} is given by

$$L^{-1}(.) = \int_0^x (.)dx, \tag{1.75}$$

so that

$$L^{-1}Lu = u(x) - u(0). \tag{1.76}$$

However, if L is a second order differential operator given by

$$L = \frac{d^2}{dx^2}, \tag{1.77}$$

then the inverse operator L^{-1} is a two-fold integration operator given by

$$L^{-1}(.) = \int_0^x \int_0^x (.)\, dxdx, \tag{1.78}$$

Hence, we have

$$L^{-1}Lu = u(x) - u(0) - xu'(0). \tag{1.79}$$

In a parallel manner, if L is a third order differential operator, we can easily show that

$$L^{-1}Lu = u(x) - u(0) - xu'(0) - \frac{1}{2!}x^2u''(0). \tag{1.80}$$

For higher order ODEs it the latter equation can be generalized in a similar fashion.

Now, to implement the ADM, we proceed by first applying L^{-1} to both sides of (1.73) and after rearranging the terms we get

$$u(x) = \Phi_0 + L^{-1}g(x) - L^{-1}Ru, \tag{1.81}$$

where, as explained above, we have

$$\Phi_0 = \begin{cases} u(0), & L = \dfrac{d}{dx} \\ u(0) + xu'(0), & L = \dfrac{d^2}{dx^2} \\ u(0) + xu'(0) + \dfrac{1}{2!}x^2u''(0), & L = \dfrac{d^3}{dx^3} \\ u(0) + xu'(0) + \dfrac{1}{2!}x^2u''(0) + \dfrac{1}{3!}x^3u'''(0), & L = \dfrac{d^4}{dx^4} \end{cases} \tag{1.82}$$

and so on. The Adomian decomposition method admits the decomposition of u in the form of an infinite series of components

$$u(x) = \sum_{n=0}^{\infty} u_n, \tag{1.83}$$

where $u_n(x)$, $n \geq 0$ are the components of $u(x)$ that will be determined recursively. Substituting (1.82) into (1.81) gives

$$\sum_{n=0}^{\infty} u_n = \Phi_0 + L^{-1}g(x) - L^{-1}R\left(\sum_{n=0}^{\infty} u_n\right). \tag{1.84}$$

The various components u_n of the solution u can be easily determined by using the recursive relation

$$\begin{aligned} u_0 &= \Phi_0 + L^{-1}g(x), \\ u_{n+1} &= -L^{-1}Ru_n, \qquad n \geq 0. \end{aligned} \tag{1.85}$$

It is worth mentioning that the determination of the u_0 term depends on the specified initial conditions $u(0), u'(0), u''(0), \ldots$.

1.4.2 Nonlinear ODEs

Consider the following nonlinear ordinary differential equation written in operator form:

$$Lu + Ru + N(u) = g(x), \tag{1.86}$$

where the linear operator L is the highest order derivative, R is the remainder of the differential operator, $N(u)$ is the nonlinear terms and $g(x)$ expresses an inhomogeneous term. Without loss of generality, let L be the first order differential operator

$$L = \frac{d}{dx}, \tag{1.87}$$

then, assuming that L is invertible, then its inverse L^{-1} is given by

$$L^{-1}(.) = \int_0^x (.)dx. \tag{1.88}$$

Therefore,

$$L^{-1}Lu = u(x) - u(0). \tag{1.89}$$

On the other hand, if L is a second derivative operator given by

$$L = \frac{d^2}{dx^2}, \tag{1.90}$$

then the inverse operator L^{-1} is given by

$$L^{-1}(.) = \int_0^x \int_0^x (.)dxdx, \tag{1.91}$$

which means that

$$L^{-1}Lu = u(x) - u(0) - xu'(0). \tag{1.92}$$

While, if L is a third order differential operator, we can show that

$$L^{-1}Lu = u(x) - u(0) - xu'(0) - \frac{x^2}{2!}u''(0), \tag{1.93}$$

and so forth. In general, if L is a differential operator of order $n+1$, we can easily show that

$$L^{-1}Lu = u(x) - u(0) - xu'(0) - \frac{x^2}{2!}u''(0) - \frac{x^3}{3!}u'''(0) - \cdots - \frac{x^n}{n!}u^{(n)}(0). \tag{1.94}$$

Applying L^{-1} to both sides of (1.75) gives

$$u = \Phi_0 - L^{-1}Ru - L^{-1}N(u) + L^{-1}g(x), \tag{1.95}$$

where

$$\Phi_0 = \begin{cases} u(0), & if\ L = \frac{d}{dx} \\ u(0) + xu'(0), & if\ L = \frac{d^2}{dx^2} \\ u(0) + xu'(0) + \frac{x^2}{2!}u''(0), & if\ L = \frac{d^3}{dx^3} \\ u(0) + xu'(0) + \frac{x^2}{2!}u''(0) + \frac{x^3}{3!}u'''(0), & if\ L = \frac{d^4}{dx^4} \\ \vdots & \vdots \\ u(0) + xu'(0) + \frac{x^2}{2!}u''(0) + \frac{x^3}{3!}u'''(0) + \cdots . + \frac{x^n}{n!}u^{(n)}(0) & if\ L = \frac{d^{n+1}}{dx^{n+1}} \end{cases} \tag{1.96}$$

The decomposition technique consists of decomposing the solution into a sum of an infinite number of terms defined by the decomposition series

$$u = \sum_{n=0}^{\infty} u_n, \tag{1.97}$$

while the nonlinear term $N(u)$ is to be expressed by an infinite series of polynomials

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{1.98}$$

where the $A_n{}'$s are the Adomian polynomials. Substituting (1.97) and (1.98) into (1.95) yields

$$\sum_{n=0}^{\infty} u_n = \Phi_0 - L^{-1}R\left(\sum_{n=0}^{\infty} u_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) + L^{-1}g(x). \tag{1.99}$$

To construct the iterative scheme, we match both sides so that the $u_n$term is expressed in terms of the previously determined terms. More specifically, the Adomian decomposition method gives the following iterative algorithm:

$$u_0 = \Phi_0 + L^{-1}g(x),$$

$$u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n\,, \quad n \geq 0. \tag{1.100}$$

This in turn gives

$$u_0 = \Phi_0 + L^{-1}g(x),$$

$$u_1 = -L^{-1}Ru_0 - L^{-1}A_0,$$

$$u_2 = -L^{-1}Ru_1 - L^{-1}A_1,$$

$$u_3 = -L^{-1}Ru_2 - L^{-1}A_2,$$

$$\ldots. \tag{1.101}$$

In (1.13), we calculated the first few Adomian polynomials for general form of nonlinearity that may arise in any ordinary or partial differential equation.

1.4.3 Initial Value Problems

In this section, we apply the ADM to initial value problems for both linear and nonlinear ordinary differential equations.

In the following, we consider some examples for the illustration of the technique and to conform its applicability and efficiency.

Example 1.5 Consider the second order linear ordinary differential equation

$$u'' - u = 1, \tag{1.102}$$

subject to the initial conditions

$$u(0) = 0, \quad u'(0) = 1. \tag{1.103}$$

Solution:
In operator form, Eq. (1.102) can be written as

$$Lu = 1 + u, \qquad u(0) = 0, \; u'(0) = 1, \tag{1.104}$$

where L is the second order differential operator $Lu = u''$. It is clear that L^{-1} is invertible and is given by

$$L^{-1}(.) = \int_0^x \int_0^x (.)\, dt\, dt. \tag{1.105}$$

Applying L^{-1} to both sides of (1.104) and using the initial conditions into (1.96) gives

$$u = u(0) + xu'(0) + L^{-1}1 = x + \frac{x^2}{2} + L^{-1}\, u. \tag{1.106}$$

Upon using the decomposition series for the solution $u(x)$, results

$$\sum_{n=0}^{\infty} u_n = x + \frac{x^2}{2} + L^{-1}\left(\sum_{n=0}^{\infty} u_n\right). \tag{1.107}$$

Upon matching both sides, this leads to the recursive relation

$$\begin{aligned} u_0 &= x + \frac{x^2}{2}, \\ u_{n+1} &= L^{-1}\,(u_n), \quad n \geq 0. \end{aligned} \tag{1.108}$$

The first few components are thus determined as follows:

$$\begin{aligned} u_0 &= x + \frac{x^2}{2}, \\ u_1 &= \frac{x^3}{6} + \frac{x^4}{24}, \\ u_2 &= \frac{x^5}{5!} + \frac{x^6}{6!}. \end{aligned} \tag{1.109}$$

Consequently, the solution in a series form is given by

$$u(x) = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots, \tag{1.110}$$

and clearly in a closed form is given by

$$u(x) = e^x - 1, \tag{1.111}$$

which the exact solution of the problem. This is a case where the ADM converges to the solution.

x	ADOMIAN	EXACT	ABSOLUTE ERROR
0.0	0.0	0.0	0.0
0.1	0.1051709181	0.105170918	1.0×10^{-10}
0.2	0.2214027556	0.221402758	2.4×10^{-9}
0.3	0.3498587625	0.349858808	4.6×10^{-8}
0.4	0.4918243556	0.491824698	3.4×10^{-7}
0.5	0.6487196181	0.648721271	1.7×10^{-6}
0.6	0.8221128000	0.822118800	6.0×10^{-6}
0.7	1.0137348180	1.013752707	4.6×10^{-5}
0.8	1.2254947560	1.225540928	1.8×10^{-5}
0.9	1.4594963620	1.459603111	1.1×10^{-4}
1.0	1.7180555560	1.718281828	2.3×10^{-4}

Table 1.3 Comparison between the (ADM) solution and the exact solutions using three iterations.

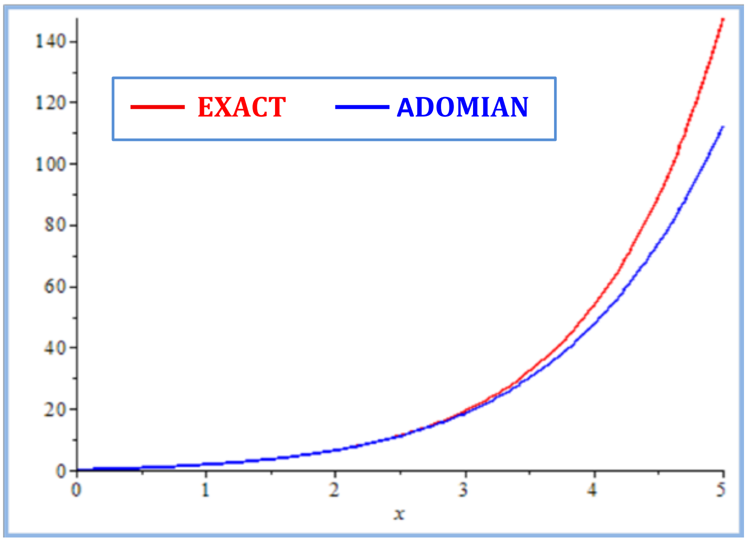

Figure 1.1 Comparison between the exact solution $y = e^x - 1$ and the ADM approximation using three iterations.

It must be stated here that (ADM) will only work for this particular test problem if $0 < x < 1$ and that the efficiency of the approach can be enhanced by computing further terms of the series. Comparison between the numerical solution using ADM and the exact solution are depicted in Figure 1.1 and Table 1.3 Note form the figure that the scheme yields highly accurate solution close to 0 but as we move away from this left end point the approximation deteriorates. This is a deficiency of the ADM as it gives highly accurate local approximation. In later, section, we will suggest domain decomposition strategy in order to improve the approximation as we move away from 0.

Example 1.6 Consider the first order nonlinear ordinary differential equation

$$u' + u^2 = 1, \tag{1.112}$$

subject to the initial condition

$$u(0) = 0. \tag{1.113}$$

Solution:

Applying L^{-1} we obtain to both sides of the equation and using the initial conditions into (1.96) gives

$$u = u(0) + L^{-1}1 - L^{-1}\, u^2 = -L^{-1}\, u^2 + x. \tag{1.114}$$

Using the decomposition series for u and the Adomian polynomial representation for the nonlinear term u^2, gives

$$\sum_{n=0}^{\infty} u_n = -L^{-1} \sum_{n=0}^{\infty} A_n + x, \tag{1.115}$$

where the $A_n{}'$s are the Adomian polynomials for u^2 as shown above. Matching both sides of the equation results in the following ADM iterative scheme:

$$\begin{aligned} u_0 &= x, \\ u_{n+1} &= -L^{-1}(\,A_n). \end{aligned} \tag{1.116}$$

This in turn gives

$$\begin{aligned} u_0 &= x, \\ u_1 &= -L^{-1}(\,A_0) = -L^{-1}(\,{u_0}^2) = -\frac{x^3}{3}, \\ u_2 &= -L^{-1}(\,A_1) = -L^{-1}(\,2u_0u_1) = \frac{2x^5}{15}, \\ u_3 &= -L^{-1}(\,A_2) = -L^{-1}(2u_0u_2 + {u_1}^2) = -\frac{17x^7}{315}, \\ &\vdots \end{aligned} \tag{1.117}$$

The solution in a series form is thus given by

$$u(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots, \tag{1.118}$$

which clearly converges to the exact solution

$$u(x) = \tanh(x). \tag{1.119}$$

In Figure 1.2 and Table 1.4 we compare the solution obtained by ADM and the exact solution, using only three iterations of the scheme. It is easy to see that the standard decomposition method converges to the exact solution very slowly. It is to be noted that only three iterates were needed to obtain an error of less than 10^{-6} for values close to 0.The overall errors can be made even much smaller by adding new terms of the decomposition. However, the solution deteriorates, as we mentioned earlier as we take values away from 0. The further the values from 0, the worst the approximation. Actually for $x \geq 1$ the method diverges as is clear from Figure 1.2.

x	ADOMIAN	EXACT	ABSOLUTE ERROR
0.0	0.0	0.0	0.0
0.1	0.09966799460	0.09966799462	2.0×10^{-11}
0.2	0.1973753092	0.1973753202	1.0×10^{-8}
0.3	0.2913121971	0.2913126125	4.2×10^{-7}
0.4	0.3799435784	0.3799489623	5.4×10^{-6}
0.5	0.4620783730	0.4621171573	3.9×10^{-5}
0.6	0.5368572343	0.5370495670	1.9×10^{-4}
0.7	0.6036314822	0.6043677771	7.3×10^{-4}
0.8	0.6617060368	0.6640367703	2.3×10^{-3}
0.9	0.7099191514	0.7162978702	6.4×10^{-3}
1.0	0.7460317460	0.7615941560	1.6×10^{-2}

Table 1.4 Comparison between the exact solution and the approximate solution $u(x)$ obtained using decomposition method with three iterations.

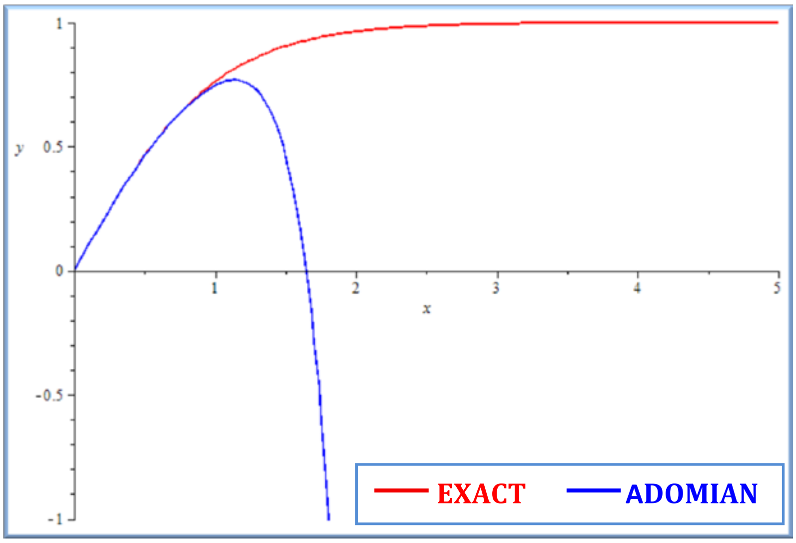

Figure 1.2 Comparison between the exact solution of Example 1.6 and the approximate solution using ADM.

Example 1.7 Consider the Bratu-type initial value problem which arises in many applications such as radiative heat transfer, chemical reaction theory and nanotechnology

$$u'' - 2e^{u} = 0, \qquad 0 < x < 1, \tag{1.120}$$

subject to the initial conditions

$$u(0) = u'(0) = 0. \tag{1.121}$$

Solution:

Applying L^{-1} we obtain to both sides of the equation and using the initial conditions into (1.96) gives

$$u = 2L^{-1}e^{u}. \tag{1.122}$$

Using the decomposition series for u and the Adomian polynomial representation for the nonlinear term e^{u}, gives

$$\sum_{n=0}^{\infty} u_n = 2L^{-1} \sum_{n=0}^{\infty} A_n, \tag{1.123}$$

where the $A_n{}'$s are the Adomian polynomials for e^{u} as shown above. Matching both sides of the equation results in the following ADM iterative scheme:

$$u_0 = 0,$$
$$u_{n+1} = 2L^{-1}(\,A_n). \tag{1.124}$$

This in turn gives

$$u_0 = 0,$$

$$u_1 = 2L^{-1}(\,A_0) = 2L^{-1}(\,e^{u_0}) = x^2,$$

$$u_2 = 2L^{-1}(\,A_1) = 2L^{-1}(\,u_1 e^{u_0}) = \frac{x^4}{6},$$

$$u_3 = 2L^{-1}(\,A_2) = L^{-1}\left(u_2 e^{u_1} + \frac{1}{2}u_1^2 e^{u_1}\right) = \frac{2x^6}{45},$$

$$u_4 = 2L^{-1}(\,A_3) = L^{-1}\left(e^{u_0}u_3 + e^{u_0}u_1u_2 + \frac{1}{6}u_1{}^3 e^{u_0}\right) = \frac{17x^8}{1260} \tag{1.125}$$

$$\vdots$$

The solution in a series form is thus given by

$$u(x) = x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \frac{17x^8}{1260} + \cdots. \quad (1.126)$$

x	ABSOLUTE ERROR
0.0	0.0
0.1	4.39×10^{-14}
0.2	4.54×10^{-11}
0.3	2.66×10^{-9}
0.4	4.85×10^{-7}
0.5	4.67×10^{-6}
0.6	3.01×10^{-6}
0.7	1.48×10^{-5}
0.8	6.00×10^{-4}
0.9	2.11×10^{-4}
1.0	6.65×10^{-4}

Table 1.5 Errors of the ADM for initial value problem of the Bratu-type

1.4.4 Boundary Value Problems

In this section, we apply the Adomian decomposition method to obtain numerical and/or exact solutions to a number of linear/nonlinear boundary value problems.

Example 1.8 Consider the following seventh order linear boundary value problem

$$u^{(7)}(x) = xu + e^x(x^2 - 2x - 6), \qquad 0 \le x \le 1, \quad (1.127)$$

which complimented with the boundary conditions

$$\begin{aligned} &u(0) = 1, && u(1) = 0, \\ &u'(0) = 0, && u'(1) = -e, \\ &u''(0) = -1, && u''(1) = -2e, \\ &u'''(0) = -2. \end{aligned} \quad (1.128)$$

The exact solution of this problem (1.127) is

$$u(x) = (1 - x)e^x. \quad (1.129)$$

Solution:
In an operator form, Equation (1.127) becomes

$$Lu = xu + e^x(x^2 - 2x - 6), \quad (1.130)$$

where the differential operator L is given by

$$L = \frac{d^7}{dx^7}, \tag{1.131}$$

and therefore the inverse operator L^{-1} will be defined by

$$L^{-1}[.] = \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x [.] \; dx\,dx\,dx\,dx\,dx\,dxdx\,. \tag{1.132}$$

Operating with L^{-1} on both sides of (1.130) and using the boundary conditions (1.128) at $x = 0$ we obtain

$$u(x) = -63 - 64x - \frac{35}{2!}x^2 - 4x^3 + \left(-\frac{1}{2} + \frac{\alpha}{24}\right)x^4 + \left(-\frac{1}{30} + \frac{\beta}{120}\right)\beta x^5 + \frac{1}{360}(2+\gamma)x^6 + e^x(-8+x)^2 + L^{-1}\big(xu(x)\big), \tag{1.133}$$

where α, β and γ are constants and

$$u^{(4)}(0) = \alpha, \; u^{(5)}(0) = \beta, \quad \text{and} \quad u^{(6)}(0) = \gamma. \tag{1.134}$$

Substituting the series assumption (1.3) into both sides of (1.133) yields

$$\sum_{n=0}^{\infty} u_n(x) = -63 - 64x - \frac{35}{2!}x^2 - 4x^3 + \left(-\frac{1}{2} + \frac{\alpha}{24}\right)x^4 + \left(-\frac{1}{30} + \frac{\beta}{120}\right)\beta x^5 + \frac{1}{360}(2+\gamma)x^6 + e^x(-8+x)^2 + L^{-1}\left(x\sum_{n=0}^{\infty} u_n(x)\right). \tag{1.135}$$

Following the decomposition method we obtain the following recursive relation

$$u_0(x) = -63 - 64x - \frac{35}{2!}x^2 - 4x^3 + \left(-\frac{1}{2} + \frac{\alpha}{24}\right)x^4 + \left(-\frac{1}{30} + \frac{\beta}{120}\right)\beta x^5 + \frac{1}{360}(2+\gamma)x^6 + e^x(-8+x)^2,$$

$$u_{n+1}(x) = L^{-1}\big(xu_n(x)\big), \quad n = 0,1,2,\dots. \tag{1.136}$$

To find the unknown constants α, β and γ we have to use the boundary conditions at $x = 1$ on the first four terms given by

$$u = u_0 + u_1 + u_2 + u_3. \tag{1.137}$$

Upon solving the resulting equations, the values of the constants α, β and γ are determined to be

$$\alpha = -3.0000001, \quad \beta = -3.9999991, \quad \gamma = -5.0000021. \tag{1.138}$$

Thus, the series solution can be written as

$$u(x) = 1 - \frac{x^2}{2} - \frac{x^3}{3} - 0.125x^4 - 0.03333x^5 - 0.00694x^6 - \frac{x^7}{840} - \frac{x^8}{5760} - \frac{x^9}{45360} - \frac{x^{10}}{403200} - \frac{x^{11}}{3991680} - (2.296 \times 10^{-8})x^{12} - (1.606 \times 10^{-10})x^{13} + \cdots. \quad (1.139)$$

x	ADM	EXACT	ABSOLUTE ERROR
0.0	1.0	1.0	0.0
0.1	0.9946538264	0.9946538262	2.0×10^{-10}
0.2	0.9771222079	0.9771222064	1.5×10^{-9}
0.3	0.9449011766	0.9449011656	1.1×10^{-8}
0.4	0.8950948709	0.8950948188	5.2×10^{-8}
0.5	0.8243608089	0.8243606355	1.7×10^{-7}
0.6	0.7288479868	0.7288475200	4.7×10^{-7}
0.7	0.6041268954	0.6041258121	1.1×10^{-6}
0.8	0.4451104431	0.4451081856	2.3×10^{-6}
0.9	0.2459646419	0.2459603111	4.3×10^{-6}
1.0	0.0000077797	0	7.8×10^{-6}

Table 1.6 Absolute error for Example 1.8 resulting from ADM using three iterations.

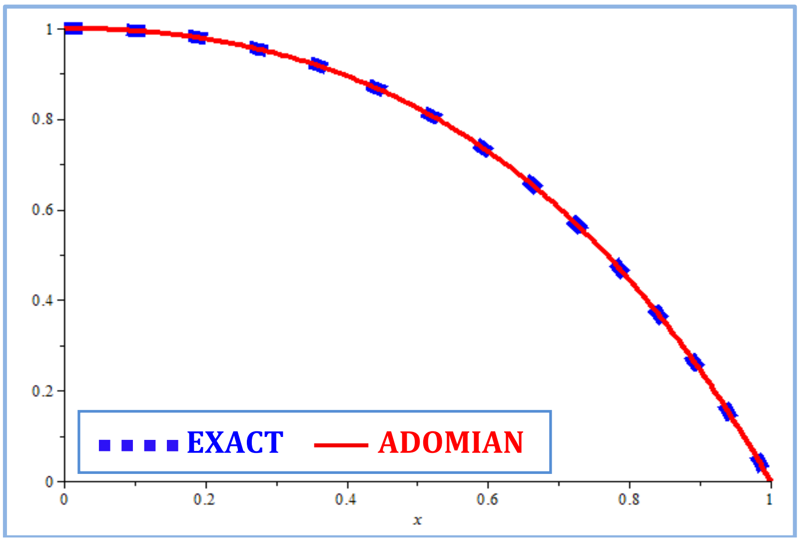

Figure 1.3 Exact solution for Example 1.8 compared with the approximate solution using ADM.

Table 1.6 and Figure 1.3 compare the exact solution with the numerical solution obtained by ADM. Clearly, the absolute error is extremely small using only few iterates. The error starts worsening as we consider values away from 0, but highly accurate in a neighborhood of 0.

Example 1.9 Consider the following nonlinear sixth order BVP:

$$u^{(6)}(x) = e^{-x}u^2(x), \qquad 0 < x < 1, \tag{1.140}$$

subject to the boundary condition

$$u(0) = u''(0) = u''''(0) = 1,$$
$$u(1) = u''(1) = u''''(0) = e. \tag{1.141}$$

The exact solution to this problem is

$$u(x) = e^x. \tag{1.142}$$

Solution:
In operator form, Eq. (1.140) can be written as

$$Lu(x) = e^{-x}u^2(x), \qquad 0 < x < 1, \tag{1.143}$$

where L is a first order differential operator. It is clear that L^{-1} is invertible and given by

$$L^{-1}[.] = \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x [.] \; dx\,dx\,dx\,dx\,dx\,dx\,. \tag{1.144}$$

Operating with L^{-1}, and using the boundary conditions at $x = 0$, we obtain

$$u(x) = 1 + \alpha x + \frac{1}{2}x^2 + \frac{1}{6}\beta x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \mathrm{L}^{-1}\left(e^{-x}u^2(x)\right), \tag{1.145}$$

where $\alpha, \beta, \&\, \gamma$ are constants and

$$\alpha = u'(0), \quad \beta = u'''(0), \quad \gamma = u^{(5)}(0). \tag{1.146}$$

Substituting the decomposition series (1.3) for $u(x)$ and the series of polynomials (1.5) for the nonlinear term $u^2(x)$ into (1.145) gives

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \alpha x + \frac{1}{2}x^2 + \frac{1}{6}\beta x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + L^{-1}\left(e^{-x}\sum_{n=0}^{\infty} A_n(x)\right), \tag{1.147}$$

where the $A_n{}'s$ are the Adomian polynomials. Consequently, the components of $u(x)$ can be elegantly determined by using the recursive relation

$$u_0(x) = 1,$$

$$u_1(x) = \alpha x + \frac{1}{2}x^2 + \frac{1}{6}\beta x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + L^{-1}(e^{-x}A_0),$$

$$u_{n+1}(x) = L^{-1}(e^{-x}A_n), \qquad n \geq 1. \tag{1.148}$$

To determine the components recurrently, we can use the scheme in (1.13). Using these polynomials into (1.148), the first few components can be determined recursively by

$$u_0(x) = 1,$$

$$u_1(x) = \alpha x + \frac{1}{2}x^2 + \frac{1}{6}\beta x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + L^{-1}(e^{-x}A_0)$$

$$= -1 + (\alpha + 1)x + \left(\frac{1}{6}\beta + \frac{1}{6}\right)x^3 + \left(\frac{1}{120}\gamma + \frac{1}{120}\right)x^5 + e^{-x}. \tag{1.149}$$

Consequently, the solution in a series form is given by

$$u(x) = (\alpha + 1)x + \left(\frac{1}{6}\beta + \frac{1}{6}\right)x^3 + \left(\frac{1}{120}\gamma + \frac{1}{120}\right)x^5 + e^{-x}. \tag{1.150}$$

We expand e^{-x} to obtain the approximation of $u(x)$ as

$$u(x) = 1 + \alpha x + \frac{1}{2}x^2 + \frac{1}{6}\beta x^3 + \frac{1}{24}x^4 + \frac{1}{120}\gamma x^5 + \cdots. \tag{1.151}$$

Now we use the boundary conditions at $x = 1$ on this 2-term approximant in order to determine the values of the constants α, β and γ. We get

$$\alpha = 1.006979226, \quad \beta = 0.9319015233, \quad \gamma = 1.718281828. \tag{1.152}$$

Consequently, the series solution becomes

$$u(x) = 1 + 1.006979226x + \frac{1}{2}x^2 + 0.1553169206x^3 + \frac{1}{24}x^4 + 0.01431901523x^5 + \cdots. \tag{1.153}$$

Table 1.7 depicts the numerical results obtained by ADM using only two iterations. The error seems this time to be uniformly distributed on the interval $[0, 1]$.

x	ADM	EXACT	ABSOLUTE ERROR
$\mathbf{0.0}$	1	1	0.0
$\mathbf{0.1}$	1.105857550	1.105170918	6.9×10^{-4}
$\mathbf{0.2}$	1.222709629	1.221402758	1.3×10^{-3}
$\mathbf{0.3}$	1.351659620	1.349858808	1.8×10^{-3}
$\mathbf{0.4}$	1.493945267	1.491824698	2.1×10^{-3}
$\mathbf{0.5}$	1.650955864	1.648721271	2.2×10^{-3}
$\mathbf{0.6}$	1.824249438	1.822118800	2.1×10^{-3}
$\mathbf{0.7}$	2.015569926	2.013752707	1.8×10^{-3}
$\mathbf{0.8}$	2.226864366	2.225540928	1.3×10^{-3}
$\mathbf{0.9}$	2.460300073	2.459603111	7.0×10^{-4}
$\mathbf{1.0}$	2.718281829	2.718281828	1.0×10^{-9}

Table 1.7 Absolute errors for Example 1.9 using two iterations of the ADM.

1.4.5 Singular Boundary Value Problems

In this section, we will consider differential equations which possess a singularity. To start, consider the singular boundary value problem of order $n+1$ given by

$$u^{n+1} + \frac{m}{x} u^n + Nu = g(x), \tag{1.154}$$

subject to the boundary conditions

$$u(0) = a_0, \qquad u'(0) = a_1, \qquad \dots, \qquad u^{(n-1)}(0) = a_{n-1}, \qquad u(b) = c, \tag{1.155}$$

where N is a nonlinear differential operator of order less than n, $g(x)$ is a given function and $a_0, a_1, \dots, a_{n-1}, b, c$ are constants. Consider the following operator L, defined as below,

$$L(.) = x^{-1} \frac{d^n}{dx^n} x^{n+1-m} \frac{d}{dx} x^{m-n}(.), \tag{1.156}$$

where $m \leq n$. Thus, in operator form, Eq. (1.135) becomes

$$Lu = g(x) - Nu. \tag{1.157}$$

We propose the inverse operator L^{-1}, as defined below

$$L^{-1}(.) = x^{n-m} \int_0^x x^{m-n-1} \int_0^x \int_0^x \dots \int_0^x x(.) dx \dots dx. \tag{1.158}$$

Applying the inverse operator L^{-1} to both sides of equation (1.157), we have

$$u(x) = \Phi(x) + L^{-1}g(x) - L^{-1}Nu, \tag{1.159}$$

where $L\Phi(x) = 0$. By Adomian decomposition method applied to Eq. (1.159), we have the following resulting equation:

$$\sum_{n=0}^{\infty} u_n = \Phi(x) + L^{-1}g(x) - L^{-1}\sum_{n=0}^{\infty} A_n, \tag{1.160}$$

where the $A_n's$ are the Adomian polynomials that can be evaluated for different forms of nonlinearity. Matching both sides, via application of the ADM, gives the recursive relation

$$\begin{aligned} u_0 &= \Phi(x) + L^{-1}g(x), \\ u_1 &= -L^{-1}A_0, \\ u_2 &= -L^{-1}A_1, \\ u_3 &= -L^{-1}A_2, \\ &\vdots \end{aligned} \tag{1.161}$$

There are various research papers dealing with differential equations that possess singularities. The way to tackle such problems is to construct a tailored integral operator that overcomes the singular point. The choice of integral operator differs depending on the type of singularity.

In the following, several examples will be discussed for the illustration of the above iterative schemes for problems with similar singularity.

Example 1.10 Consider the inhomogeneous Bessel equation

$$u'' + \frac{1}{x}u' + u = 4 - 9x + x^2 - x^3, \tag{1.162}$$

complimented with the boundary conditions

$$u(0) = 0 \text{ and } u(1) = 0. \tag{1.163}$$

Equation (1.162) has a singular point at $x = 0$ and the differential operator L, as stated above employs the first two derivatives in the form

$$L = x^{-1}\frac{d}{dx}\left(x\frac{d}{dx}\right). \tag{1.164}$$

In view of (1.163), the inverse operator L^{-1}we shall consider is the twofold integral operator defined by

$$L^{-1}(.) = \int_1^x x^{-1} \int_0^x x(.)dxdx \tag{1.165}$$

Applying L^{-1} defined in (1.165), to the first two terms $u'' + \frac{1}{x}u'$ of Eq. (1.162), will lead to the following:

$$\begin{aligned} L^{-1}\left(u'' + \frac{1}{x}u'\right) &= \int_1^x x^{-1} \int_0^x x\left(u'' + \frac{1}{x}u'\right) dxdx \\ &= \int_1^x x^{-1}\left[xu' - \int_0^x u'\,dx + \int_0^x u'\,dx\right] dx \\ &= \int_1^x u'\,dx = u(x) - u(1). \end{aligned} \tag{1.166}$$

Operating with L^{-1} on both sides of (1.162) and applying the decomposition method, it then follows that

$$\sum_{n=0}^{\infty} u_n = L^{-1}(4 - 9x + x^2 - x^3) - L^{-1}\sum_{n=0}^{\infty} u_n, \tag{1.167}$$

or

$$\sum_{n=0}^{\infty} u_n = -\frac{9}{400} + x^2 - x^3 + \frac{x^4}{16} - \frac{x^5}{25}x^7 - L^{-1}\sum_{n=0}^{\infty} u_n. \tag{1.168}$$

The various terms $u_n(x)$ of the solution $u(x)$ can be easily determined by using the recursive relation

$$\begin{aligned} u_0(x) &= -\frac{9}{400} + x^2 - x^3 + \frac{x^4}{16} - \frac{x^5}{25}x^7, \\ u_1(x) &= -L^{-1}u_0 = \frac{3139}{176400} + \frac{9x^2}{1600} - \frac{x^4}{16} + \frac{x^5}{25} - \frac{x^6}{576} + \frac{x^7}{1225}, \\ u_2(x) &= -L^{-1}u_1 = \frac{314039}{81285120} - \frac{3139x^2}{176400} - \frac{9x^4}{25600} + \frac{x^5}{25} - \frac{x^6}{576} + \frac{x^7}{1225} \\ &+ \frac{x^8}{36864} - \frac{x^9}{99225}, \end{aligned} \tag{1.169}$$

and so forth. Based on these calculations, the solution in a series form is given by

$$u(x) = \frac{-1955}{2322432} + \frac{70643}{70560}x^2 - x^3 - \frac{9x^4}{25600} + \frac{x^8}{36864} - \frac{x^9}{99225}. \tag{1.170}$$

The exact solution for this problem is

$$u(x) = x^2 - x^3. \tag{1.171}$$

x	ADM	EXACT	ABSOLUTE ERROR
0.0	-0.00084	0.000	8.4×10^{-4}
0.1	0.008170	0.009	8.3×10^{-4}
0.2	0.03120	0.032	8.0×10^{-4}
0.3	0.06226	0.063	7.4×10^{-4}
0.4	0.09534	0.960	6.6×10^{-4}
0.5	0.12443	0.125	5.7×10^{-4}
0.6	0.14354	0.144	4.6×10^{-4}
0.7	0.14665	0.147	3.5×10^{-4}
0.8	0.12777	0.128	2.3×10^{-4}
0.9	0.08089	0.081	1.1×10^{-4}
1.0	0.00000	0.000	0.00

Table 1.8 Absolute errors for Example 1.10 using ADM with three iterations.

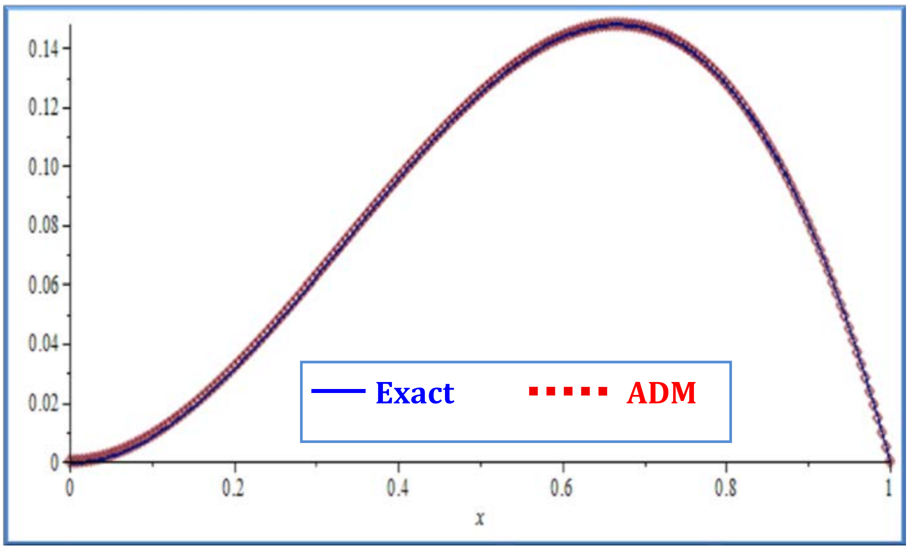

Figure 1.4 Exact solution for Example 1.10 compared with the approximate solution using ADM.

In order to verify numerically whether the proposed approach (ADM) leads to accurate solutions, we used the Computer Algebra System *MAPLE* to evaluate the decomposition series solutions using 3-terms approximation. The numerical results show that a good approximation is achieved using only few terms of the iterative scheme.

Moreover, comparison between the exact solution and the approximate solution $u(x)$ obtained using the decomposition method with three iterations is summarized in Table 1.8 and Figure 1.4.

The numerical experiments show that the absolute error is small using only few terms and thus the proposed approach is highly accurate.

Example 1.11 Consider the nonlinear BVP

$$u''' - \frac{2}{x}u'' - u - u^2 = g(x), \tag{1.172}$$

where

$$g(x) = 7x^2e^x + 6xe^x - 6e^x - x^6e^{2x}, \tag{1.173}$$

and complimented with the boundary conditions

$$u(0) = u'(0) = 0, \ \ u(1) = e. \tag{1.174}$$

Solution:

We define the new integral operator L as follows:

$$L[.] = x^{-1}\frac{d^2}{dx^2}x^5\frac{d}{dx}x^{-4}[.], \tag{1.175}$$

and thus its inverse operator, L^{-1}, is given by

$$L^{-1}[.] = x^4\int_1^x x^{-5}\int_0^x\int_0^x x[.] \ dx\,dx\,dx. \tag{1.176}$$

Applying L^{-1} to the first two terms $u''' - \frac{2}{x}u''$of Eq. (1.172) we find

$$L^{-1}\left(u''' - \frac{2}{x}u''\right) = x^4\int_1^x x^{-5}\int_0^x\int_0^x x\left(u''' - \frac{2}{x}u''\right)dx\,dx\,dx,$$

$$= x^4\int_1^x x^{-5}\left[\int_0^x\int_0^x xu'''dxdx - \int_0^x\int_0^x 2u''\,dx\,dx\right]dx,$$

$$= x^4\int_1^x x^{-5}[xu' - 4u]\,dx = x^4[x^{-4}u(x) - u(1)] = u(x) - x^4e. \tag{1.177}$$

Operating with L^{-1}on (1.172), it then follows that

$$u(x) = x^4e - L^{-1}u - L^{-1}u^2 + L^{-1}g(x), \tag{1.178}$$

Using the Adomian decomposition strategy to equation (1.159), gives

$$\sum_{n=0}^{\infty}u_n = x^4e - L^{-1}\left(\sum_{n=0}^{\infty}u_n\right) - L^{-1}\left(\sum_{n=0}^{\infty}A_n\right) + L^{-1}(7x^2e^x + 6xe^x - 6e^x - x^6e^{2x}). \tag{1.179}$$

The Adomian polynomials $A_n's$ for u^2 have been derived and used before. Following the decomposition method we get from the recursive relation the following three terms:

$$u_0 = x^4 e,$$

$$u_1 = x^3 + \frac{1}{2}x^5 + 0.15x^6 + 0.0215x^7 + 0.0095x^{11} - 1.6810x^4,$$

$$u_2 = \frac{1}{60}x^6 + 0.0082x^7 + 0.0022x^8 + 0.0004x^9 + 0.0101x^{10} - 0.0375x^4. \qquad (1.180)$$

Based on these calculations, the solution in a series form is given by

$$u(x) = x^3 + 0.9997x^4 + \frac{x^5}{2} + 0.1666x^6 + 0.0297x^7 + 0.0095x^{11} + 0.0022x^8 + 0.0004x^9 + 0.0101x^{10}. \qquad (1.181)$$

The closed form solution for this problem is given by

$$u(x) = x^3 e^x. \qquad (1.182)$$

The numerical results obtained by ADM are given in Table 1.9 and Figure 1.5 .It is important to mention that the ADM starts diverging, as is clear from Figure 1.5, for values of $x > 1$.

x	ADM	EXACT	ABSOLUTE ERROR
0.0	−0.00084	0.0	8.4×10^{-4}
0.1	0.008170	0.009	8.3×10^{-4}
0.2	0.03120	0.032	8.0×10^{-4}
0.3	0.06226	0.063	7.4×10^{-4}
0.4	0.09534	0.960	6.6×10^{-4}
0.5	0.12443	0.125	5.7×10^{-4}
0.6	0.14354	0.144	4.6×10^{-4}
0.7	0.14665	0.147	3.5×10^{-4}
0.8	0.12777	0.128	2.3×10^{-4}
0.9	0.08089	0.081	1.1×10^{-4}
1.0	0.0	0.0	0.0

Table 1.9 Absolute errors obtained for Example 1.11 using ADM with three iterations.

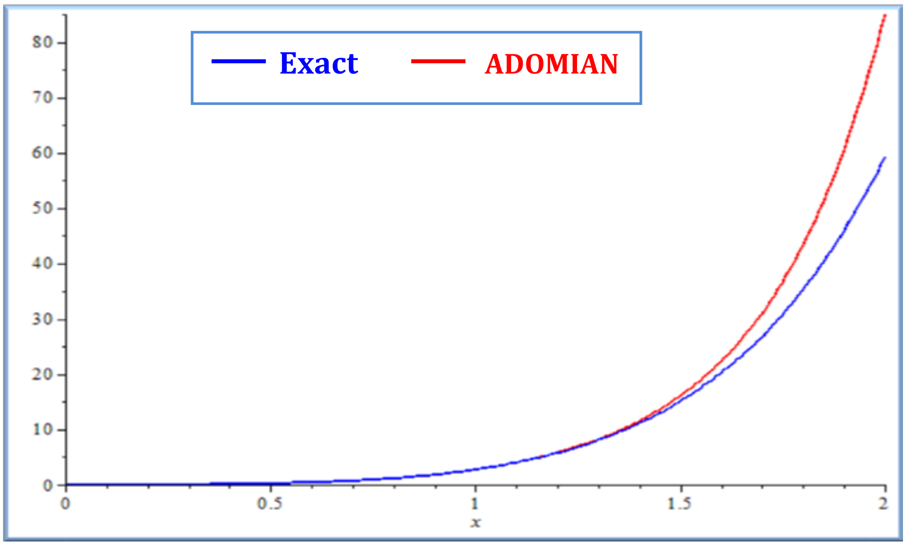

Figure 1.5 Comparison of the exact solution for Example 1.11 with ADM's approximate solution.

1.4.6 Boundary Value Problems over an Infinite Domain

In this section, we apply Adomian decomposition method to a boundary value problem on an infinite domain. Padé approximations are crucial for such problems and will be manipulated to handle the condition at infinity.

Example 1.12 Consider the nonlinear boundary value problem

$$u'''(x) + \frac{1}{2}u(x)u'(x) = 0, \tag{1.183}$$

subject to the boundary conditions

$$u(0) = 0, \qquad u'(0) = 1, \qquad u'(\infty) = 0, \tag{1.184}$$

where $0 < x < \infty$.

Solution:

Since it is necessary to have three initial conditions to apply the ADM, we set $u''(0) = \alpha$. The value of α can then be found using the condition at infinity. To start, we write (1.183) in operator form as

$$Lu = -\frac{1}{2}uu', \tag{1.185}$$

where L is a third order differential operator, and hence its inverse L^{-1} is defined by

$$L^{-1}(.) = \int_0^x \int_0^x \int_0^x (.)dx\, dx\, dx. \tag{1.186}$$

Applying L^{-1} to both sides of (1.185) and using the initial conditions we obtain

$$u(x) = x + \frac{1}{2}\alpha x^2 - \frac{1}{2}L^{-1}[uu'']. \tag{1.187}$$

Using the decomposition series $u(x)$ and the polynomial representation for uu'', we have

$$\sum_{n=0}^{\infty} u(x) = x + \frac{1}{2}\alpha x^2 - \frac{1}{2}L^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \tag{1.188}$$

where the $A_n{'s}$ are the Adomian polynomials that represent the nonlinear terms uu''. This leads to the recursive relation

$$u_0 = x$$

$$u_1 = \frac{1}{2}\alpha x^2 - \frac{1}{2}L^{-1}A_0$$

$$u_{n+1} = -\frac{1}{2}L^{-1}A_n, \qquad n \geq 1. \tag{1.189}$$

Next we calculate the first few terms of the Adomian polynomials.

$$A_0 = u_0 u_0'',$$

$$A_1 = u_0 u_1'' + u_0'' u_1,$$

$$A_2 = u_0 u_2'' + u_1'' u_1 + u_0'' u_2,$$

$$A_3 = u_0 u_3'' + u_1'' u_2 + u_1 u_2'' + u_0'' u_3, \tag{1.190}$$

....

Applying the decomposition algorithm to equation (1.189), yields the following iterates

$$u_0 = x,$$

$$u_1 = \frac{1}{2}\alpha x^2 - \frac{1}{2}L^{-1}A_0,$$

$$u_2 = -\frac{1}{2}L^{-1}A_1,$$

$$u_3 = -\frac{1}{2}L^{-1}A_2, \tag{1.191}$$

....

This in turn gives

$$u_0 = x,$$

$$u_1 = \frac{1}{2}\alpha x^2 - \frac{x^4}{24},$$

$$u_2 = -\frac{1}{24}\alpha x^4 + \frac{x^6}{240},$$

$$u_3 = -\frac{1}{120}\alpha x^5 - \frac{1}{144}\alpha x^6 + \frac{1}{7}\left(-\frac{7\alpha}{720} - \frac{1}{180}\right)x^7 - \frac{1}{3360}x^8 - \frac{1}{24192}x^9, \tag{1.192}$$

....

Hence we obtain the following solution:

$$u_0 = x,$$

$$u_1 = \frac{1}{2}\alpha x^2 - \frac{x^4}{24},$$

$$u_2 = -\frac{1}{24}\alpha x^4 + \frac{x^6}{240},$$

$$u_3 = -\frac{1}{120}\alpha x^5 - \frac{1}{144}\alpha x^6 + \frac{1}{7}\left(-\frac{7\alpha}{720} - \frac{1}{180}\right)x^7 - \frac{1}{3360}x^8 - \frac{1}{24192}x^9, \tag{1.193}$$

$$\vdots$$

Now, in order to find the value of α we use the condition

$$\lim_{\alpha\to\infty} u'(\alpha) = 0, \tag{1.194}$$

which is obtained from the boundary condition at infinity, namely, $u'(\infty) = 0$. We apply Padé approximation on the derivative of the solution, that is on u'. This will convert the series into a rational function and thus it will become possibly to evaluate the limit at infinity, unlike the failure to evaluate it in case we have infinite series expressed in powers of x. After applying the [2,2] Padé approximant, the boundary condition gives

$$\lim_{\alpha\to\infty} \frac{3\left(4 + 3\alpha x + \left(-\alpha^2 + \frac{1}{3}\right)x^2\right)}{12 - 3\alpha x + x^2} = -3\alpha^2 + 1. \tag{1.195}$$

Solving the equation $-3\alpha^2 + 1 = 0$, we get that $\alpha = 0.5773502692$. If we apply the [3,3] Padé approximant, we get this equation

$$-\frac{1}{5}\frac{184 + 525\alpha^2}{5\alpha^2 + 12} = 0, \tag{1.196}$$

so we get $\alpha = 0.5920102959$. For [4,4] Padé approximant, we get that $\alpha = 0.5163977795$. The $[n, n]$ Padé approximants seem to converge.

Generally speaking, boundary conditions at infinity pose a problem when applying the various numerical solution methods. So in order to tackle this problem and avoid such a difficulty, Padé approximations with the ADM present a potential and effective answer to the condition at infinity.

1.4.7 Systems of Differential Equations

Now, we will demonstrate how one can apply the ADM for systems. Let us consider the following system of ordinary differential equations:

$$\begin{aligned} y_1' &= f_1(x, y_1, y_2, \dots, y_n) + g_1, \\ y_2' &= f_2(x, y_1, y_2, \dots, y_n) + g_2, \\ &\vdots \\ y_n' &= f_n(x, y_1, y_2, \dots, y_n) + g_n, \end{aligned} \tag{1.197}$$

where $f_1, f_2, \dots, f_n$ are nonlinear functions, $g_1, g_2, \dots, g_n$ are known functions, and we are seeking the solution $y_1, y_2, \dots, y_n$ satisfying (1.197).
Rewrite (1.197) in operator form by using the nth equation as:

$$Ly_n = N_n(x, y_1, y_2, \dots, y_k) + g_n, \qquad n = 1,2, \dots, m. \tag{1.198}$$

where $L = \frac{d}{dx}$ is the linear operator and $N_n(x, y_1, y_2, \dots, y_k) = f_n(x, y_1, y_2, \dots, y_k)$ represent the nonlinear operators. Appling the inverse operator of L (namely, $L^{-1}[.] = \int_0^t [.]\, dt$) gives

$$y_n = y_n(0) + L^{-1}N_n(x, y_1, y_2, \dots, y_k) + L^{-1}g_n. \tag{1.199}$$

The Adomian technique consists of approximating the solution of (1.178) as an infinite series

$$y_n = \sum_{k=0}^{\infty} y_{n,k}, \qquad n = 1,2, \dots, m, \tag{1.200}$$

and decomposing the nonlinear operator N_n as

$$N_n(x, y_1, y_2, \ldots, y_k) = \sum_{k=0}^{\infty} A_{n,k}, \qquad n = 1,2,..,m, \tag{1.201}$$

where $A_{n,k}$ are called Adomian polynomials of $y_0, y_1, \ldots, y_k$. Substituting (1.200) and (1.201) into (1.199) we get:

$$\sum_{k=0}^{\infty} y_{n,k} = y_n(0) + L^{-1} \sum_{k=0}^{\infty} A_{n,k} + L^{-1} g_n. \tag{1.202}$$

The various terms $y_{n,k}$ of the solution y_n can be easily determined by using the recursive relation

$$y_{n,0} = y_n(0) + L^{-1} g_n,$$

$$y_{n,k+1} = L^{-1} A_{n,k}\,(y_0, y_1, \ldots, y_k), \qquad n = 1,2, \ldots,\ k = 0,1,2, \ldots. \tag{1.203}$$

Two examples are solved next to show the applicability of the method for systems of ODEs.

Example 1.13 Consider the following system of ODEs, with initial values $y_1(0) = 1$, $y_2(0) = 0$, and $y_3(0) = 2$.

$$y'_1 = y_3 - cos(x),$$

$$y'_2 = y_3 - e^x,$$

$$y'_3 = y_1 - y_2. \tag{1.204}$$

Solution:

Applying the inverse operator $L^{-1} = \int_0^x (.)dx$ to both sides of (1.204) we get

$$y_1 = 1 - L^{-1}\cos(x) + L^{-1} y_3,$$

$$y_2 = -L^{-1} e^x + L^{-1} y_3,$$

$$y_3 = 2 + L^{-1}(y_1 - y_2). \tag{1.205}$$

The Adomian decomposition method gives

$$y_{1,0} = 1 - \sin(x), \qquad y_{1,k+1} = L^{-1} y_{3,k},$$

$$y_{2,0} = 1 - e^x, \qquad y_{2,k+1} = L^{-1} y_{3,k},$$

$$y_{3,0} = 2, \qquad y_{3,k+1} = L^{-1}\left(y_{1,k} - y_{2,k}\right), \qquad k = 0,1,2, \ldots. \tag{1.206}$$

After finding the first few terms we get the exact solutions:

$$y_1 = e^x, \quad y_2 = sin(x), \quad y_3 = e^x + cos(x). \tag{1.207}$$

Example 1.14 Consider the following nonlinear system of ordinary differential equation, with the initial conditions $u_1(0) = 1$, $u_2(0) = 1$, and $u_3(0) = 0$ and with exact solutions $u_1(x) = e^{2x}$, $u_2(x) = e^x$ and $u_3(x) = xe^x$.

$$\begin{aligned} u'_1 &= 2u_2^2, \\ u'_2 &= e^{-x}u_1, \\ u'_3 &= u_2 + u_3. \end{aligned} \tag{1.208}$$

Solution:
Using the inverse operator $L^{-1} = \int_0^x [.]\, dx$ we get:

$$\begin{aligned} u_1 &= 1 + 2L^{-1}u_2^2, \\ u_2 &= 1 + L^{-1}e^{-x}u_1, \\ u_3 &= L^{-1}(u_2 + u_3). \end{aligned} \tag{1.209}$$

Using the scheme (1.13) to compute the Adomian polynomials, the decomposition procedure would be as follows:

$$\begin{aligned} y_{1,0} &= 1, \qquad & y_{1,k+1} &= 2L^{-1}A_{2,k}, \\ y_{2,0} &= 1, \qquad & y_{2,k+1} &= L^{-1}e^{-x}u_{1,k}, \\ y_{3,0} &= 0, \qquad & y_{3,k+1} &= L^{-1}\left(y_{2,k} - y_{3,k}\right), \quad k = 0,1,2,.... \end{aligned} \tag{1.210}$$

1.4.8 ADM and Domain Decomposition

We have seen earlier, from the various numerical experiments that we have conducted, one key setback of the ADM. The error worsens as we move away from the specified initial condition, that is, the convergence is local and is highly accurate mainly in a neighborhood of the initial point and deteriorates as we move far away from it, that is, as the applicable domain increases. In this section, we will overcome this setback by applying a domain decomposition technique that will improve the error for large values of the independent variable.

The main thrust of the DD is to decompose the domain of the problem into a union of disjoint subintervals in such a way that the error is uniformly distributed. The spirit of the DD is to decompose one large global problem into many smaller subdomain problems. The computational domain is initially partitioned into a number M of non-overlapping subdomains $X_i = [x_i, x_{i+1}], i = 0,1, \dots, M-1$ with overlap at the mesh point x_{i+1} between neighboring regions X_i and X_{i+1}. The ADM, which converges fast locally near the left endpoint $x = 0$, is applied at first in a small neighborhood of thickness δ about the origin. Then, from the resulting numerical solution on the first subinterval, an initial condition is estimated at $x = \delta$, that approximates the value of the true solution at $x = \delta$, then the ADM is applied again on the second subdomain.

Therefore the ADM is applied on the first subdomain and the values on inter-domain right endpoint boundary are calculated, that is the original problem is solved by computing sub-problems in parallel. In comparison with the standard ADM approach, non-overlapping domain decomposition approach is more efficient especially if a highly accurate numerical solution with uniform error distribution is required.

To explain the DD more precisely, assume we have a BVP on [0,1] subject to the boundary conditions $y(0) = a, y(1) = b$. First, we solve the problem directly by the ADM over the interval [0,1], then obtain the first value of x from resulting numerical solution, say $x = \delta_1$, that satisfies the condition

$$|y_{n1}(\delta_1) - y_{n1-1}(\delta_1)| < Tol,$$

where Tol is an assigned tolerance and $n1$ denotes the number of ADM terms that are needed to satisfy the later condition. Set $Y_1 = y_{n1}(x)$ on $[\delta_0, \delta_1]$, where $\delta_0 = 0$. The next step is to apply the ADM again on the same problem on the domain $[\delta_1, 1]$ subject to the boundary conditions

$$y(\delta_1) = Y_1(\delta_1), \qquad y(1) = b,$$

where $Y_1(\delta_1)$ is the numerical approximation obtain by the first application of the ADM. From this we get the value of δ_2 and hence the solution on $[\delta_1, \delta_2]$. The procedure is repeated in a similar fashion till we get the approximate solution on $[\delta_n, 1]$.

Example 1.15 We will apply the domain decomposition (DD) combined with the ADM on Example 1.6 which is given by:

$$u' + u^2 = 1, \qquad u(0) = 0. \tag{1.211}$$

Solution:

To solve our example, we will subdivide the domain into two sub-domains, $[0, 0.5]$ and the second is $[0.5, 1]$. Applying the ADM on $[0, 0.5]$ first, then from equation (1.118) we can get an estimate of the value of the solution at $x = 0.5$, in particular, we get the following value:

$$u(0.5) = \frac{18631}{40320}. \tag{1.212}$$

This value is now used as the initial condition when applying the ADM on the sub-interval $[0.5, 1]$. Applying the inverse operator $L^{-1}$and using this initial condition gives

$$u = u(0.5) + x - 0.5 - L^{-1}(u^2). \tag{1.213}$$

By ADM, we can represent the nonlinear term u^2 by an infinite series of Adomian polynomials A_n and the term $u(x)$ by decomposition series. We have

$$\sum_{n=0}^{\infty} u_n(x) = -\frac{1529}{40320} + x - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{1.214}$$

Upon matching both sides of the latter equation, and after computing the Adomian polynomials A_n for u^2, we get the following recursive relation:

$$\begin{aligned} u_0 &= -\frac{1529}{40320} + x, \\ u_{n+1} &= -L^{-1}A_n, \qquad n \geq 0. \end{aligned} \tag{1.215}$$

Upon solving we get the first few iterates:

$$\begin{aligned}
u_0 &= -0.0379216270 + x, \\
u_1 &= -L^{-1}(A_0) \\
&= 0.03290528481 - 0.001438049794x + 0.03792162700x^2 \\
&\quad - 0.3333333x^3, \\
u_2 &= -L^{-1}(A_1) \\
&= 0.004165858722 + 0.002495643874x - 0.03295981800x^2 \\
&\quad + 0.001917399726x^3 - 0.02528108467x^4 \\
&\quad + 0.1333333333x^5, \\
u_3 &= -L^{-1}(A_2) \\
&= 0.0004964057573 - 0.0002254266030x \\
&\quad - 0.004047560127x^2 - 0.002913307842x^3 \\
&\quad + 0.01927200482x^4 - 0.001390114801x^5 \\
&\quad + 0.01221919092x^6 - 0.04603174602x^7.
\end{aligned} \tag{1.216}$$

Based on these calculations, the solution in a series form is given by

$$u(x) = -0.00035407771 + 1.000832167x + 0.00091424887x^2 - 0.3343292414x^3 - 0.00600907985x^4 + 0.1319432185\, x^5 + 0.01221919092x^6 - 0.04603174602x^7 + \cdots. \tag{1.217}$$

In Table 1.10 we give a comparison between the solution obtained solely by the ADM and the second using DD. It is obvious that for larger values of x, the approximate solution starts improving after applying the DD. For smaller values of x, both yield similar results since the ADM converges fast locally and hence such a DD improvement is not necessary.

x	ADM	EXACT	Error using ADM and DD	Error using ADM
0.0	0.0	0.0	0.0	0.0
0.1	0.100334672	0.100334672	1.0×10^{-10}	1.0×10^{-10}
0.2	0.202710036	0.202710024	1.1×10^{-8}	1.1×10^{-8}
0.3	0.309335803	0.309336250	4.5×10^{-7}	4.5×10^{-7}
0.4	0.422787088	0.422793219	6.1×10^{-6}	6.1×10^{-6}
0.5	0.546254960	0.546302490	3.9×10^{-5}	4.8×10^{-5}
0.6	0.683878766	0.684136808	2.8×10^{-5}	2.6×10^{-4}
0.7	0.841187184	0.842288381	1.3×10^{-5}	1.1×10^{-3}
0.8	1.025675296	1.029638557	6.9×10^{-6}	4.0×10^{-3}
0.9	1.247544849	1.260158218	4.4×10^{-4}	1.3×10^{-2}
1.0	1.520634921	1.557407725	2.4×10^{-3}	3.8×10^{-2}
1.3	2.866033456	3.602102448	5.1×10^{-2}	0.73607
1.5	4.559598214	14.10141995	0.20634	9.54182
1.7	7.445335506	−7.696602139	0.64497	15.1419
2.0	15.84126984	−2.185039863	2.617691	18.0263

Table 1.10 Comparison of the absolute errors obtained by ADM and those by DD for Example 1.6 of subsection (1.4.3) using four iterations for both methods.

1.5 Partial Differential Equations

In this previous section, we applied the Adomian Decomposition Method to linear and nonlinear ordinary differential equations (ODEs). Now, we will show how the method can be implemented to partial differential equations (PDEs) as well.

1.5.1 Linear PDEs

First, we will employ the ADM for solving linear partial differential equations. Consider the general linear partial differential equation written in operator form:

$$L_x u + L_t u + Ru = g, \tag{1.218}$$

where L_x is the highest order differential in x, L_t is the highest order differential in u, R is the remainder of the differential operator consisting of lower derivatives, and g is an inhomogeneous term.

Applying the inverse operator ${L_x}^{-1}$ to the equation (1.218) yields

$$u = \Phi(0) - {L_x}^{-1}L_t u - {L_x}^{-1}Ru + {L_x}^{-1}g, \tag{1.219}$$

where

$$\Phi(x) = \begin{cases} u(0,t) & , if\ L = \frac{\partial}{\partial x} \\ u(0,t) + xu_x(0,t) & , if\ L = \frac{\partial^2}{\partial x^2} \\ u(0,t) + xu_x(0,t) + \frac{x^2}{2!}u_{xx}(0,t) & , if\ L = \frac{\partial^3}{\partial x^3} \\ u(0,t) + xu_x(0,t) + \frac{x^2}{2!}u_{xx}(0,t) + \frac{x^3}{3!}u_{xxx}(0,t) & , if\ L = \frac{\partial^4}{\partial x^4} \\ \vdots & \vdots \\ u(0,t) + xu_x(0,t) + \frac{x^2}{2!}u_{xx}(0,t) + \frac{x^3}{3!}u_{xxx}(0,t) + + \frac{x^n}{n!}u_{xxx\ldots n(times)\ldots x}(0,t) & , if\ L = \frac{\partial^{n+1}}{\partial x^{n+1}} \\ \vdots & \vdots \end{cases} \tag{1.220}$$

The Adomian decomposition method suggests that the linear terms $u(x,t)$ be decomposed by an infinite series of components of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{1.221}$$

where $u_n(x,t)$ is the components of $u(x,t)$ that will be elegantly determined in a recursive manner. Substituting (1.221) into (1.219) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = \Phi(0) - {L_x}^{-1}L_t\left(\sum_{n=0}^{\infty} u_n(x,t)\right) - {L_x}^{-1}R\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + {L_x}^{-1}g. \tag{1.222}$$

Following the decomposition analysis strategy, equation (1.222) is transformed into a set of recursive relations given by

$$\begin{aligned} &u_0(x,t) = \Phi(0) + \ {L_x}^{-1}g, \\ &u_{n+1}(x,t) = -{L_x}^{-1}\big(L_t u_n(x,t)\big) - {L_x}^{-1}Ru_n(x,t), \qquad n \geq 0. \end{aligned} \tag{1.223}$$

This latter algorithm will be later used on a number of test examples to show the efficiency and applicability of the technique.

1.5.2 Non-Linear PDEs

We will generalize the ideas of the previous subsection to general nonlinear partial differential equation of the form. To start the ADM, we need to rewrite the PDE in operator form as:

$$L_x u + L_t u + Ru + F(u) = g, \tag{1.224}$$

where L_x is the highest order differential in x, L_t is the highest order differential in u, R is the remainder of differential operator consisting of lower order derivatives, $F(u)$ is an analytic nonlinear term, and g is the specified inhomogeneous term.
Applying the inverse operator $L_x{}^{-1}$, the equation (1.224) becomes

$$u = \Phi(0) - L_x{}^{-1} L_t u - L_x{}^{-1} Ru - L_x{}^{-1} F(u) + L_x{}^{-1} g, \tag{1.225}$$

where

$$\Phi(x) = \begin{cases} u(0,t) & , if\ L = \dfrac{\partial}{\partial x} \\ u(0,t) + x u_x(0,t) & , if\ L = \dfrac{\partial^2}{\partial x^2} \\ u(0,t) + x u_x(0,t) + \dfrac{x^2}{2!} u_{xx}(0,t) & , if\ L = \dfrac{\partial^3}{\partial x^3} \\ u(0,t) + x u_x(0,t) + \dfrac{x^2}{2!} u_{xx}(0,t) + \dfrac{x^3}{3!} u_{xxx}(0,t) & , if\ L = \dfrac{\partial^4}{\partial x^4} \\ \vdots & \vdots \\ u(0,t) + x u_x(0,t) + \dfrac{x^2}{2!} u_{xx}(0,t) + \dfrac{x^3}{3!} u_{xxx}(0,t) + + \dfrac{x^n}{n!} u_{xxx\ldots n(times)\ldots x}(0,t) & , if\ L = \dfrac{\partial^{n+1}}{\partial x^{n+1}} \\ \vdots & \vdots \end{cases} \tag{1.226}$$

The method admits the decomposition of $u(x,t)$ into an infinite series of terms expressed as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{1.227}$$

and the nonlinear term $F(u)$ is to be equated to an infinite series of polynomials

$$F\big(u(x,t)\big) = \sum_{n=0}^{\infty} A_n, \tag{1.228}$$

where A_n are the Adomian polynomials that represent the nonlinear term $F\big(u(x,t)\big)$. Inserting (1.226) and (1.227) into (1.225) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = \Phi(0) - L_x^{-1} L_t \left(\sum_{n=0}^{\infty} u_n(x,t)\right) - L_x^{-1} R \left(\sum_{n=0}^{\infty} u_n(x,t)\right) - L_x^{-1}\left(\sum_{n=0}^{\infty} A_n\right) + L_x^{-1} g. \tag{1.229}$$

The various terms $u_n(x,t)$ of the solution $u(x,t)$ can be easily determined by using the recursive relation

$$u_0 = \Phi(0) + L_x^{-1} g,$$
$$u_{n+1} = -L_x^{-1} L_t u_n(x,t) - L_x^{-1} R u_n(x,t) - L_x^{-1} A_n. \tag{1.230}$$

Consequently, the first few terms of the solution are given by

$$\begin{aligned} u_0 &= \Phi(0) + L_x^{-1} g \\ u_1 &= -L_x^{-1} L_t u_0(x,t) - L_x^{-1} R u_0(x,t) - L_x^{-1} A_0, \\ u_2 &= -L_x^{-1} L_t u_1(x,t) - L_x^{-1} R u_1(x,t) - L_x^{-1} A_1, \\ u_3 &= -L_x^{-1} L_t u_2(x,t) - L_x^{-1} R u_2(x,t) - L_x^{-1} A_2. \end{aligned} \tag{1.231}$$

Examples will be given next sections to illustrate this algorithm.

1.5.3 Initial Value Problems

In this current unit, we will implement the strategy behind the ADM algorithm described previously apply to some examples in which the linear and nonlinear partial differential equations are subjected only to initial conditions.

Example 1.16 Consider the initial value problem of nonlinear partial differential equation

$$u_{xx} + \frac{1}{4} u_t^2 = u(x,t), \quad u(0,t) = 1 + t^2, \qquad u_x(0,t) = 1. \tag{1.232}$$

Solution:

We first rewrite equation (1.232) in an operator form as

$$L_x u = u - \frac{1}{4} u_t^2, \tag{1.233}$$

where L_x is a second order partial differential operator. Operating with L_x^{-1} on both sides of the PDE and using the initial conditions gives

$$u = 1 + t^2 + x + L_x{}^{-1}\, u - \frac{1}{4} L_x{}^{-1} u_t^2, \tag{1.234}$$

so that

$$\sum_{n=0}^{\infty} u_n(x,t) = 1 + t^2 + x + L_x{}^{-1} \left(\sum_{n=0}^{\infty} u_n(x,t) \right) - \frac{1}{4} L_x{}^{-1} \left(\sum_{n=0}^{\infty} A_n \right). \tag{1.235}$$

Equations (1.234) and (1.235) imply that the various iterates are given by

$$\begin{aligned} u_0(x,t) &= 1 + t^2 + x, \\ u_{n+1}(x,t) &= L_x{}^{-1} u_n(x,t) - \frac{1}{4} L_x{}^{-1} A_n, \qquad n \geq 0, \end{aligned} \tag{1.236}$$

where the A_n are the Adomian polynomials. The first few polynomials for the nonlinear quadratic term u_t^2 are given by

$$\begin{aligned} A_0 &= u_{0_t}{}^2, \\ A_1 &= 2u_{0_t} u_{1_t}, \\ A_2 &= 2u_{0_t} u_{2_t} + u_{1_t}{}^2. \end{aligned} \tag{1.237}$$

Consequently, the first three terms of the solution $u(x,t)$ are given by

$$\begin{aligned} u_0(x,t) &= 1 + t^2 + x, \\ u_1(x,t) &= L_x{}^{-1} u_0(x,t) - \frac{1}{4} L_x{}^{-1} A_0 = L_x{}^{-1}(1 + x) = \frac{x^2}{2!} + \frac{x^3}{3!}, \\ u_2(x,t) &= L_x{}^{-1} u_1(x,t) - \frac{1}{4} L_x{}^{-1} A_1 = L_x{}^{-1} \left(\frac{x^2}{2!} + \frac{x^3}{3!} \right) = \frac{x^4}{4!} + \frac{x^5}{5!}. \end{aligned} \tag{1.238}$$

Thus, the infinite solution in a series form is given by

$$u(x,t) = t^2 + \left(1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \right). \tag{1.239}$$

Note that infinite series is the McLaurin series expansion of e^x. Indeed, the latter equation leads to the exact solution of our IVP which is given by

$$u(x,t) = t^2 + e^x. \tag{1.240}$$

x	$t = 0.1$	$t = 0.2$	$t = 0.3$
0.0	0.0	0.0	0.0
0.2	9.1×10^{-8}	9.1×10^{-8}	9.1×10^{-8}
0.4	6.0×10^{-6}	6.0×10^{-6}	6.0×10^{-6}
0.6	7.1×10^{-5}	7.1×10^{-5}	7.1×10^{-5}
0.8	4.1×10^{-4}	4.1×10^{-4}	4.1×10^{-4}
1.0	1.6×10^{-3}	1.6×10^{-3}	1.6×10^{-3}

Table 1.11 Absolute error obtained using ADM with three iterations.

The numerical results are depicted in Table 1.11. The absolute error is very small for small values of x and t, however the error starts worsening for larger values. Thus, more iterates are obviously needed to improve the error.

Example 1.17 Consider the following nonlinear initial value problem:

$$u_t + \frac{1}{36} x {u^2}_{xx} = x^3, \qquad u(x,0) = 0. \tag{1.241}$$

Solution:

According to the scheme applied to the PDE in Equation (1.241), we have

$$u(x,t) = x^3 t - \frac{1}{36} {L_t}^{-1}(x {u^2}_{xx}). \tag{1.242}$$

Using the decomposition assumptions for the linear and the nonlinear terms we find

$$\sum_{n=0}^{\infty} u_n(x,t) = x^3 t - \frac{1}{36} {L_t}^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \tag{1.243}$$

where A_n are the Adomian polynomials that represent the nonlinear term $x{u^2}_{xx}$.
Equations (1.243) and (1.13) imply that the various iterates are given by:

$$\begin{aligned} u_0(x,t) &= x^3 t, \\ u_1(x,t) &= -\frac{1}{3} x^3 t^3, \\ u_2(x,t) &= \frac{2}{15} x^3 t^5, \\ &\dots \end{aligned} \tag{1.244}$$

Upon summing these iterates we get

$$u(x,t) = x^3\left(t - \frac{1}{3}t^3 + \frac{2}{15}t^5 + \cdots\right). \tag{1.245}$$

If we proceed with iterating, we will notice that the term in brackets turns out to be the McLaurin series expansion of the $\tanh t$. Actually, this way we obtain the latter equation we get the closed form solution of the problem which is

$$u(x,t) = x^3 \tanh t. \tag{1.246}$$

t	$x = 0.1$	$x = 0.2$	$x = 0.3$
0.0	0.0	0.0	0.0
0.1	5.4×10^{-12}	4.3×10^{-11}	1.5×10^{-10}
0.3	1.1×10^{-8}	9.1×10^{-8}	3.1×10^{-7}
0.5	3.8×10^{-7}	3.1×10^{-6}	1.0×10^{-5}
0.7	3.7×10^{-6}	3.0×10^{-5}	1.0×10^{-4}
0.9	1.9×10^{-5}	1.6×10^{-4}	5.2×10^{-3}

Table 1.12 Absolute error obtained using ADM with three iterations.

1.5.4 Boundary Value Problems

In this section, we will tackle BVPs and in particular we will apply the ADM to the two dimensional Laplace's equation with specified boundary conditions.

Consider the following Laplace's equation of the form

$$u_{xx} + u_{tt} = 0, \qquad 0 < x < a, \qquad 0 < t < b, \tag{1.247}$$

Subject to the boundary conditions

$$u(0,t) = 0, \qquad u(a,t) = f(t),$$
$$u(x,0) = 0, \qquad u(x,b) = 0, \tag{1.248}$$

where $u = u(x,t)$ is the solution of Laplace's equation. We can write the equation (1.247) in operator form as

$$L_t u(x,t) = -L_x u(x,t), \tag{1.249}$$

where

$$L_x = \frac{\partial^2}{\partial x^2}, \quad L_t = \frac{\partial^2}{\partial t^2}, \tag{1.250}$$

and hence ${L_x}^{-1}$and ${L_t}^{-1}$ are the inverse operators defined by

$$L_x^{-1}(.) = \int_0^x \int_0^x (.) dx\, dx,$$

$$L_t^{-1}(.) = \int_0^t \int_0^t (.) dt\, dt. \tag{1.251}$$

Applying the inverse operator ${L_t}^{-1}$ to the operator form of our problem (1.13), and using the proper boundary conditions and assuming that $g(x) = u_t(x, 0)$, we find that

$$u(x,t) = tg(x) - {L_t}^{-1} L_x u(x,t). \tag{1.252}$$

The decomposition method assumes a series solution given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{1.253}$$

Substituting (1.253) into both sides of (1.252) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = tg(x) - {L_t}^{-1} L_x \left(\sum_{n=0}^{\infty} u_n(x,t) \right). \tag{1.254}$$

This gives the recursive relation

$$u_0(x,t) = tg(x),$$

$$u_{n+1}(x,t) = -{L_t}^{-1} L_x u_n(x,t), \qquad n \geq 0. \tag{1.255}$$

Thus,

$$u_0(x,t) = tg(x),$$

$$u_1(x,t) = -{L_t}^{-1} L_x u_0(x,t) = -\frac{1}{3!} t^3 g''(x),$$

$$u_2(x,t) = -{L_t}^{-1} L_x u_1(x,t) = \frac{1}{5!} t^5 g^{(4)}(x),$$

$$\cdots. \tag{1.256}$$

So the solution is given by

$$u(x,t) = tg(x) - \frac{1}{3!}t^3 g''(x) + \frac{1}{5!}t^5 g^{(4)}(x) - \cdots. \tag{1.257}$$

We should find $g(x)$ in order to complete the solution $u(x,t)$. We can find it using the boundary condition $u(a,t) = f(t)$. After substituting x by a, using the Taylor expansion for $f(t)$, and comparing the coefficients in both sides we can determine $g(x)$.

In the following we apply the decomposition procedure described above to some particular boundary value problems.

Example 1.18 Consider the boundary value problem

$$\begin{aligned} &u_{xx} + u_{tt} = 0, \qquad 0 < x, t < \pi, \\ &u(0,t) = 0, \;\; u(\pi,t) = \sinh \pi \;\; sin\, t, \\ &u(x,0) = 0, \;\; u(x,\pi) = 0. \end{aligned} \tag{1.258}$$

Solution:
We first rewrite (1.258) in an operator form as

$$L_t u(x,t) = -L_x u(x,t). \tag{1.259}$$

Applying the inverse operator ${L_t}^{-1}$ to the operator form of (1.259), and using the proper boundary conditions, we find

$$u(x,t) = tg(x) - {L_t}^{-1} L_x u(x,t), \tag{1.260}$$

where

$$g(x) = u_t(x,0). \tag{1.261}$$

Using the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{1.262}$$

into both sides of (1.260) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = tg(x) - {L_t}^{-1}\left(L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right). \tag{1.263}$$

Decomposition analysis admits the use of the recursive relation

$$u_0(x,t) = tg(x),$$

$$u_{n+1}(x,t) = -L_t^{-1} L_x u_n(x,t), \;\; n \geq 0. \tag{1.264}$$

This leads to

$$\begin{aligned}
&u_0(x,t) = tg(x),\\
&u_1(x,t) = -L_t^{-1} L_x u_0(x,t) = -\frac{1}{3!} t^3 g''(x),\\
&u_2(x,t) = -L_t^{-1} L_x u_1(x,t) = \frac{1}{5!} t^5 g^{(4)}(x),\\
&u_3(x,t) = -L_t^{-1} L_x u_2(x,t) = -\frac{1}{7!} t^7 g^{(6)}(x),\\
&\cdots.
\end{aligned} \tag{1.265}$$

In view of (1.265), we can write

$$u(x,t) = tg(x) - \frac{1}{3!} t^3 g''(x) + \frac{1}{5!} t^5 g^{(4)}(x) - \frac{1}{7!} t^7 g^{(6)}(x) + \cdots. \tag{1.266}$$

To find the function $g(x)$, we have to use the boundary condition $u(\pi,t) = \sinh\pi \, \sin t$; using also the Taylor expansion of $\sin t$ we get

$$\begin{aligned}
tg(\pi) - \frac{1}{3!} t^3 g''(\pi) + \frac{1}{5!} t^5 g^{(4)}(\pi) - \frac{1}{7!} t^7 g^{(6)}(\pi) + \cdots \\
= \sinh\pi \left(t - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 - \frac{1}{7!} t^7 + \cdots \right).
\end{aligned} \tag{1.267}$$

Equating the coefficients of like terms on both sides gives

$$g(\pi) = g''(\pi) = g^{(4)}(\pi) = g^{(6)}(\pi) = \cdots = \sinh\pi \tag{1.268}$$

Thus,

$$g(x) = \sinh x. \tag{1.269}$$

Consequently, the solution in a series form is given by

$$u(x,t) = \sinh x \left(t - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 - \frac{1}{7!} t^7 + \cdots \right). \tag{1.270}$$

This obviously leads to the exact solution which is given by

$$u(x,t) = \sinh x \sin t. \tag{1.271}$$

1.5.5 Systems of Equations

We will now consider the numerical solution of systems of nonlinear partial differential equations and examine them using the decomposition method.

Consider the following system:

$$u_t + v_x + N_1(u,v) = g_1,$$
$$v_t + u_x + N_2(u,v) = g_2, \qquad (1.272)$$

with initial conditions

$$u(x,0) = f_1(x),$$
$$v(x,0) = f_2(x). \qquad (1.273)$$

We rewrite a system (1.272) in operator form as

$$L_t u + L_x v + N_1(u,v) = g_1, \qquad (1.274)$$
$$L_t v + L_x u + N_2(u,v) = g_2,$$

where L_t and L_x are considered, without loss of generality, first order partial differential operators, N_1 and N_2 are nonlinear operators, and g_1 and g_2 are source terms.

Applying the inverse operator L_t^{-1} to the system (1.274) and using the initial conditions (1.273) yields

$$u(x,t) = f_1(x) - L_t^{-1} L_x v - L_t^{-1} N_1(u,v) + L_t^{-1} g_1,$$
$$v(x,t) = f_2(x) - L_t^{-1} L_x u - L_t^{-1} N_2(u,v) + L_t^{-1} g_2. \qquad (1.275)$$

The linear terms $u(x,t)$ and $v(x,t)$ can be represented by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$
$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t), \qquad (1.276)$$

and the nonlinear terms $N_1(u,v)$and $N_2(u,v)$ by an infinite series of polynomials

$$N_1(u,v) = \sum_{n=0}^{\infty} A_n(x,t),$$

$$N_2(u,v) = \sum_{n=0}^{\infty} B_n(x,t), \tag{1.277}$$

where A_n and B_n are the Adomian polynomials. Substituting (1.276) and (1.277) into (1.275) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = f_1(x) - L_t^{-1} L_x \left(\sum_{n=0}^{\infty} v_n(x,t)\right) - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n(x,t)\right) + L_t^{-1} g_1,$$

$$\sum_{n=0}^{\infty} v_n(x,t) = f_2(x) - L_t^{-1} L_x \left(\sum_{n=0}^{\infty} u_n(x,t)\right) - L_t^{-1}\left(\sum_{n=0}^{\infty} B_n(x,t)\right) + L_t^{-1} + L_t^{-1} g_2. \tag{1.278}$$

This results in the recursive relation

$$u_0(x,t) = f_1(x) L_t^{-1} g_1,$$

$$u_{n+1}(x,t) = -L_t^{-1} L_x v_n(x,t) - L_t^{-1} A_n(x,t), \quad n \geq 0.$$

and

$$v_0(x,t) = f_2(x) L_t^{-1} g_1,$$

$$v_{n+1}(x,t) = -L_t^{-1} L_x u_n(x,t) - L_t^{-1} B_n(x,t), \quad n \geq 0. \tag{1.279}$$

Next, we will consider particular examples.

Example 1.19 Use Adomian decomposition method to solve the nonlinear system:

$$u_t + v u_x + u = 1,$$

$$v_t + u v_x - v = 1, \tag{1.280}$$

with initial conditions

$$u(x,0) = e^x, \quad v(x,0) = e^{-x}. \tag{1.281}$$

Solution:

Applying the inverse operator L_t^{-1} to the system (1.280) and using the initial conditions (1.281) yields

$$u(x,t) = e^x + t - L_t{}^{-1}(vu_x + u), \quad (1.282)$$
$$v(x,t) = e^{-x} + t + L_t{}^{-1}(uv_x + v).$$

The linear terms $u(x,t)$ and $v(x,t)$ can be represented by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$
$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t), \quad (1.283)$$

and the nonlinear terms $vu_x$and uv_x by an infinite series of polynomials

$$N_1(u,v) = \sum_{n=0}^{\infty} A_n(x,t),$$
$$N_2(u,v) = \sum_{n=0}^{\infty} B_n(x,t), \quad (1.284)$$

where A_n and B_n are Adomian polynomials. Substituting (1.283) and (1.284) into (1.282) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = e^x + t - L_t{}^{-1}\left(\sum_{n=0}^{\infty} A_n(x,t) + \sum_{n=0}^{\infty} u_n(x,t)\right),$$
$$\sum_{n=0}^{\infty} v_n(x,t) = e^{-x} + t + L_t{}^{-1}\left(\sum_{n=0}^{\infty} B_n(x,t) + \sum_{n=0}^{\infty} v_n(x,t)\right). \quad (1.285)$$

The decomposition method defines the recursive relations in the form

$$u_0(x,t) = e^x + t,$$
$$u_{n+1}(x,t) = -L_t{}^{-1}\big(A_n + u_n(x,t)\big), \quad n \geq 0,$$

and

$$v_0(x,t) = e^{-x} + t,$$
$$v_{n+1}(x,t) = L_t{}^{-1}\big(B_n + u_n(x,t)\big), \quad n \geq 0. \quad (1.286)$$

We can use the derived Adomian polynomials (1.13) into (1.286) to get the pairs of components.

1.6 Delay Differential Equations

In this section, we will present the solution of linear and nonlinear delay differential equations using Adomian decomposition method (ADM).

Consider the delay differential equation written in general form

$$Lu(x) = f\left(x, u(x), u\big(g(x)\big)\right), \quad 0 \leq x \leq 1, \tag{1.287}$$

with initial conditions

$$u^{i}(0) = a_i, \quad \text{for} \quad i = 0,1,2,\dots,n-1, \tag{1.288}$$

where L is n order operator defined by

$$L(.) = \frac{d^n(.)}{dx^n}. \tag{1.289}$$

As a consequence, the inverse operator L^{-1} is regarded an n-fold integration operator defined by

$$L^{-1}(.) = \int_0^x \int_0^x \int_0^x \dots \int_0^x (.)\, dxdxdx \dots dx \;, \qquad (n \text{ times}). \tag{1.290}$$

Applying the L^{-1} to both sides of (1.287) gives

$$u(x) = \Phi_0 + L^{-1} f\left(x, u(x), u\big(g(x)\big)\right), \tag{1.291}$$

where

$$\Phi_0 = \begin{cases} u(0), & if\ L = \dfrac{d}{dx} \\ u(0) + xu'(0), & if\ L = \dfrac{d^2}{dx^2} \\ u(0) + xu'(0) + \dfrac{x^2}{2!}u''(0), & if\ L = \dfrac{d^3}{dx^3} \\ u(0) + xu'(0) + \dfrac{x^2}{2!}u''(0) + \dfrac{x^3}{3!}u'''(0), & if\ L = \dfrac{d^4}{dx^4} \\ \quad\vdots & \quad\vdots \\ u(0) + xu'(0) + \dfrac{x^2}{2!}u''(0) + \dfrac{x^3}{3!}u'''(0) + \cdots . + \dfrac{x^{n-1}}{(n-1)!}u^{(n-1)}(0) & if\ L = \dfrac{d^n}{dx^n} \end{cases} \tag{1.292}$$

The decomposition technique consists of decomposing the solution into a sum of an infinite number of terms defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{1.293}$$

while the nonlinear term $f\left(x, u(x), u\big(g(x)\big)\right)$ is to be expressed by an infinite series of polynomials as

$$f\left(x, u(x), u\big(g(x)\big)\right) = \sum_{n=0}^{\infty} A_n, \tag{1.294}$$

where the $A_n{}'$s are the Adomian polynomials. Substituting (1.293) and (1.294) into (1.291) yields

$$\sum_{n=0}^{\infty} u_n = \Phi_0 + L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{1.295}$$

The various components u_n of the solution y can be easily determined by using the recursive relation

$$u_0 = \Phi_0, \tag{1.296}$$

$$u_{n+1} = L^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \quad n \geq 0.$$

Having determined the components u_n, $n \geq 0$, the solution u in a series form follows immediately. As stated before, the series may be summed to provide the solution in closed form.

In the following, a number of examples will be discussed for illustration.

Example 1.20 Consider the linear differential delay equation of the first order

$$u'(x) = \frac{1}{2} e^{\frac{x}{2}} u\left(\frac{x}{2}\right) + \frac{1}{2} u(x), \qquad 0 \leq x \leq 1, \quad u(0) = 1. \tag{1.297}$$

The exact solution is $u(x) = e^x$.

Solution:
In an operator form, Eq. (1.297) can be written as

$$u(x) = 1 + L^{-1}\left(\frac{1}{2} e^{\frac{x}{2}} u\left(\frac{x}{2}\right) + \frac{1}{2} u(x)\right), \tag{1.298}$$

where $L^{-1}(.) = \int_0^x [.]dx$.

The decomposition method suggests that the solution $u(x)$ be expressed by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{1.299}$$

Inserting (1.299) into (1.298) yields

$$\sum_{n=0}^{\infty} u_n(x) = 1 + L^{-1}\left(\frac{1}{2}e^{x/2}\sum_{n=0}^{\infty} u_n\left(\frac{x}{2}\right) + \frac{1}{2}\sum_{n=0}^{\infty} u_n(x)\right). \tag{1.300}$$

This leads to the recursive relation

$$u_0(x) = 1,$$

$$u_{n+1}(x) = L^{-1}\left(\frac{1}{2}e^{x/2}u_n\left(\frac{x}{2}\right) + \frac{1}{2}u_n(x)\right). \tag{1.301}$$

Consequently, the first few components of the solution are given by

$$u_0(x) = 1,$$

$$u_1(x) = L^{-1}\left(\frac{1}{2}e^{x/2}u_0\left(\frac{x}{2}\right) + \frac{1}{2}u_0(x)\right) = -1 + e^{\frac{x}{2}} + \frac{1}{2}x, \tag{1.302}$$

$$\vdots$$

If we take four terms of the series, we get the absolute error given in Table 1.13. Note that the error deteriorates as we move away from the initial point $x = 0$. In order to improve the accuracy and overcome this setback; we will subdivide the domain into three subintervals $[0,1] = [0,0.1] \cup [0.1,0.2] \cup [0.2,0.3] \cup [0.3,1]$ using Domain Decomposition method (DDM) that discussed earlier in Section 1.4.8.

Applying the ADM on $[0, 0.1]$ first, then we can get an estimate of the value of the solution at $x = 0.1$. In particular, we get the following value:

$$u(0.1) = 1.10517090608515. \tag{1.303}$$

This value is now used as the initial condition when applying the ADM on the sub-interval $[0.1, 0.2]$. Then, again applying the ADM on $[0.1, 0.2]$ and therefore we can get an estimate of the value of the solution at $x = 0.2$, we get the following value:

$$u(0.2) = 1.22140268244442 \tag{1.304}$$

Same as before, this value is now used as the initial condition when applying the ADM on the sub-interval $[0.2, 1]$. Applying the inverse operator $L^{-1}$and using this initial condition gives

$$u = u(0.2) + L^{-1}\left(\frac{1}{2}e^{\frac{x}{2}}u\left(\frac{x}{2}\right) + \frac{1}{2}u(x)\right). \tag{1.305}$$

By ADM, we can represent the term $u(x)$ by decomposition series. We have

$$\sum_{n=0}^{\infty} u_n(x) = 1.22140268244442 + L^{-1}\left(\frac{1}{2}e^{\frac{x}{2}}\sum_{n=0}^{\infty} u_n\left(\frac{x}{2}\right) + \frac{1}{2}\sum_{n=0}^{\infty} u_n(x)\right) \tag{1.306}$$

Upon matching both sides of the latter equation, we get the following recursive relation:

$$\begin{aligned} u_0 &= 1.22140268244442, \\ u_{n+1} &= L^{-1}\left(\frac{1}{2}e^{\frac{x}{2}}u_n\left(\frac{x}{2}\right) + \frac{1}{2}u_n(x)\right), \quad n \geq 0. \end{aligned} \tag{1.307}$$

Taking four terms of the series, we get the absolute error as given in Table 1.13. The table shows a comparison between ADM and DDM approaches and clearly the accuracy improves when we decompose the domain.

x	ADM	DDM
0.1	1.2×10^{-8}	1.2×10^{-8}
0.2	3.9×10^{-7}	7.6×10^{-8}
0.3	3.1×10^{-6}	2.9×10^{-6}
0.4	1.3×10^{-5}	2.6×10^{-6}
0.5	4.2×10^{-5}	1.9×10^{-6}
0.6	1.1×10^{-4}	4.2×10^{-6}
0.7	2.4×10^{-4}	1.5×10^{-5}
0.8	4.8×10^{-4}	4.6×10^{-5}
0.9	8.9×10^{-4}	1.2×10^{-4}

Table 1.13 Comparison between ADM and DDM for the same number of terms.

x	$h = 0.001$[13]	$h = 0.001$[14]	ADM[12]
0.2	1.37×10^{-11}	3.10×10^{-15}	0.0
0.4	3.27×10^{-11}	7.54×10^{-15}	2.23×10^{-16}
0.6	5.86×10^{-11}	1.39×10^{-14}	2.22×10^{-16}
0.8	9.54×10^{-11}	2.13×10^{-14}	1.33×10^{-15}
1.0	1.43×10^{-10}	3.19×10^{-14}	4.88×10^{-15}

Table 1.14 Comparison between ADM [12] using 13 terms and other methods [13,14].

Example 1.21 Consider the nonlinear differential delay equation of the third order

$$u'''(x) = -1 + 2u^2\left(\frac{x}{2}\right), \qquad 0 \leq x \leq 1, \tag{1.308}$$

with initial conditions

$$u(0) = 0, u'(0) = 1, u''(0) = 0. \tag{1.309}$$

The exact solution is $u(x) = \sin(x)$.

Solution:

Using the ADM, we represent the linear term as the decomposition series of components and equating the nonlinear term u^2 by the series of Adomian polynomials A_n. Then, we get the following recurrence relations

$$\begin{aligned} u_0(x) &= x - \frac{x^3}{6}, \\ u_{n+1}(x) &= 2L^{-1}(A_n), \qquad n \geq 0, \end{aligned} \tag{1.310}$$

where $L^{-1}(.) = \int_0^x \int_0^x \int_0^x (.)\, dxdxdx$.

The Adomian polynomials A_n for u^2 have been derived and used before. Following the first few components of the solution, we get

$$\begin{aligned} u_0(x) &= x - \frac{x^3}{6}, \\ u_1(x) &= 2L^{-1}(A_0) = 2L^{-1}\left({u_0}^2\left(\frac{x}{2}\right)\right) = 2L^{-1}\left(\left(\frac{x}{2}\right) - \frac{\left(\frac{x}{2}\right)^3}{6}\right) \\ &= \frac{1}{129024}x^9 - \frac{1}{1440}x^7 + \frac{1}{48}x^5, \\ u_2(x) &= 2L^{-1}(A_1) = 2L^{-1}\left(2u_0\left(\frac{x}{2}\right)u_1\left(\frac{x}{2}\right)\right). \\ &\vdots \end{aligned} \tag{1.311}$$

Then, the solution in series form is given by

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots. \tag{1.312}$$

x	ADM	Exact
0.0	0.0	0.0
0.2	0.19866933079506122	0.19866933079506122
0.4	0.3894183423086505	0.3894183423086505
0.6	0.56464224733950355	0.56464224733950355
0.8	0.7173560908995227	0.7173560908995228
1.0	0.84147109848078966	0.84147109848078965

Table 1.15 Comparison between the exact solution and approximation solution (ADM).

1.7 Integral Equations

In this section, we will tackle integral equations and demonstrate how they can be handled pretty efficiently using the ADM. We will consider both Fredholm and Volterra integral equations. As anticipated, in the case of a nonlinear integral equation, the linear term $u(x)$ is represented by an infinite sum of components, but the nonlinear terms such as u^2, u^5, $\cos u$, e^u, etc that arise in the equation should be expressed in terms of Adomian polynomial A_n.While for linear integral equation, the linear term $u(x)$ is represented by an infinite sum of components.

To start with, recall that an integral equation is an equation in which the unknown function $u(x)$ appears under an integral sign. A standard integral equation in $u(x)$ is of the form:

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)F\big(u(t)\big)dt, \tag{1.313}$$

where $F\big(u(x)\big)$ is a nonlinear function of $u(x)$, $g(x)$ and $h(x)$ are the limits of the integral, λ is a constant parameter, and $K(x,t)$ is a function of two variables x and t called the kernel or the nucleus of the integral equation. We have to mention that the limits of integration $g(x)$ and $h(x)$ can be variables, constants, or mixed.

By the decomposition method, assume the series solution for the unknown function $u(x)$ to be in the form

$$u(x) = \sum_{n=0}^{\infty} u_n(x,t), \tag{1.314}$$

while writing $F(u(x))$ in terms of Adomian polynomials as

$$F(u(x)) = \sum_{n=0}^{\infty} A_n(x,t). \tag{1.315}$$

From (1.314) and (1.315), we get

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t) \left(\sum_{n=0}^{\infty} A_n \right) dt. \tag{1.316}$$

Assuming the nonlinear function is $F(u(x))$, therefore by using (1.13), the Adomian polynomials A_n can be easily determined. Applying the decomposition method for the equation (1.316), the terms are given by the iterative scheme

$$u_0 = f(x),$$

$$u_{n+1} = \lambda \int_{g(x)}^{h(x)} K(x,t) A_n dt, \qquad n \geq 0, \tag{1.317}$$

or equivalently,

$$u_0 = f(x), \qquad u_1 = \lambda \int_{g(x)}^{h(x)} K(x,t) A_0 dt,$$
$$u_2 = \lambda \int_{g(x)}^{h(x)} K(x,t) A_1 dt, \qquad u_3 = \lambda \int_{g(x)}^{h(x)} K(x,t) A_2 dt, \tag{1.318}$$

and so on.

From (1.318), it is clear that the terms $u_0, u_1, u_2, \ldots$ are totally determined.

Example 1.22 We will apply ADM to solve the Fredholm integral equation

$$u(x) = 2 + \cos x + \int_0^{\pi} t\, u(t) dt. \tag{1.319}$$

Solution:

Using the Adomian decomposition method we find

$$\sum_{n=0}^{\infty} u_n(x) = 2 + \cos x + \int_0^{\pi} t \sum_{n=0}^{\infty} u_n(t)\, dt. \tag{1.320}$$

The Adomian decomposition method admits the use of the recurrence relation:

$$u_0(x) = 2 + \cos x,$$

$$u_1(x) = \int_0^{\pi} t\, u_0(t)dt = -2 + \pi^2,$$

$$u_2(x) = \int_0^{\pi} t\, u_1(t)dt = -\pi^2 + \frac{1}{2}\pi^4. \tag{1.321}$$

Using (1.321) gives the series solution

$$u(x) = \cos x + \frac{1}{2}\pi^4 + \cdots. \tag{1.322}$$

The exact solution is given by

$$u(x) = \cos x. \tag{1.323}$$

Example 1.23 We will now use the Adomian decomposition method to solve the following nonlinear Volterra integral equation:

$$u(x) = x + \int_0^x u^2(t)dt. \tag{1.324}$$

Solution:

Substituting the series (1.295) and the Adomian polynomials (1.296) into the left side and the right side of (1.305) respectively gives

$$\sum_{n=0}^{\infty} u_n(x) = x + \int_0^x \sum_{n=0}^{\infty} A_n(t)\, dt, \tag{1.325}$$

where the $A_n's$ are the Adomian polynomials for $u^2(x)$ as shown previously. Using the ADM strategy, we set

$$u_0 = x,$$

$$u_{n+1} = \int_0^x A_n dt\,, \qquad n = 0,1,2,.... \tag{1.326}$$

This gives

$$u_0 = x,$$

$$u_1 = \int_0^x A_0 dt = \int_0^x u_0^2\,(t)dt = \frac{1}{3}x^3,$$

$$u_2 = \int_0^x A_1 dt = \int_0^x 2u_0\,(t)u_1(t)dt = \frac{2}{15}x^5, \tag{1.327}$$

$$\vdots$$

Using (1.327) yields the series solution

$$u(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots. \tag{1.328}$$

This is basically the McLaurin series expansion of the exact solution of the integral equation which is given by

$$u(x) = \tan x. \tag{1.329}$$

1.8 Integro-Differential Equations

Finally, in this last section we will handle integro-differential equations. Recall that an integro-differential equation is an equation that contains $u^{(n)}(x)$, which is the nth derivative of $u(x)$, and an unknown function $u(x)$ that appears under an integral sign. A standard integro-differential equation is of the form:

$$u^{(n)}(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)F\big(u(t)\big)dt, \tag{1.330}$$

where $F\big(u(x)\big)$is a nonlinear function of $u(x)$, $g(x)$ and $h(x)$ are the limits of the integral , λ is a constant parameter, $K(x,t)$ is a function of two variables x and t called the kernel or the nucleus of the equation. We have to mention that the limits of integration $g(x)$ and $h(x)$ can be variables, constants, or mixed.

Without loss of generality, we may assume and consider a second order integro-differential equation given by

$$u''(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)F\big(u(t)\big)dt\,, \qquad u(0) = a, \quad u'(0) = b. \tag{1.331}$$

Integrating both sides of equation (1.331) twice from 0 to x and then using the initial conditions $u(0) = a,\ u'(0) = b$ gives

$$u(x) = a + bx + L^{-1}(f(x)) + \lambda L^{-1}\left(\int_{g(x)}^{h(x)} K(x,t)F\big(u(t)\big)dt\right), \tag{1.332}$$

where $L^{-1} = \int_0^x \int_0^x (.)dx\, dx$. Then use the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{1.333}$$

and the Adomian polynomials for the nonlinear term

$$F\big(u(t)\big) = \sum_{n=0}^{\infty} A_n \tag{1.334}$$

into both sides of (1.332) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = a + bx + L^{-1}(f(x)) + \lambda L^{-1}\left(\int_{g(x)}^{h(x)} K(x,t)\left(\sum_{n=0}^{\infty} A_n\right)dt\right). \tag{1.335}$$

This in turn is equivalent to

$$\begin{aligned} u_0 + u_1 + u_2 + \cdots = a + bx + L^{-1}\big(f(x)\big) \\ +\lambda L^{-1}\left(\int_{g(x)}^{h(x)} K(x,t)A_0 dt\right)\lambda L^{-1}\left(\int_{g(x)}^{h(x)} K(x,t)A_1 dt\right) \\ +\lambda L^{-1}\left(\int_{g(x)}^{h(x)} K(x,t)A_2 dt\right) + \cdots. \end{aligned} \tag{1.336}$$

Here the $A_n's$ are the Adomian polynomials and upon utilizing the scheme (1.13) we can find these polynomials easily. To determine the terms $u_0(x),\ u_1(x),\ u_2(x), \ldots$ of the solution $u(x)$, we construct and set the recurrence relation

$$\begin{aligned} u_0 &= a + bx + L^{-1}(f(x)), \\ u_{n+1} &= \lambda L^{-1}\left(\int_{g(x)}^{h(x)} K(x,t)A_n dt\right), \quad n \geq 0. \end{aligned} \tag{1.337}$$

The terms $u_0(x), u_1(x), u_2(x), \ldots$ are completely determined. The series solution converges to the exact solution if such a solution exists.

Example 1.24 Use the Adomian method to solve the Volterra integro-differential equation

$$u'(x) = 1 - \int_0^x u^2(t)dt, \quad u(0) = 0. \tag{1.338}$$

Solution:
Applying the one-fold integral operator L^{-1} defined by

$$L^{-1}(.) = \int_0^x (.)dx. \tag{1.339}$$

to both sides of (1.339), and using the initial condition we obtain

$$u(x) = x - L^{-1}\left(\int_0^x u^2(t)dt\right). \tag{1.340}$$

Using the decomposition series (1.333), Adomian polynomials (1.334), and using the recurrence relation (1.337) we obtain

$$\begin{aligned} u_0(x) &= x \\ u_1(x) &= -L^{-1}\left(\int_0^x A_0(t)dt\right) = -\frac{1}{12}x^4 \\ u_2(x) &= -L^{-1}\left(\int_0^x A_1(t)dt\right) = \frac{1}{252}x^8 \\ &\vdots \end{aligned} \tag{1.341}$$

This gives the solution in a series form

$$u(x) = x - \frac{1}{12}x^4 + \frac{1}{252}x^8 - \cdots. \tag{1.342}$$

Example 1.25 Using Adomian method we will solve the Volterra integro-differential equation

$$u'''(x) = -1 + x - \int_0^x (x-t)u(t)dt, \quad u(0) = 1, \quad u'(0) = -1, \quad u''(0) = 1. \tag{1.343}$$

Solution:
Applying the three-fold integral operator L^{-1} defined by

$$L^{-1}(.) = \int_0^x \int_0^x \int_0^x (.)\, dxdxdx, \tag{1.344}$$

to both sides of (1.324), and using the initial conditions we obtain

$$u(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - L^{-1}\left(\int_0^x (x-t)u(t)dt\right). \tag{1.345}$$

Using the decomposition series (1.333), and the recurrence relation (1.337) we obtain

$$u_0(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4,$$

$$u_1(x) = -L^{-1}\left(\int_0^x (x-t)u_0(t)dt\right) = -\frac{1}{5!}x^5 + \frac{1}{6!}x^6 - \frac{1}{7!}x^7 + \frac{1}{8!}x^8 - \frac{1}{9!}x^9. \tag{1.346}$$

Note that the decomposition solution given above, namely $u = u_0 + u_1 + \cdots$ is clearly the McLaurin series expansion of the true solution which is given by

$$u(x) = e^{-x}. \tag{1.347}$$

An important conclusion can made here. The Adomian Decomposition method has many advantages and disadvantages. The main advantage of ADM is that it can be applied directly for all types of differential and integral equations, homogeneous or inhomogeneous. Another important advantage is that it is capable of reducing the size of computational work while still maintaining high accuracy of the numerical solution. The effectiveness and the usefulness of the method are demonstrated by finding exact solutions to the models that will be investigated. However, one major deficiency is that the ADM requires finding and evaluating the Adomian polynomials for the nonlinear terms, and this is costly as it needs extensive calculations. The error in ADM is not uniform across the interval. Further, the convergence is accurate locally, mainly in a neighborhood of the boundary point(s).

More specifically, the ADM yields a series solution which has to be truncated for practical applications. Furthermore, the rate and region of convergence are likely deficiencies and limitations. Though in certain situations the series converges very rapidly in a very small region or neighborhood of the boundary points, it has very slow convergence rate in the wider and/or outer region, where he truncated series solution is an inaccurate solution in that region, which will greatly restrict the application area of the method.

CHAPTER 2: THE VARIATIONAL ITERATION METHOD

2.1 Method Description

The variation iteration method (VIM), first introduced by J. H. He, is a scheme that in many instances gives rapidly convergent successive approximations of the exact solution if such a solution exists. If convergence is assured, the obtained approximations by this technique are of high accuracy level even if some iterations are used.

Consider the nonlinear differential equation

$$Lu + Nu = g(x), \tag{2.1}$$

where L and N are linear and nonlinear operators respectively, and $g(x)$ is analytical function. We can construct a correction functional according to the variational iteration method for Eq. (2.1) in the form

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\big(Lu_n(s) + N\tilde{u}_n(s) - g(s)\big)ds, \qquad n \geq 0, \tag{2.2}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, u_n is the nth approximate solution and $\tilde{u}_n$ is a restricted variation, which means $\delta\tilde{u}_n = 0$.

It is clear that the main steps of the He's variational iteration method is to determine the Lagrange multiplier $\lambda(s)$. Integration by parts is usually used to determine $\lambda(s)$. More specifically, we can use

$$\begin{aligned} \int \lambda(s)\, u'_n(s)ds &= \lambda(s)u_n(s) - \int \lambda'(s)u_n(s)ds, \\ \int \lambda(s)\, u''_n(s)ds &= \lambda(s)u'_n(s) - \lambda'(s)u_n(s) + \int \lambda''(s)\, u_n(s)ds, \end{aligned} \tag{2.3}$$

and so forth. The Successive approximations $u_{n+1}(x)$ of the solution $u(x)$ will be readily obtained upon using selective function $u_0(x)$. However, for fast convergence, the function $u_0(x)$ should be selected by using the initial conditions as follows:

$$\begin{aligned} u_0(x) &= u(0), && \text{for the first order } u'_n, \\ u_0(x) &= u(0) + xu'(0), && \text{for the second order } u''_n, \\ u_0(x) &= u(0) + xu'(0) + \tfrac{1}{2!}x^2u''(0), && \text{for the third order } u'''_n. \end{aligned} \tag{2.4}$$

Consequently, the solution is given by

$$u = \lim_{n\to\infty} u_n. \tag{2.5}$$

2.2 Derivation of Iteration Schemes:

In this section, we derive some useful iteration formulas for certain classes of first order and higher order differential equations and determine the Lagrange multiplier $\lambda(s)$ for each class as well.

Summary of Iteration Formulas: First, we give the following summary for some useful iteration formulas that correspond to certain classes of differential equations:

$$\textbf{(I)}\quad \begin{cases} u' + f(u,u') = 0: \\ u_{n+1}(x) = u_n(x) - \int_0^x [u'_n(s) + f(u_n,u'_n)]ds. \end{cases} \tag{2.6}$$

$$\textbf{(II)}\quad \begin{cases} u' + \alpha u + f(u,u') = 0: \\ u_{n+1}(x) = u_n(x) - \int_0^x e^{\alpha(s-x)}[u'_n(s) + \alpha u_n(s) + f(u_n,u'_n)]ds. \end{cases} \tag{2.7}$$

$$\textbf{(III)}\quad \begin{cases} u'' + f(u,u',u'') = 0: \\ u_{n+1}(x) = u_n(x) + \int_0^x (s-x)[u''_n(s) + f(u_n,u'_n,u''_n)]ds. \end{cases} \tag{2.8}$$

$$\textbf{(IV)}\quad \begin{cases} u'' + \beta^2 u + f(u,u',u'') = 0: \\ u_{n+1}(x) = u_n(x) + \frac{1}{\beta}\int_0^x \sin\beta(s-x)\,[u''_n(s) + \beta^2 u_n(s) + f(u_n,u'_n,u''_n)]ds. \end{cases} \tag{2.9}$$

$$\textbf{(V)}\quad \begin{cases} u'' - \alpha^2 u + f(u,u',u'') = 0: \\ u_{n+1}(x) = u_n(x) + \frac{1}{2\alpha}\int_0^x \left(e^{\alpha(s-x)} - e^{\alpha(x-s)}\right)[u''_n - \alpha^2 u_n(s) + f(u_n,u'_n,u''_n)]ds. \end{cases} \tag{2.10}$$

$$\textbf{(VI)}\quad \begin{cases} u''' + f(u,u',u'',u''') = 0: \\ u_{n+1}(x) = u_n(x) - \frac{1}{2}\int_0^x (s-x)^2[u'''_n(s) + f(u_n,u'_n,u''_n,u'''_n)]ds. \end{cases} \tag{2.11}$$

$$
\textbf{(VII)} \begin{cases} u^{(4)} + f\left(u, u', u'', u''', u^{(4)}\right) = 0: \\ u_{n+1}(x) = u_n(x) + \dfrac{1}{6}\displaystyle\int_0^x (s-x)^3 \left[u_n^{(4)}(s) + f\left(u_n, u'_n, u''_n, u'''_n, u_n^{(4)}\right)\right] ds. \end{cases} \quad (2.12)
$$

$$
\textbf{(VIII)} \begin{cases} u^{(n)} + f\left(u, u', u'', u''', \dots, u^{(n)}\right) = 0: \\ u_{n+1}(x) = u_n(x) + (-1)^n \displaystyle\int_0^x \dfrac{1}{(n-1)!}(s-x)^{n-1}\left[u_n^{(n)}(s) + f\left(u_n, u'_n, u''_n, u'''_n, \dots, u_n^{(n)}\right)\right] ds. \end{cases} \quad (1.13)
$$

Derivations of Iteration Formulas: Next, we will show the derivation of the above formulas; other formulas can be proved in an analogous fashion.

(I) Consider the first order equation ordinary differential equation of the form

$$
u' + f(u, u') = 0, \qquad u(0) = a. \tag{2.14}
$$

Proof: The VIM employs the correction functional

$$
u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left[(u_n)_s + \tilde{f}_n(u, u')\right] ds, \qquad n \geq 0, \tag{2.15}
$$

where $\tilde{f}_n$ is a restricted variation, $\left(\delta \tilde{f}_n = 0\right)$.

To find the value of $\lambda(s)$, we start by taking the variation with respect to $u_n(x)$, which yields

$$
\frac{\delta u_{n+1}}{\delta u_n} = 1 + \frac{\delta}{\delta u_n}\left(\int_0^x \lambda(s)\left[(u_n)_s + \tilde{f}_n(u, u')\right] ds\right), \tag{2.16}
$$

or equivalently,

$$
\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda(s)\left[(u_n)_s + \tilde{f}_n(u, u')\right] ds\right). \tag{2.17}
$$

Applying the variation to Eq. (2.176) gives

$$
\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda(s)(u_n)_s ds\right). \tag{2.18}
$$

Integrating the integral in Eq. (2.18) by parts we have

$$\int_0^x \lambda(s)(u_n)_s(s)ds = [\lambda(x)(u_n)(x) - \lambda(0)(u_n)(0)] - \int_0^x \lambda'(s)u_n(s)ds. \tag{2.19}$$

Replacing the integral in Eq. (2.19) by its value in Eq. (2.18) we obtain

$$\delta u_{n+1} = \delta u_n + \delta[\lambda(x)(u_n)(x)] - \delta\left(\int_0^x \lambda'(s)u_n(s)ds\right) = 0. \tag{2.20}$$

By Simplifying Eq. (2.20) we get

$$\delta u_{n+1} = [1 + \lambda(x)]\delta u_n - \delta\left(\int_0^x \lambda'(s)u_n(s)ds\right) = 0. \tag{2.21}$$

The last equation is satisfied if the following '*stationary conditions'* are satisfied:

$$\lambda'(s) = 0,$$

$$1 + \lambda(s)|_{s=x} = 0. \tag{2.22}$$

By solving (2.22) for $\lambda(s)$ we have $\lambda(s) = -1$.
Substituting this value of $\lambda(s)$ into Eq. (2.15) gives the corresponding iterative scheme.

$$u_{n+1}(x) = u_n(x) - \int_0^x [(u_n)_s + f_n(u, u')]\, ds, \tag{2.23}$$

Thus, in general, the differential equation of the form $u'(x) + f(u, u') = 0$ has this iteration formula

$$u_{n+1}(x) = u_n(x) - \int_0^x [u'_n(s) + f(u_n, u'_n)]ds. \tag{2.24}$$

(II) Consider the first order equation ordinary differential equation of the form

$$u' + \alpha u + f(u, u') = 0, \qquad u(0) = a, \tag{2.25}$$

where α is a constant.

Proof: The (VIM) admits the construction of the correction functional for equation (2.25) given by

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left[(u_n)_s + \alpha u_n(x) + \widetilde{f_n}(u,u')\right]ds, \quad n \geq 0, \tag{2.26}$$

where $\widetilde{f_n}$ is a restricted variation $\left(\delta\tilde{f}_n = 0\right)$.

To find the optimal value of $\lambda(s)$, we proceed as follows. Take the variation with respect to $u_n(x)$; this leads to

$$\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda(s)\left[(u_n)_s + \alpha u_n(s) + \widetilde{f_n}(u,u')\right]ds\right). \tag{2.27}$$

Applying the variation to the integrand yields

$$\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda(s)(u_n)_s + \alpha\int_0^x \lambda(s)u_n(s)\,ds\right). \tag{2.28}$$

Integrating the later integral by parts we get

$$\int_0^x \lambda(s)(u_n)_s ds = [\lambda(x)(u_n)(x) - \lambda(0)(u_n)(0)] - \int_0^x \lambda'(s)u_n(s)ds. \tag{2.29}$$

Replacing the integral in (2.28) by its equivalent in (2.29) then operating the variation we have

$$\begin{aligned}\delta u_{n+1} = \delta u_n(x) + \lambda(x)\delta u_n(x) - \delta\left(\int_0^x \lambda'(s)u_n(s)ds\right)\\ + \delta\left(\alpha\int_0^x \lambda(s)u_n(s)ds\right).\end{aligned} \tag{2.30}$$

After simplifying, the last equation can be written as

$$\delta u_{n+1} = [1+\lambda(x)]\delta u_n(x) - \delta\left(\int_0^x \lambda'(s)u_n(s)ds\right) + \delta\left(\alpha\int_0^x \lambda(s)u_n(s)ds\right). \tag{2.31}$$

Hence, we obtain the stationary conditions

$$\begin{aligned}\lambda'(s) - \alpha\lambda(s) = 0,\\ 1 + \lambda(s)|_{s=x} = 0.\end{aligned} \tag{2.32}$$

Solving (2.30) for $\lambda(s)$ yields $\lambda(s) = -e^{\alpha(s-x)}$.

Then according to (2.32), we have the following VIM iteration formulation:

$$u_{n+1}(x) = u_n(x) - \int_0^x e^{\alpha(s-x)} [(u_n)_s + \alpha u_n(s) + f_n(u,u')]ds, \tag{2.33}$$

where $n \geq 0$.

Therefore, in general, the differential equation of the form $u' + \alpha u + f(u,u') = 0$ has this iteration formula

$$u_{n+1}(x) = u_n(x) - \int_0^x e^{\alpha(s-x)} [(u_n)_s + \alpha u_n(s) + f(u_n, u'_n)]ds. \tag{2.34}$$

(**III**) Consider the second order equation ordinary differential equation of the form

$$u'' + f(u,u',u'') = 0, \qquad u(0) = a, \qquad u'(0) = b. \tag{2.35}$$

Proof: The VIM employs the correction functional

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left[(u_n)_{ss} + \tilde{f}_n(u,u',u'')\right]ds, \qquad n \geq 0. \tag{2.36}$$

where $\tilde{f}_n$ is a restricted variation, $\left(\delta\tilde{f}_n = 0\right)$.

To find the value of $\lambda(s)$, start with taking the variation with respect to $u_n(x)$. This yields

$$\frac{\delta u_{n+1}}{\delta u_n} = 1 + \frac{\delta}{\delta u_n}\left(\int_0^x \lambda(s)\left[(u_n)_{ss} + \tilde{f}_n(u,u',u'')\right]ds\right), \tag{2.37}$$

or equivalently

$$\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda(s)\left[(u_n)_{ss} + \tilde{f}_n(u,u',u'')\right]ds\right). \tag{2.38}$$

Applying the variation to Eq. (2.38) gives

$$\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda(s)(u_n)_{ss}ds\right). \tag{2.39}$$

Integrating the integral in Eq. (2.39) by parts we have

$$\int_0^x \lambda(s)(u_n)_{ss}(s)ds = [\lambda(x)(u_n)_s(x) - \lambda(0)(u_n)_s(0)]$$

$$-[\lambda'(x)(u_n)(x) - \lambda'(0)(u_n)(0)] + \int_0^x \lambda''(s)u_n(s)ds. \qquad (2.40)$$

Substituting the integral in Eq.(2.39) by the value of the integral (2.40) we obtain

$$\delta u_{n+1} = \delta u_n + \delta[\lambda(x)(u_n)_s(x)] - \delta[\lambda'(x)(u_n)(x)] + \delta\left(\int_0^x \lambda''(s)u_n(s)ds\right) = 0. \qquad (2.41)$$

By simplifying Eq. (2.41) we get

$$\delta u_{n+1} = [1-\lambda'(x)]\delta u_n + \delta[\lambda(x)(u_n)_s(x)] + \delta\left(\int_0^x \lambda''(s)u_n(s)ds\right) = 0. \qquad (2.42)$$

So, the following stationary conditions are obtained

$$\lambda''(s) = 0,$$

$$1 - \lambda'(s)|_{s=x} = 0,$$

$$\lambda(s)|_{s=x} = 0. \qquad (2.43)$$

By solving (2.41) for $\lambda(s)$ we have $\lambda(s) = (s - x)$.

Thus, in general, the differential equation of the form $u''(x) + f(u, u', u'') = 0$ has this iteration formula:

$$u_{n+1}(x) = u_n(x) + \int_0^x (s - x)\,[(u_n)_{ss} + f(u_n, u'_n, u''_n)]ds. \qquad (2.44)$$

(**IV**) Consider the second order equation ordinary differential equation of the form

$$u'' + \beta^2 u + f(u, u', u'') = 0, \qquad u(0) = a, \quad u'(0) = b. \qquad (2.45)$$

Proof: The VIM employs the correction functional

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\big[(u_n)_{ss} + \beta^2 u_n(s) + \tilde{f}_n(u, u', u'')\big]ds, \qquad n \geq 0 \qquad (2.46)$$

where $\tilde{f}_n$ is a restricted variation, $(\delta\tilde{f}_n = 0)$.

To find the value of $\lambda(s)$, start with taking the variation with respect to $u_n(x)$ yields

$$\frac{\delta u_{n+1}}{\delta u_n} = 1 + \frac{\delta}{\delta u_n}\left(\int_0^x \lambda\,(s)\big[(u_n)_{ss} + \beta^2 u_n(s) + \widetilde{f_n}(u,u',u'')\big]ds\right), \tag{2.47}$$

which is the same as

$$u'' + f(u,u',u'') = 0, \qquad u(0) = a, \qquad u'(0) = b. \tag{2.48}$$

Applying the variation to Eq. (2.48) gives

$$\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda\,(s)(u_n)_{ss} + \int_0^x \beta^2\lambda(s)u_n(s)\,ds\right). \tag{2.49}$$

Integrating the integral in Eq. (2.47) by parts we have

$$\int_0^x \lambda\,(s)(u_n)_{ss}(s)ds = [\lambda(x)(u_n)_s(x) - \lambda(0)(u_n)_s(0)]$$

$$-[\lambda'(x)(u_n)(x) - \lambda'(0)(u_n)(0)] + \int_0^x \lambda''(s)u_n(s)ds. \tag{2.50}$$

Substituting the integral in Eq. (2.49) by the value of the integral (2.50) we obtain

$$\delta u_{n+1} = \delta u_n + \delta[\lambda(x)(u_n)_s(x)] - \delta[\lambda'(x)(u_n)(x)] + \delta\left(\int_0^x \lambda''(s)u_n(s)ds\right)$$

$$+\,\delta\left(\int_0^x \beta^2\lambda(s)u_n(s)\,ds\right) = 0. \tag{2.51}$$

By simplifying Eq. (2.51) we get

$$\delta u_{n+1} = [1 - \lambda'(x)]\delta u_n + \delta[\lambda(x)(u_n)_s(x)] + \delta\left(\int_0^x \lambda''(s)u_n(s)ds\right)$$

$$+\,\delta\left(\int_0^x \beta^2\lambda(s)u_n(s)\,ds\right) = 0. \tag{2.52}$$

So, the following stationary conditions are obtained

$$\lambda''(s) + \beta^2\lambda(s) = 0,$$

$$1 - \lambda'(s)|_{s=x} = 0,$$

$$\lambda(s)|_{s=x} = 0. \tag{2.53}$$

By solving (2.53) for $\lambda(s)$ we have $\lambda(s) = \frac{1}{\beta}\sin(\beta(s-x))$

In general, the differential equation of the form $u'' + \beta^2 u + f(u, u', u'') = 0$ has this iteration formula

$$u_{n+1}(x) = u_n(x) + \frac{1}{\beta}\int_0^x \sin(\beta(s-x))\,[(u_n)_{ss} + \beta^2 u_n + f(u_n, u'_n, u''_n)]ds. \tag{2.54}$$

(**V**) Consider the second order equation ordinary differential equation of the form

$$u'' - \alpha^2 u + f(u, u', u'') = 0, \quad u(0) = a, \quad u'(0) = b. \tag{2.55}$$

Proof: The VIM employs the correction functional

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left[(u_n)_{ss} - \alpha^2 u_n + \tilde{f}_n(u, u', u'')\right]ds, \quad n \geq 0, \tag{2.56}$$

where $\tilde{f}_n$ is a restricted variation, $\left(\delta \tilde{f}_n = 0\right)$.

To find the value of $\lambda(s)$, start with taking the variation with respect to $u_n(x)$yields

$$\frac{\delta u_{n+1}}{\delta u_n} = 1 + \frac{\delta}{\delta u_n}\left(\int_0^x \lambda(s)\left[(u_n)_{ss} - \alpha^2 u_n \tilde{f}_n(u, u', u'')\right]ds\right), \tag{2.57}$$

which is the same as

$$\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda(s)\left[(u_n)_{ss} - \alpha^2 + \tilde{f}_n(u, u', u'')\right]ds\right). \tag{2.58}$$

Applying the variation to Eq. (2.56) gives

$$\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda(s)(u_n)_{ss} ds\right) - \delta\left(\int_0^x \lambda(s)\alpha^2 u_n ds\right). \tag{2.59}$$

Integrating the integral in Eq. (2.59) by parts we have

$$\int_0^x \lambda(s)(u_n)_{ss}(s)ds = [\lambda(x)(u_n)_s(x) - \lambda(0)(u_n)_s(0)]$$

$$-[\lambda'(x)(u_n)(x) - \lambda'(0)(u_n)(0)] + \int_0^x \lambda''(s)u_n(s)ds. \tag{2.60}$$

Substituting the integral in Eq. (2.59) by the value of the integral (2.60) we obtain

$$\delta u_{n+1} = \delta u_n + \delta[\lambda(x)(u_n)_s(x)] - \delta[\lambda'(x)(u_n)(x)] + \delta\left(\int_0^x \lambda''(s)u_n(s)ds\right) - \delta\left(\int_0^x \lambda(s)\alpha^2 u_n ds\right) = 0. \quad (2.61)$$

By simplifying Eq. (2.61) we get

$$\delta u_{n+1} = [1-\lambda'(x)]\delta u_n + \delta[\lambda(x)(u_n)_s(x)] + \delta\left(\int_0^x \lambda''(s)u_n(s)ds\right) - \delta\left(\int_0^x \lambda(s)\alpha^2 u_n(s)ds\right) = 0. \quad (2.62)$$

So, the following stationary conditions are obtained

$$\lambda''(s) - \alpha^2\lambda(s) = 0,$$

$$1 - \lambda'(s)|_{s=x} = 0,$$

$$\lambda(s)|_{s=x} = 0. \quad (2.63)$$

By solving (2.63) for $\lambda(s)$ we have $\lambda(s) = \frac{1}{2\alpha}\left(e^{\alpha(s-x)} - e^{\alpha(x-s)}\right)$.

Thus, in general, the differential equation of the form $u'' - \alpha^2 u + f(u,u',u'') = 0$ has this iteration formula

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{1}{2\alpha}\left(e^{\alpha(s-x)} - e^{\alpha(x-s)}\right)\,[(u_n)_{ss} - \alpha^2 u_n + f(u_n u'_n, u''_n)]\,ds. \quad (2.64)$$

(**VI**) Consider the third order equation ordinary differential equation of the form

$$u'''(x) + f(u,u',u'',u''') = 0, \quad u(0) = a,\; u'(0) = b,\; u''(0) = c. \quad (2.65)$$

Proof: The VIM employs the correction functional

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left[(u_n)_{sss} + \tilde{f}_n(u,u',u'',u''')\right]ds, \qquad n \geq 0, \quad (2.66)$$

where $\tilde{f}_n$ is a restricted variation $\left(\delta\tilde{f}_n = 0\right)$.

To find the value of $\lambda(s)$, start with taking the variation with respect to $u_n(x)$ yields

$$\frac{\delta u_{n+1}}{\delta u_n} = 1 + \frac{\delta}{\delta u_n}\left(\int_0^x \lambda\,(s)\big[(u_n)_{sss} + \tilde{f}_n(u, u', u'', u''')\big]ds\right), \tag{2.67}$$

which is the same as

$$\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda\,(s)\big[(u_n)_{sss} + \tilde{f}_n(u, u', u'', u''')\big]ds\right). \tag{2.68}$$

Applying the variation to Eq. (2.68) gives

$$\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda\,(s)(u_n)_{sss} ds\right). \tag{2.69}$$

Integrating the integral in Eq. (2.69) by parts we have

$$\int_0^x \lambda\,(s)(u_n)_{sss}(s)ds = [\lambda(x)(u_n)_{ss}(x) - \lambda(0)(u_n)_{ss}(0)] - [\lambda'(x)(u_n)_s(x) - \lambda'(0)(u_n)_s(0)]$$

$$+[\lambda''(x)(u_n)(x) - \lambda''(0)(u_n)(0)] - \int_0^x \lambda'''\,(s)u_n(s)ds. \tag{2.70}$$

Substituting the integral in Eq.(2.69) by the value of the integral (2.70) we obtain

$$\begin{aligned}\delta u_{n+1} = \delta u_n + \delta[\lambda(x)(u_n)_{ss}(x)] - \delta[\lambda'(x)(u_n)_s(x)] + \delta[\lambda''(x)(u_n)(x)] \\ - \delta\left(\int_0^x \lambda'''\,(s)u_n(s)ds\right) = 0.\end{aligned} \tag{2.71}$$

By Simplifying Eq. (2.71) we get

$$\begin{aligned}\delta u_{n+1} = [1 + \lambda''(x)]\delta u_n - \delta[\lambda'(x)(u_n)_s(x)] + \delta[\lambda(x)(u_n)_{ss}(x)] \\ - \delta\left(\int_0^x \lambda'''\,(s)u_n(s)ds\right) = 0.\end{aligned} \tag{2.72}$$

So, the following stationary conditions are obtained

$$\lambda'''(s) = 0,$$

$$1 + \lambda''(s)|_{s=x} = 0,$$

$$\lambda'(s)|_{s=x} = 0,$$

$$\lambda(s)|_{s=x} = 0. \tag{2.73}$$

By solving (2.73) for $\lambda(s)$ we have $\lambda(s) = -\frac{1}{2}(s-x)^2$.

In general, the differential equation of the form $u'''(x) + f(u, u', u'', u''') = 0$ has this iteration formula

$$u_{n+1}(x) = u_n(x) + \int_0^x -\frac{1}{2}(s-x)^2 \; [(u_n)_{sss} + f(u_n u'_n, u''_n, u'''_n)]ds. \tag{2.74}$$

(**VII**) Consider the fourth order equation ordinary differential equation of the form

$$\begin{gathered} u''''(x) + f(u, u', u'', u''', u'''') = 0, \\ u(0) = a, u'(0) = b, u''(0) = c, u'''(0) = d. \end{gathered} \tag{2.75}$$

Proof: The VIM employs the correction functional

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\big[(u_n)_{ssss} + \tilde{f}_n(u, u', u'', u''', u'''')\big]ds, \quad n \geq 0. \tag{2.76}$$

To find $\lambda(s)$ we can follow the same steps in (VI), we get

$$\lambda(s) = \frac{1}{6}(s-x)^3. \tag{2.77}$$

and the correction functional for equation (2.77) is thus given by

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{1}{6}(s-x)^3 \, [(u_n)_{ssss} + f(u_n u'_n, u''_n, u'''_n, u''''_n)]ds, \quad n \geq 0. \tag{2.78}$$

In general, using similar steps as before, the differential equation of the form

$$u^{(n)}(x) + f(u, u', u'', \dots, u^{(n)}) = 0, \tag{2.79}$$

where $f\big(u, u', u'', \dots, u^{(n)}\big)$ is the linear or nonlinear term, gives the correction functional of the form

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\Big[(u_n)_{sss\dots s} + \tilde{f}_n(u_n, u'_n, u''_n, \dots, u_n^{(n)})\Big]ds, \qquad n \geq 0 \tag{2.80}$$

with $\lambda(s) = (-1)^n \frac{1}{(n-1)!}(s-x)^n$. Hence the correction functional for equation (2.80) is given by

$$u_{n+1}(x) = u_n(x) + (-1)^n \int_0^x \frac{1}{(n-1)!}(s-x)^n \left[(u_n)_{sss....s} + f\left(u_n, u'_n u''_n, \dots, u_n^{(n)}\right)\right] ds, \quad n \geq 0. \tag{2.81}$$

2.3 Implementation of the Method

The variational iteration method (VIM) handles nonlinear problems and linear problems in a parallel manner. Unlike Adomian decomposition method, the variational iteration method does not need specific treatment for the nonlinear operator. There is no need for Adomian polynomials. As stated before, the main step in the variational iteration method is to determine the Lagrange multiplier $\lambda(s)$. In this section, we will apply the VIM for certain classes of nonlinear ordinary differential equations and show the resulting iterative formula. Numerical results will be given in later sections.

In what follows we summarize the Lagrange multipliers as derived in section 2.2, and the selective zeroth approximations:

$$\begin{array}{lll}
u' + f(u,u') = 0, & \lambda(s) = -1, & u_0(x) = u(0), \\
u'' + f(u,u',u'') = 0, & \lambda(s) = s - x, & u_0(x) = u(0) + xu'(0), \\
u''' + f(u,u',u'',u''') = 0, & \lambda(s) = -\frac{1}{2!}(s-x)^2, & u_0(x) = u(0) + xu'(0) + \frac{1}{2!}x^2u''(0), \\
\vdots & &
\end{array} \tag{2.82}$$

Consequently, the solution is given by

$$u = \lim_{n\to\infty} u_n. \tag{2.83}$$

The VIM will be illustrated by studying the following examples.

Example 2.1 Consider the second order nonlinear ordinary differential equation

$$u''(x) + u^2(x) = 0, \qquad u(0) = a, \qquad u'(0) = b. \tag{2.84}$$

Following the discussion presented above we find that $\lambda = (s - x)$. Therefore, the iteration formula is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)[(u_n(s))_{ss} + u_n^2(s)]\, ds. \tag{2.85}$$

Using the Taylor expansion and the specified initial conditions, we can choose $u_0(x) = u(0) + u'(0)x = a + bx$. Using $u_0(x) = a + bx$ we have

$$u_0(x) = a + bx,$$

$$u_1(x) = a + bx + \int_0^x (s - x)\left[\left(u_0(s)\right)_{ss} + u_0^2(s)\right] ds,$$

$$u_2(x) = u_1(x) + \int_0^x (s - x)\left[\left(u_1(s)\right)_{ss} + u_1^2(s)\right] ds. \tag{2.86}$$

Consequently, the solution can be obtained from

$$u = \lim_{n\to\infty} u_n. \tag{2.87}$$

Example 2.2 Consider the third order nonlinear ordinary differential equation

$$u'''(x) + u^2(x) = 0, \qquad u''(0) = c, u'(0) = b, u(0) = a. \tag{2.88}$$

Following the discussion presented above we find that $\lambda = -\frac{1}{2}(s - x)^2$. Therefore, the iteration formula is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \frac{1}{2}(s - x)^2[(u_n(s))_{sss} + u_n^2(s)]\, ds. \tag{2.89}$$

Using the Taylor expansion, we can choose $u_0(x) = u(0) + u''(0)x + \frac{u''(0)}{2}x^2 = a + bx + \frac{c}{2}x^2$ from the given initial conditions. Using $u_0(x) = a + bx + \frac{c}{2}x^2$ we have

$$u_0(x) = a + bx + \frac{c}{2}x^2,$$

$$u_1(x) = a + bx + \frac{c}{2}x^2 - \int_0^x \frac{1}{2}(s - x)^2\left[\left(u_0(s)\right)_{sss} + u_0^2(s)\right] ds,$$

$$u_2(x) = u_1(x) - \int_0^x \frac{1}{2}(s - x)^2\left[\left(u_1(s)\right)_{sss} + u_1^2(s)\right] ds. \tag{2.90}$$

Consequently, the solution can be obtained from

$$u = \lim_{n\to\infty} u_n. \tag{2.91}$$

Example 2.3 Consider the first order nonlinear ordinary differential equation

$$u'(x) + \alpha u(x) + u^3(x) = 0, \quad u(0) = a, \tag{2.92}$$

where α is a constant.

Following the discussion presented above we find that $\lambda = -e^{\alpha(s-x)}$. Therefore, the iteration formula is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x e^{\alpha(s-x)}[(u_n(s))_s + \alpha u_n(x) + u_n^3(s)]\, ds. \tag{2.93}$$

By Taylor expansion, we can choose $u_0(x) = u(0) = a$ from the given initial condition. Using $u_0(x) = a$, we have

$$\begin{aligned}
&u_0(x) = a,\\
&u_1(x) = a - \int_0^x e^{\alpha(s-x)}\left[\left(u_0(s)\right)_s + \alpha u_0(x) + u_0^3(s)\right] ds,\\
&u_2(x) = u_1(x) - \int_0^x e^{\alpha(s-x)}\left[\left(u_1(s)\right)_s + \alpha u_1(x) + u_1^3(s)\right] ds.
\end{aligned} \tag{2.94}$$

Consequently, the solution can be obtained from

$$u = \lim_{n\to\infty} u_n. \tag{2.95}$$

Example 2.4 Consider the first order nonlinear partial differential equation

$$\frac{\partial}{\partial x}u(x,t) + \frac{\partial}{\partial t}u(x,t) = u^2(x,t), \quad u(0,t) = a. \tag{2.96}$$

Following the discussion presented above we find that $\lambda = -1$. Therefore, the iteration formula is given by

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^x \left[\frac{\partial}{\partial s}u_n(s,t) + \frac{\partial}{\partial t}u_n(s,t) - {u_n}^2(s,t)\right] ds. \tag{2.97}$$

Using the Taylor expansion, we can choose $u_0(x,t) = u(0,t) = a$ from the given condition. Using $u_0(x,t) = a$, we have

$$\begin{aligned}
&u_0(x,t) = a,\\
&u_1(x,t) = a - \int_0^x \left[\frac{\partial}{\partial s}u_0(s,t) + \frac{\partial}{\partial t}u_0(s,t) - {u_0}^2(s,t)\right] ds,\\
&u_2(x,t) = u_1(x,t) - \int_0^x \left[\frac{\partial}{\partial s}u_1(s,t) + \frac{\partial}{\partial t}u_1(s,t) - {u_1}^2(s,t)\right] ds.
\end{aligned} \tag{2.98}$$

Consequently, the solution can be obtained from

$$u = \lim_{n \to \infty} u_n. \tag{2.99}$$

Example 2.5 Consider the second order partial differential equation

$$\frac{\partial^2}{\partial x^2} u(x,t) + \frac{\partial}{\partial t} u(x,t) = 0, \quad u(0,t) = a, \qquad u^{'}(0,t) = b. \tag{2.100}$$

Following the discussion presented above we find that $\lambda = s - x$. Therefore, the iteration formula is given by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x (s-x) \left[\frac{\partial^2}{\partial s^2} u_n(s,t) + \frac{\partial}{\partial t} u_n(s,t) \right] ds. \tag{2.101}$$

Using the Taylor expansion, we can choose $u_0(x,t) = a + bx$ from the given initial conditions. Using $u_0(x,t) = a + bx$, we have

$u_0(x,t) = a + bx$

$u_1(x,t) = a + bx + \int_0^x (s-x) \left[\frac{\partial^2}{\partial s^2} u_0(s,t) + \frac{\partial}{\partial t} u_0(s,t) \right] ds$

$u_2(x,t) = u_1(x) + \int_0^x (s-x) \left[\frac{\partial^2}{\partial s^2} u_1(s,t) + \frac{\partial}{\partial t} u_1(s,t) \right] ds$

$\vdots$ (2.102)

Consequently, the solution can be obtained from

$$u = \lim_{n \to \infty} u_n. \tag{2.103}$$

2.4 Convergence of the Method

The variational iteration formula creates a recurrence sequence $\{u_n(x)\}_{n=1}^{\infty}$. Obviously, the limit of the sequence will be the solution $u(x)$, (2.5) if the sequence is convergent. In this section, we will discuss the convergence of the variational iteration method.

Theorem 2.1 The sequence $\{u_n(x)\}_{n=1}^{\infty}$ defined by (2.85) with $u_0(x) = a + bx$ (a, b are real constant) converges to the solution, $u(x)$, of problem (2.84) provided that u and the iterates $u_n's$ are bounded.

Proof:
By subtracting $u(x)$ from both sides of (2.85), the equation can be rewritten as

$$u_{n+1}(x) - u(x) = u_n(x) - u(x) + \int_0^x \lambda(s)\,[(u_n - u)_{ss} + u_{ss} + u^2{}_n(s)]ds, \quad (2.104)$$

where $\lambda = s - x$. Since $u(x)$ is the exact solution of (2.84) then the term u_{ss} in the integrand can be replaced by $-u^2(s)$. By letting

$E_n(x) = u_n(x) - u(x)$, equation (2.104) becomes

$$E_{n+1}(x) = E_n(x) + \int_0^x \lambda(s)\left[\left(E_n(s)\right)_{ss} - u^2(s) + u^2{}_n(s)\right]ds. \quad (2.105)$$

Integrating the first term in the integrand twice by parts we have

$$E_{n+1}(x) = E_n(x) + \lambda(x)(E_n)_s(x) - \lambda'(x)E_n(x) + \int_0^x \lambda''(s)\,E_n(s)ds + \int_0^x \lambda(s)\,[-u^2(s) + u^2{}_n(s)]ds. \quad (2.106)$$

Upon using the three stationary conditions (2.43) into the equation (2.106) we obtain

$$E_{n+1}(x) = \int_0^x \lambda(s)\,[-u^2(s) + u^2{}_n(s)]ds. \quad (2.107)$$

Operating with the L^2-norm on both sides of the last equation we get

$$\begin{aligned}\|E_{n+1}(x)\|_{L^2} &\le \int_0^x \lambda(s)\,\|-u^2(s) + u^2{}_n(s)\|_{L^2}ds \\ &\le \|\lambda(s)\|_\infty \int_0^x \|-u^2(s) + u^2{}_n(s)\|_{L^2}\,ds,\end{aligned} \quad (2.108)$$

where $\|\lambda(s)\|_\infty = \max_{s\in[0,T]}|\lambda(s)|$. Clearly, $\lambda(s)$ is bounded since

$$\|\lambda(s)\|_\infty = \|s - x\|_\infty \le \|s\|_\infty + \|x\|_\infty = 2T. \quad (2.109)$$

Applying the Mean Value Theorem to the integrand in (2.109), therefore equation (2.109) becomes

$$\|E_{n+1}(x)\|_{L^2} \le \|\lambda(s)\|_\infty \int_0^x 2\|\bar{u}(s)\|_{L^2}\|u_n(s) - u(s)\|_{L^2}\,ds$$

$$\leq 2\|\lambda(s)\|_\infty \int_0^x \|\bar{u}(s)\|_{L^2}\|E_n(s)\|_{L^2}\, ds. \tag{2.110}$$

Let

$L = \max_{s\in[0,T]}|\lambda(s)|$ and $P = \max_{s\in[0,T]}|\bar{u}(s)|$.

Then, from inequality (2.110) we get

$$\|E_{n+1}(x)\|_{L^2} \leq 2LP \int_0^x \|E_n(s)\|_{L^2}\, ds. \tag{2.111}$$

By induction and by letting $M = 2LP$, we get

$$\|E_1(x)\|_{L^2} \leq M \int_0^x \|E_0(s)\|_{L^2}\, ds \leq M\|E_0(s)\|_\infty \int_0^x ds = M\|E_0(s)\|_\infty\, x,$$

$$\|E_2(x)\|_{L^2} \leq M \int_0^x \|E_1(s)\|_{L^2}\, ds \leq M^2\|E_0(s)\|_\infty \int_0^x s\, ds = M^2\|E_0(s)\|_\infty \frac{x^2}{2},$$

...

$$\|E_{n+1}(x)\|_{L^2} \leq M^n \int_0^x \|E_n(s)\|_{L^2}\, ds \leq M^{n+1}\|E_0(x)\|_\infty \int_0^x \frac{s^n}{n!}\, ds = M^{n+1}\|E_0(x)\|_\infty \frac{x^{n+1}}{(n+1)!}, \tag{2.112}$$

where $\|E_0(x)\|_\infty = \max_{x\in[0,T]}|E_0(x)|$.

We have

$$\begin{aligned}\|E_0(x)\|_\infty &= \|u_0(x) - u(x)\|_\infty = \|a + bx - u(x)\|_\infty \\ &\leq \|a\|_\infty + \|bx\|_\infty + \|u(x)\|_\infty = |a| + |bL| + \max_{x\in[0,T]}|u(x)|.\end{aligned} \tag{2.113}$$

Where $L = \max_{x\in[0,T]}|x|$. Since $u(x)$ is the exact solution of equation (2.84) then it belongs to $C^2[0,T]$. Therefore, it is bounded and consequently $E_0(x)$ is bounded as well. Let $c = \max_{x\in[0,T]}|u(x)|$. Then we have from (2.112) and (2.113):

$$\|E_{n+1}(x)\|_{L^2} \leq M^{n+1}(|a| + |b|L + c)\frac{x^{n+1}}{(n+1)!}, \tag{2.114}$$

As $n \to \infty$. Therefore, the sequence $\left\{M^{n+1}(|a| + |b|L + c)\frac{x^{n+1}}{(n+1)!}\right\}$ converges uniformly to 0 and thus from (2.114) it follows that $\|E_{n+1}(x)\|_{L^2} \to 0$, which means $u_n(x)$ converges uniformly to $u(x)$.

Theorem 2.2 The sequences $\{u_n(x)\}_{n=1}^{\infty}$ defined by (2.89) with $u_0(x) = a + bx + cx^2$ $(a, b, c$ are real constants) converges to the solution $u(x)$, of the problem (2.88), provided that u and the iterates $u_n's$ are bounded.

Proof:
By subtracting $u(x)$ from both sides of (2.88), the equation (2.88) can be rewritten as

$$u_{n+1}(x) - u(x) = u_n(x) - u(x) + \int_0^x \lambda(s)\,[(u_n - u)_{sss} + u_{sss} + u^2{}_n(s)]ds, \tag{2.115}$$

where $\lambda = -\frac{1}{2}(s - x)^2$. Since $u(x)$ is the exact solution of (2.89) then the term u_{sss} in the integrand can be replaced by $-u^2(s)$. By letting $E_n(x) = u_n(x) - u(x)$, equation (2.115) becomes

$$E_{n+1}(x) = E_n(x) + \int_0^x \lambda(s)\left[\big(E_n(s)\big)_{sss} - u^2(s) + u^2{}_n(s)\right] ds. \tag{2.116}$$

Integrating the first term in the integrand three times by parts we have

$$\begin{aligned} E_{n+1}(x) = E_n(x) + \lambda(x)(E_n)_{ss}(x) - \lambda'(x)(E_n)_s(x) + \lambda''(x)E_n(x) \\ - \int_0^x \lambda'''(s)\,E_n(s)ds + \int_0^x \lambda(s)\,[-u^2(s) + u^2{}_n(s)]ds. \end{aligned} \tag{2.117}$$

Upon using the four stationary conditions (2.73) into the equation (2.117) we obtain

$$E_{n+1}(x) = \int_0^x \lambda(s)\,[-u^2(s) + u^2{}_n(s)]ds. \tag{2.118}$$

Operating with the L^2-norm on both sides of the last equation we get

$$\begin{aligned} \|E_{n+1}(x)\|_{L^2} &\le \int_0^x \lambda(s)\,\|-u^2(s) + u^2{}_n(s)\|_{L^2} ds \\ &\le \|\lambda(s)\|_\infty \int_0^x \|-u^2(s) + u^2{}_n(s)\|_{L^2}\, ds. \end{aligned} \tag{2.119}$$

where $\|\lambda(s)\|_\infty = \max_{s\in[0,T]}|\lambda(s)|$ .Clearly, $\lambda(s)$ is bounded since

$$\|\lambda(s)\|_\infty = \|s - x\|_\infty \le \|s\|_\infty + \|x\|_\infty = 2T. \tag{2.120}$$

Applying the Mean Value Theorem to the integrand in (2.120), therefore equation (2.120) becomes

$$\begin{aligned} \|E_{n+1}(x)\|_{L^2} &\le \|\lambda(s)\|_\infty \int_0^x 2\|\bar{u}(s)\|_{L^2}\|u_n(s) - u(s)\|_{L^2}\, ds \\ &\le 2\|\lambda(s)\|_\infty \int_0^x \|\bar{u}(s)\|_{L^2}\|E_n(s)\|_{L^2}\, ds. \end{aligned} \tag{2.121}$$

Let

$L = \max_{s\in[0,T]}|\lambda(s)|$ and $P = \max_{s\in[0,T]}|\bar{u}(s)|$. Then, from inequality (2.121) we get

$$\|E_{n+1}(x)\|_{L^2} \le 2LP\int_0^x \|E_n(s)\|_{L^2}\, ds. \tag{2.122}$$

By mathematical induction and letting $M = 2LP$, we conclude that

$$\|E_1(x)\|_{L^2} \le M\int_0^x \|E_0(s)\|_{L^2}\, ds \le M\|E_0(s)\|_\infty \int_0^x ds = M\|E_0(s)\|_\infty\, x,$$

$$\|E_2(x)\|_{L^2} \le M\int_0^x \|E_1(s)\|_{L^2}\, ds \le M^2\|E_0(s)\|_\infty \int_0^x s\, ds = M^2\|E_0(s)\|_\infty \frac{x^2}{2},$$

...

$$\|E_{n+1}(x)\|_{L^2} \le M^n\int_0^x \|E_n(s)\|_{L^2}\, ds \le M^{n+1}\|E_0(x)\|_\infty \int_0^x \frac{s^n}{n!} ds$$
$$= M^{n+1}\|E_0(x)\|_\infty \frac{x^{n+1}}{(n+1)!}, \tag{2.123}$$

where $\|E_0(x)\|_\infty = \max_{x\in[0,T]}|E_0(x)|$.

We have that

$$\|E_0(x)\|_\infty = \|u_0(x) - u(x)\|_\infty = \|a + bx + cx^2 - u(x)\|_\infty$$
$$\le \|a\|_\infty + \|bx\|_\infty + \|cx^2\|_\infty + \|u(x)\|_\infty = |a| + |bL| + |cL^2| + \max_{x\in[0,T]}|u(x)|, \tag{2.124}$$

where $L = \max_{x\in[0,T]}|x|$ and $L^2 = \max_{x\in[0,T]}|x^2|$. Since $u(x)$ is the exact solution of equation (2.88), then this implies that it belongs to $C^2[0,T]$ and so it is bounded and hence $E_0(x)$ is bounded by the latter inequality. Let $d = \max_{x\in[0,T]}|u(x)|$.

Then we have from (2.123) and (2.124)

$$\|E_{n+1}(x)\|_{L^2} \le M^{n+1}(|a| + |bL| + |cL^2| + d)\frac{x^{n+1}}{(n+1)!}, \tag{2.125}$$

as $n \to \infty$. Therefore, the sequence $\left\{M^{n+1}(|a| + |bL| + |cL^2| + d)\frac{x^{n+1}}{(n+1)!}\right\}$ converges uniformly to 0 and from (2.125) it follows that $\|E_{n+1}(x)\|_{L^2} \to 0$ and hence $u_n(x)$ converges uniformly to $u(x)$.

Theorem 2.3 The sequences $\{u_n(x)\}_{n=1}^{\infty}$ defined by (2.93) with $u_0(x) = a$ (a is a real constant) converges to the solution ,$u(x)$, of the problem (2.92), provided that u and the iterates $u_n's$ are bounded.

Proof:

By subtracting $u(x)$ from both sides of (2.93), the equation (2.93) can be rewritten as

$$u_{n+1}(x) - u(x) = u_n(x) - u(x)$$
$$+ \int_0^x \lambda(s)\left[\left(u_n(s) - u(s)\right)_s + u_s + \alpha u_n(s) + u^3{}_n(s)\right] ds, \tag{2.126}$$

where $\lambda = -e^{\alpha(s-x)}$. Since $u(x)$ is the exact solution of (2.92), then the term u_s in the integrand can replaced by $[-\alpha u(s) - u^3(s)]$. By letting $E_n(x) = u_n(x) - u(x)$, equation (2.126) becomes

$$E_{n+1}(x) = E_n(x) + \int_0^x \lambda(s) \left[\left(E_n(s)\right)_s - u^3(s) - \alpha u(s) + \alpha u_n(s) + u^3{}_n(s) \right] ds. \tag{2.127}$$

Integrating the first term in the integrand once by parts we have

$$E_{n+1}(x) = E_n(x) + \lambda(x)(E_n)(x) - \int_0^x \lambda'(s)\, E_n(s) ds$$
$$+ \int_0^x \lambda(s)\, [u^3{}_n(s) - u^3(s)] ds + \alpha \int_0^x \lambda(s)\, E_n(s) ds. \tag{2.128}$$

Upon using the two stationary conditions (2.32) into the equation (2.128) we obtain

$$E_{n+1}(x) = \int_0^x \lambda(s)\, [u^3{}_n(s) - u^3(s)] ds. \tag{2.129}$$

Operating with the L^2-norm on both sides of the last equation we get

$$\|E_{n+1}(x)\|_{L^2} \le \int_0^x \lambda(s)\, \|u^3{}_n(s) - u^3(s)\|_{L^2} ds$$
$$\le \|\lambda(s)\|_\infty \int_0^x \|u^3{}_n(s) - u^3(s)\|_{L^2}\, ds, \tag{2.130}$$

where $\|\lambda(s)\|_\infty = \max_{s\in[0,T]}|\lambda(s)|$. Clearly, $\lambda(s)$ is bounded since

$$\|\lambda(s)\|_\infty = \left\|-e^{\alpha(s-x)}\right\|_\infty \le \|e^{\alpha s}\|_\infty + \|e^{-\alpha x}\|_\infty = e^{\alpha T} + 1. \tag{2.131}$$

Applying the Mean Value Theorem to the integrand in (2.130), then equation (2.130) becomes

$$\|E_{n+1}(x)\|_{L^2} \le \|\lambda(s)\|_\infty \int_0^x 2\|\bar{u}(s)\|_{L^2} \|u_n(s) - u(s)\|_{L^2}\, ds$$
$$\le 2\|\lambda(s)\|_\infty \int_0^x \|\bar{u}(s)\|_{L^2} \|E_n(s)\|_{L^2}\, ds. \tag{2.132}$$

Let

$$L = \max_{s\in[0,T]}|\lambda(s)| \text{ and } P = \max_{s\in[0,T]}|\bar{u}(s)|. \tag{2.133}$$

Then, from inequality (2.132) we get

$$\|E_{n+1}(x)\|_{L^2} \le 2LP \int_0^x \|E_n(s)\|_{L^2}\, ds. \tag{2.134}$$

By induction and upon letting $M = 2LP$, we get

$$\|E_1(x)\|_{L^2} \le M\int_0^x \|E_0(s)\|_{L^2}\, ds \le M\|E_0(s)\|_\infty \int_0^x ds = M\|E_0(s)\|_\infty\, x,$$
$$\|E_2(x)\|_{L^2} \le M\int_0^x \|E_1(s)\|_{L^2}\, ds \le M^2\|E_0(s)\|_\infty \int_0^x s\, ds = M^2\|E_0(s)\|_\infty \frac{x^2}{2},$$
...
$$\|E_{n+1}(x)\|_{L^2} \le M^n\int_0^x \|E_n(s)\|_{L^2}\, ds \le M^{n+1}\|E_0(x)\|_\infty \int_0^x \frac{s^n}{n!} ds$$
$$= M^{n+1}\|E_0(x)\|_\infty \frac{x^{n+1}}{(n+1)!}, \tag{2.135}$$

where $\|E_0(x)\|_\infty = \max_{x\in[0,T]}|E_0(x)|$.

We have

$$\|E_0(x)\|_\infty = \|u_0(x) - u(x)\|_\infty = \|a - u(x)\|_\infty$$
$$\le \|a\|_\infty + \|u(x)\|_\infty = |a| + \max_{x\in[0,T]}|u(x)|. \tag{2.136}$$

Since $u(x)$ is the exact solution of equation (2.92) then it belongs to $C^2[0,T]$hence it is bounded. This means that $E_0(x)$ is also bounded. Let $c = \max_{x\in[0,T]}|u(x)|$. Then we have from (2.135) and (2.136):

$$\|E_{n+1}(x)\|_{L^2} \le M^{n+1}(|a| + c)\frac{x^{n+1}}{(n+1)!}, \tag{2.137}$$

as $n \to \infty$. Therefore, the sequence $\left\{M^{n+1}(|a| + c)\frac{x^{n+1}}{(n+1)!}\right\}$ converges uniformly to 0 and from (2.136) it follows that $\|E_{n+1}(x)\|_{L^2} \to 0$ and hence $u_n(x)$ converges uniformly to $u(x)$.

Theorem 2.4 Let $u_n(x,t)$ be the sequences $\{u_n(x,t)\}_{n=1}^{\infty}$ defined by (2.97) with $u_0(x,t) = a$ (a is a real constant). If $E_n(x,t) = u_n(x,t) - u(x,t)$ and $\left\|\frac{\partial}{\partial t}E_n(x,t)\right\|_{L^2} \le \|E_n(x,t)\|_{L^2}$ then the sequences $\{u_n(x,t)\}_{n=1}^{\infty}$ converges to the solution $u(x,t) \in (C(R))^n, (x,t) \in R = [0,T] \times [0,L]$, of the problem (2.96).

Proof:
By subtracting $u(x,t)$ from both sides of (2.97), the equation (2.97) can be rewritten as

$$\begin{aligned} u_{n+1}(x,t) - u(x,t) \\ &= u_n(x,t) - u(x,t) \\ &+ \int_0^x \lambda(s)\left[\frac{\partial}{\partial s}\big(u_n(s,t) - u(s,t)\big) + \frac{\partial}{\partial s}u(s,t) + \frac{\partial}{\partial t}u_n(s,t)\right. \\ &\left. - u_n{}^2(s,t)\ \right] ds, \end{aligned} \tag{2.138}$$

where $\lambda = -1$. Since $u(x,t)$ is the exact solution of (2.96) then the term $\frac{\partial}{\partial s}u(s,t)$ in the integrand can be written as $\left[-\frac{\partial}{\partial t}u(s,t)+u^2(s,t)\right]$ and by letting $E_n(x,t) = u_n(x,t) - u(x,t)$, equation (2.138) becomes

$$\begin{aligned} E_{n+1}(x,t) = E_n(x,t) \\ + \int_0^x \lambda(s)\left[\frac{\partial}{\partial s}\left(E_n(s,t)\right) - \frac{\partial}{\partial t}u(s,t) + u^2(s,t) + \frac{\partial}{\partial t}u_n(s,t) \right. \\ \left. - {u_n}^2(s,t)\right] ds. \end{aligned} \tag{2.139}$$

Integrating the first term in the integrand once by parts we have

$$\begin{aligned} E_{n+1}(x,t) = E_n(x,t) + \lambda(x)(E_n)(x,t) - \int_0^x \lambda'(s)\, E_n(s,t)ds \\ + \int_0^x \lambda(s)\,[u^2{}_n(s,t) - u^2(s,t)]ds + \int_0^x \lambda(s)\frac{\partial}{\partial t}\left(E_n(s,t)\right)ds. \end{aligned} \tag{2.140}$$

Upon using the two stationary conditions (2.22) into the equation (2.140) we obtain

$$E_{n+1}(x,t) = -\int_0^x [u^2{}_n(s,t) - u^2(s,t)] - \int_0^x \frac{\partial}{\partial t}\left(E_n(s,t)\right)ds. \tag{2.141}$$

Operating with the L^2-norm on both sides of the last equation we get

$$\begin{aligned} \|E_{n+1}(x,t)\|_{L^2} &\le -\int_0^x \|u^2{}_n(s,t) - u^2(s,t)\|_{L^2}\, ds - \int_0^x \left\|\frac{\partial}{\partial t}\left(E_n(s,t)\right)\right\|_{L^2} ds \\ &\le \|-1\|_\infty \left(\int_0^x \|u^2{}_n(s,t) - u^2(s,t)\|_{L^2}\, ds + \int_0^x \left\|\frac{\partial}{\partial t}\left(E_n(s,t)\right)\right\|_{L^2} ds\right), \end{aligned} \tag{2.142}$$

Where $\|-1\|_\infty = \max_{s\in[0,T],t\in[0,L]}|-1|$. Clearly, $\lambda(s)$ is bounded since

$$\|\lambda(s)\|_\infty = \|-1\|_\infty \le 1 \tag{2.143}$$

and

$$\left\|\frac{\partial}{\partial t}E_n(x,t)\right\|_{L^2} \le \|E_n(x,t)\|_{L^2}. \tag{2.144}$$

From (2.141), (2.143) and (2.144) we get

$$\|E_{n+1}(x,t)\|_{L^2} \le \int_0^x \|u^2{}_n(s,t) - u^2(s,t)\|_{L^2}\, ds + \int_0^x \|E_n(s,t)\|_{L^2} ds \tag{2.145}$$

Applying the Mean Value Theorem to the first integrand in (2.145).Therefore the equation (2.145) becomes

$$\|E_{n+1}(x,t)\|_{L^2} \le \int_0^x 2\|\bar{u}(s,t)\|_{L^2}\|u_n(s,t) - u(s,t)\|_{L^2}\, ds$$

$$\leq 2\int_0^x \|\bar{u}(s,t)\|_{L^2}\|E_n(s,t)\|_{L^2}\,ds. \tag{2.146}$$

Let $P = \max_{s\in[0,T]}|\bar{u}(s,t)|$.

Then, substituting the inequality (2.146) into (2.145) we get

$$\begin{aligned}\|E_{n+1}(x,t)\|_{L^2} &\leq 2P\int_0^x \|E_n(s,t)\|_{L^2}\,ds + \int_0^x \|E_n(s,t)\|_{L^2}ds\\ &= (1+2P)\int_0^x \|E_n(s,t)\|_{L^2}ds.\end{aligned} \tag{2.147}$$

Then by induction and letting $M = 1 + 2P$, we get

$$\begin{aligned}&\|E_1(x,t)\|_{L^2} \leq M\int_0^x \|E_0(s,t)\|_{L^2}\,ds \leq M\|E_0(s,t)\|_\infty \int_0^x ds = M\|E_0(s,t)\|_\infty\, x,\\ &\|E_2(x,t)\|_{L^2} \leq M\int_0^x \|E_1(s,t)\|_{L^2}\,ds \leq M^2\|E_0(s,t)\|_\infty \int_0^x s\,ds\\ &\qquad\qquad = M^2\|E_0(s,t)\|_\infty \frac{x^2}{2},\\ &\dots\\ &\|E_{n+1}(x,t)\|_{L^2} \leq M^n\int_0^x \|E_n(s,t)\|_{L^2}\,ds \leq M^{n+1}\|E_0(x,t)\|_\infty \int_0^x \frac{s^n}{n!}ds\\ &= \|E_0(x,t)\|_\infty \frac{(Mx)^{n+1}}{(n+1)!},\end{aligned} \tag{2.148}$$

where $\|E_0(x,t)\|_\infty = \max_{x\in[0,T],t\in[0,L]}|E_0(x,t)|$.

We have

$$\begin{aligned}\|E_0(x,t)\|_\infty &= \|u_0(x,t) - u(x,t)\|_\infty = \|a - u(x,t)\|_\infty\\ &\leq \|a\|_\infty + \|u(x,t)\|_\infty = |a| + \max_{x\in[0,T],t\in[0,L]}|u(x,t)|.\end{aligned} \tag{2.149}$$

Since $u(x,t)$ is the exact solution of equation (2.96) then it belongs to $C^2[0,T]$hence it is bounded. Let $c = \max_{x\in[0,T],t\in[0,L]}|u(x,t)|$. Then we have from (2.148) and (2.149):

$$\|E_{n+1}(x,t)\|_{L^2} \leq M^{n+1}(|a| + c)\frac{x^{n+1}}{(n+1)!}. \tag{2.150}$$

As $n\to\infty$. Therefore the sequence $\left\{M^{n+1}(|a|+c)\frac{x^{n+1}}{(n+1)!}\right\}$ converges uniformly to 0 and from (2.150) it follows that $\|E_{n+1}(x,t)\|_{L^2} \to 0$ and hence $u_n(x,t)$ converges uniformly to $u(x,t)$.

Theorem 2.5 Let $u_n(x,t)$ be the sequences $\{u_n(x,t)\}_{n=1}^{\infty}$defined by (2.101) with $u_0(x,t) = a + bx(a,b$ are real constants). If $E_n(x,t) = u_n(x,t) - u(x,t)$ and $\left\|\frac{\partial}{\partial t}E_n(x,t)\right\|_{L^2} \leq$

$\|E_n(x,t)\|_{L^2}$ then the sequences $\{u_n(x,t)\}_{n=1}^{\infty}$ converges to the solution $u(x,t) \in (C(R))^n, (x,t) \in R = [0,T] \times [0,L]$, of the problem (2.100).

Proof:

By subtracting $u(x,t)$ from both sides of (2.101), the equation (2.101) can be rewritten as

$$\begin{aligned} u_{n+1}(x,t) - u(x,t) \\ &= u_n(x,t) - u(x,t) \\ &+ \int_0^x \lambda(s) \left[\frac{\partial^2}{\partial s^2} \left(u_n(s,t) - u(s,t) \right) + \frac{\partial}{\partial s} u(s,t) \right. \\ &\left. + \frac{\partial}{\partial t} u_n(s,t) \right] ds, \end{aligned} \tag{2.151}$$

where $\lambda = s - x$. Since $u(x,t)$ is the exact solution of (2.100) then the term $\frac{\partial}{\partial s} u(s,t)$ in the integrand can be written as $\left[-\frac{\partial}{\partial t} u(s,t) \right]$, and by letting $E_n(x,t) = u_n(x,t) - u(x,t)$, equation (2.151) becomes

$$E_{n+1}(x,t) = E_n(x,t) + \int_0^x \lambda(s) \left[\frac{\partial^2}{\partial s^2} \left(E_n(s,t) \right) - \frac{\partial}{\partial t} u(s,t) + \frac{\partial}{\partial t} u_n(s,t) \right] ds. \tag{2.152}$$

Integrating the first term in the integrand once by parts we have

$$\begin{aligned} E_{n+1}(x,t) = E_n(x,t) + \lambda(x) \frac{\partial}{\partial s} (E_n)(x,t) - \lambda'(x)(E_n)(x,t) \\ + \int_0^x \lambda''(s) \, E_n(s,t) ds \\ + \int_0^x \lambda(s) \frac{\partial}{\partial t} \left(E_n(s,t) \right) ds. \end{aligned} \tag{2.153}$$

Upon using the three stationary conditions (2.43) into the equation (2.153) we obtain

$$E_{n+1}(x,t) = \int_0^x (s - x) \frac{\partial}{\partial t} \left(E_n(s,t) \right) ds. \tag{2.154}$$

Operating with the L^2-norm on both sides of the last equation we get

$$\|E_{n+1}(x,t)\|_{L^2} \leq \|s - x\|_\infty + \int_0^x \left\| \frac{\partial}{\partial t} \left(E_n(s,t) \right) \right\|_{L^2} ds, \tag{2.155}$$

where $\|-1\|_\infty = \max_{s \in [0,T], t \in [0,L]} |-1|$. Clearly, $\lambda(s)$ is bounded since

$$\|\lambda(s)\|_\infty = \|s - x\|_\infty \leq \|s\|_\infty + \|x\|_\infty = 2T, \tag{2.156}$$

and

$$\left\| \frac{\partial}{\partial t} E_n(x,t) \right\|_{L^2} \leq \|E_n(x,t)\|_{L^2}. \tag{2.157}$$

From (2.156) and (2.157) we get

$$\|E_{n+1}(x,t)\|_{L^2} \le 2T\int_0^x \|E_n(s,t)\|_{L^2} ds. \tag{2.158}$$

Then by induction and letting $M = 2T$, we get

$$\begin{aligned}
&\|E_1(x,t)\|_{L^2} \le M\int_0^x \|E_0(s,t)\|_{L^2}\, ds \le M\|E_0(s,t)\|_\infty \int_0^x ds = M\|E_0(s,t)\|_\infty\, x,\\
&\|E_2(x,t)\|_{L^2} \le M\int_0^x \|E_1(s,t)\|_{L^2}\, ds \le M^2\|E_0(s,t)\|_\infty \int_0^x s\, ds\\
&\qquad\qquad = M^2\|E_0(s,t)\|_\infty \frac{x^2}{2},\\
&\ldots\\
&\|E_{n+1}(x,t)\|_{L^2} \le M\int_0^x \|E_n(s,t)\|_{L^2}\, ds \le M^n\|E_0(x,t)\|_\infty \int_0^x \frac{s^n}{n!} ds\\
&= \|E_0(x,t)\|_\infty \frac{(Mx)^{n+1}}{(n+1)!},
\end{aligned} \tag{2.159}$$

where $\|E_0(x,t)\|_\infty = \max_{x\in[0,T],t\in[0,L]}|E_0(x,t)|$.

We have

$$\begin{aligned}
\|E_0(x,t)\|_\infty = \|u_0(x,t) - u(x,t)\|_\infty = \|a + bx - u(x,t)\|_\infty\\
\le \|a\|_\infty + \|bx\|_\infty + \|u(x,t)\|_\infty = |a| + |bN| + \max_{x\in[0,T],t\in[0,L]}|u(x,t)|.
\end{aligned} \tag{2.160}$$

where $N = \max_{x\in[0,T],t\in[0,L]}|(x,t)|$. Since $u(x,t)$ is the exact solution of equation (2.100) then it belongs to $C^2[0,T]$ hence it is bounded. Let $c = \max_{x\in[0,T],t\in[0,L]}|u(x,t)|$. Then we have from (2.159) and (2.160):

$$\|E_{n+1}(x,t)\|_{L^2} \le M^{n+1}(|a| + |b|N + c)\frac{x^{n+1}}{(n+1)!}. \tag{2.161}$$

As $n \to \infty$, the sequence $\left\{M^{n+1}(|a| + |bN| + c)\frac{x^{n+1}}{(n+1)!}\right\}$ converges uniformly to 0 and from (2.161) it follows that $\|E_{n+1}(x,t)\|_{L^2} \to 0$ and hence $u_n(x,t)$ converges uniformly to $u(x,t)$.

2.5 Ordinary Differential Equations

2.5.1 Initial Value Problems

In this section, we will apply the VIM method, as presented before, to some examples involving linear and nonlinear IVPs.

Example 2.6 Consider the following first order nonlinear ordinary differential equation subject to an initial condition:

$$u'(x) - u^2(x) = 1, \qquad u(0) = 0. \tag{2.162}$$

Solution:

Following the discussion presented above, we find that $\lambda(s) = -1$. Therefore, the iteration formula is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x [(u_n(s))_s - u_n^2(s) - 1]\, ds. \tag{2.163}$$

We can choose $u_0(x) = u(0) = 0$ and this choice is appropriate based on the given condition. Using $u_0(x) = 0$ we have

$$\begin{aligned}
u_0(x) &= 0,\\
u_1(x) &= 0 - \int_0^x \left[(u_0(s))_s - u_0^2(s) - 1\right] ds = x,\\
u_2(x) &= x - \int_0^x \left[(u_1(s))_s - u_1^2(s) - 1\right] ds = x + \frac{x^3}{3},\\
u_3(x) &= x + \frac{x^3}{3} - \int_0^x \left[(u_2(s))_s - u_2^2(s) - 1\right] ds = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{1}{63}x^7,\\
u_4(x) &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{1}{59535}x^{15} + \frac{4}{12285}x^{13} + \frac{134}{51975}x^{11}\\
&\quad + \frac{38}{2835}x^9,\\
&\cdots\\
u_n(x) &= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{1}{63}x^7 + \cdots.
\end{aligned} \tag{2.164}$$

The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{2.165}$$

Note that the infinite series solution obtained by the VIM is basically the McLaurin series expansion of the exact solution of the problem which is given by

$$u(x) = \tan x. \tag{2.166}$$

Table 2.1 shows the resulting absolute error obtained by comparing the VIM, with three iterations and four iterations, with the exact solution which is $\tan(x)$. The method yields highly accurate numerical solution using few iterates particularly for values of x in the vicinity of 0. However, it is important to mention that the error is not uniformly distributed over the entire domain and slowly deteriorates as we increase the values of x, that is, for larger values that are further away from the origin. Later, we will suggest a domain decomposition that will somewhat overcome this setback. Figure 2.1 also depicts the numerical results.

x	EXACT	$\lvert tan(x) - u_3(x) \rvert$	$\lvert tan(x) - u_4(x) \rvert$
0	0.0	0.0	0.0
0.1	0.100334672	3.9×10^{-9}	1.0×10^{-10}
0.2	0.202710036	5.0×10^{-7}	4.3×10^{-9}
0.3	0.309336250	8.8×10^{-6}	1.8×10^{-7}
0.4	0.422793219	6.9×10^{-5}	2.5×10^{-6}
0.5	0.546302490	3.5×10^{-4}	2.0×10^{-6}
0.6	0.684136808	1.3×10^{-3}	1.1×10^{-5}
0.7	0.842288380	4.2×10^{-3}	5.1×10^{-4}
0.8	1.029638557	1.2×10^{-2}	1.9×10^{-3}
0.9	1.260158218	3.1×10^{-2}	6.5×10^{-3}
1.0	1.557407725	7.5×10^{-2}	2.0×10^{-2}

Table 2.1 Error obtained using VIM with three and four iterations.

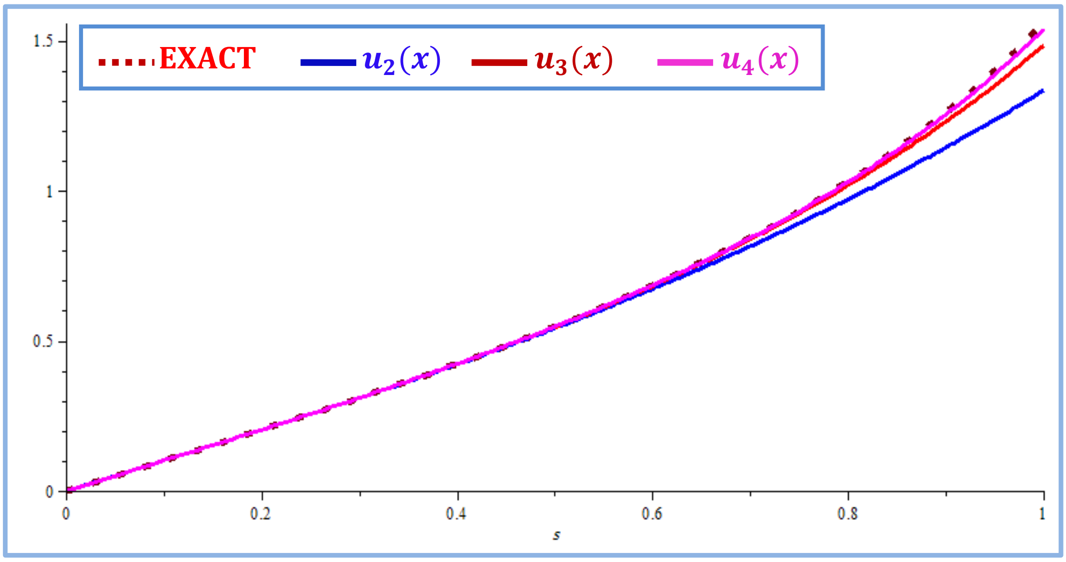

Figure 2.1 Comparison between exact and VIM solutions with two, three and four iterations.

Example 2.7 Consider the following first order nonlinear ordinary differential equation subject to an initial condition:

$$u'(x) + u^2(x) = 0, \qquad u(0) = 1. \tag{2.167}$$

Solution:
Following the discussion presented above, we find that $\lambda(s) = -1$. Therefore, the iteration formula is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x [(u_n(s))_s + u_n^2(s)]\, ds. \tag{2.168}$$

We can choose $u_0(x) = u(0) = 1$ and this choice is appropriate based on the given condition. Using $u_0(x) = 1$ we have

$$\begin{aligned}
&u_0(x) = 1,\\
&u_1(x) = 1 - \int_0^x [(u_n(s))_s + u_n^2(s)]\, ds = 1 - x,\\
&u_2(x) = x - \int_0^x [(u_n(s))_s + u_n^2(s)]\, ds = 1 - x + x^2 - \frac{x^3}{3},\\
&\dots\\
&u_n(x) = 1 - x + x^2 - \frac{x^3}{3} + \cdots.
\end{aligned} \tag{2.169}$$

The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{2.170}$$

Note that the infinite series solution obtained by the VIM is basically the McLaurin series expansion of the exact solution of the problem which is given by

$$u(x) = \frac{1}{(1+x)}. \tag{2.171}$$

In order to accelerate the convergent rate, we can differentiate both sides of equation (2.167) with respect to x, so we get

$$u'' + 2uu' = 0. \tag{2.172}$$

From equation (2.167) we can find that

$$u' = -u^2, \quad u(0) = 1, u'(0) = -1. \tag{2.173}$$

Substituting (2.173) into (2.172) we get

$$u'' - 2u^3 = 0, \quad u(0) = 1, u'(0) = -1. \tag{2.174}$$

Using VIM to solve Eq.(2.174), therefore, the iteration formula is given by

$$u_{n+1} = u_n + \int_0^x (s - x)[(u_n)_{ss}(s) - {u^3}_n(s)]\, ds. \tag{2.175}$$

We can choose $u_0(x) = u(0) + u'(0)x = 1 - x$, using the first two terms of McLaurin series and the given initial conditions

$$u_0(x) = 1 - x,$$

$$u_1(x) = 1 - x + \int_0^x (s-x)\left[\left(u_0(s)\right)_{ss} - 2u^3{}_0(s)\right] ds$$
$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \frac{x^4}{4} - \frac{x^5}{20},$$
$$u_2(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \frac{x^4}{4} - \frac{x^5}{20} + \int_0^x (s-x)\left[\left(u_1(s)\right)_{ss} - 2u^3{}_1(s)\right] ds, \quad (2.176)$$
$$\vdots$$

Table 2.2 shows the resulting absolute error obtained by comparing the VIM, before and after differentiating using only two iterations, with the exact solution which is $\frac{1}{(1+x)}$. The method yields highly accurate numerical solution after differentiating the first order equation (2.167) into the second order equation (2.174).

x	1st order	2nd order
0	0.0	0.0
0.1	5.1×10^{-3}	3.7×10^{-3}
0.2	2.0×10^{-2}	1.1×10^{-2}
0.3	4.5×10^{-2}	1.7×10^{-2}
0.4	8.0×10^{-2}	2.2×10^{-2}
0.5	1.2×10^{-1}	2.6×10^{-2}
0.6	1.8×10^{-1}	3.0×10^{-2}
0.7	2.4×10^{-1}	3.6×10^{-2}
0.8	3.1×10^{-1}	4.7×10^{-2}
0.9	3.8×10^{-1}	6.1×10^{-2}
1.0	4.6×10^{-1}	8.0×10^{-2}

Table 2.2 Comparison between the error obtained using the 1st and 2nd order equation for Example 2.7 by VIM using two iterations.

Example 2.8 Consider the second order homogenous ordinary differential equation

$$u'' + u = 0, \qquad u'(0) = 1, \quad u(0) = 1. \quad (2.177)$$

Solution:
From (2.43) we find that $\lambda = s - x$. Therefore, the iteration formula is given by

$$u_{n+1} = u_n + \int_0^x (s-x)[(u_n)_{ss}(s) + u_n(s)]\, ds. \quad (2.178)$$

We can choose $u_0(x) = 1 + x$ by using the first two terms of McLaurin series and from the given initial conditions. Using $u_0(x) = 1 + x$ we have

$$u_0(x) = 1 + x,$$

$$u_1(x) = 1 + x + \int_0^x (s-x)\left[\left(u_0(s)\right)_{ss} + u_0(s)\right] ds = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!},$$

$$u_2(x) = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \int_0^x (s-x)\left[\left(u_1(s)\right)_{ss} + u_1(s)\right] ds$$

$$= 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!},$$

$\vdots$

$$u_n(x) = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right) + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right). \quad (2.179)$$

The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x). \quad (2.180)$$

The VIM solution in series form is basically the McLaurin series expansion of the exact solution to this IVP which is given by

$$u(x) = \cos x + \sin x. \quad (2.181)$$

The numerical results are summarized in Table 2.3 and illustrated in Figure 2.2. Similar, to our analysis as in Example 2.6; the error is very small in the vicinity of 0 and worsens away from it. Only two iterations were necessary to obtain accurate results. The error can be improved by taking more terms but this is at the expense of CPU time.

x	EXACT	VIM
0	1.0	0.0
0.1	1.094837582	1.0×10^{-9}
0.2	1.178735909	9.2×10^{-8}
0.3	1.250856696	1.1×10^{-6}
0.4	1.310479336	6.0×10^{-6}
0.5	1.357008100	2.3×10^{-5}
0.6	1.389978088	7.0×10^{-5}
0.7	1.409059874	1.8×10^{-4}
0.8	1.414062800	4.0×10^{-4}
0.9	1.404936878	8.2×10^{-4}
1.0	1.094837582	1.6×10^{-3}

Table 2.3 Error obtained using VIM with two iterations.

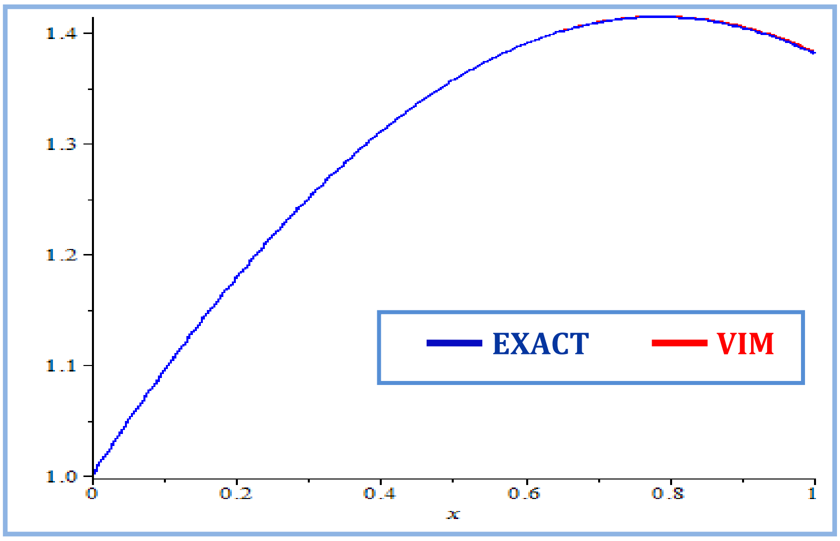

Figure 2.2 Comparison of the exact solution with VIM using two iterations.

Example 2.9 We will now apply the variational iteration method to solve the following third order linear homogeneous IVP:

$$u''' + u' = 0, \quad u(0) = 1, \; u'(0) = 0, \; u''(0) = 1. \tag{2.182}$$

Solution:

From (2.73) we find that $\lambda = -\frac{1}{2}(s - x)^2$. Therefore, the iteration formula is given by

$$u_{n+1} = u_n - \int_0^x \frac{1}{2}(s - x)^2 [(u_n)_{sss}(s) + (u_n)_s(s)]\, ds. \tag{2.183}$$

We can choose $u_0(x) = 1 + \frac{x^2}{2}$ from the given condition. Using $u_0(x) = 1 + \frac{x^2}{2}$ we have

$$\begin{aligned}
u_0(x) &= 1 + \frac{x^2}{2},\\
u_1(x) &= 1 + \frac{x^2}{2} + \int_0^x (s - x)\left[\left(u_0(s)\right)_{sss} + u'_0(s)\right] ds = 1 + \frac{x^2}{2!} + \frac{x^4}{4!},\\
u_2(x) &= 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} - \int_0^x (s - x)\left[\left(u_1(s)\right)_{ss} + u_1(s)\right] ds\\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!},\\
&\vdots\\
u_n(x) &= \left(1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots\right).
\end{aligned} \tag{2.184}$$

The latter VIM solution in series form is basically the McLaurin series expansion of the exact solution to this IVP which is given by

$$u(x) = \cosh x. \tag{2.185}$$

The numerical results are shown in Table 2.4 and Figure 2.3. We have similar observations as those in the previous two examples.

x	EXACT	VIM
0	1.0	0.0
0.1	1.005004168	0.0
0.2	1.020066756	0.0
0.3	1.045338514	2.0×10^{-9}
0.4	1.081072372	1.6×10^{-8}
0.5	1.127625965	9.7×10^{-8}
0.6	1.185465218	4.1×10^{-7}
0.7	1.255169006	1.4×10^{-6}
0.8	1.337434946	4.2×10^{-6}
0.9	1.433086385	1.1×10^{-5}
1.0	1.005004168	2.5×10^{-5}

Table 2.4 Error obtained using VIM with two iterations.

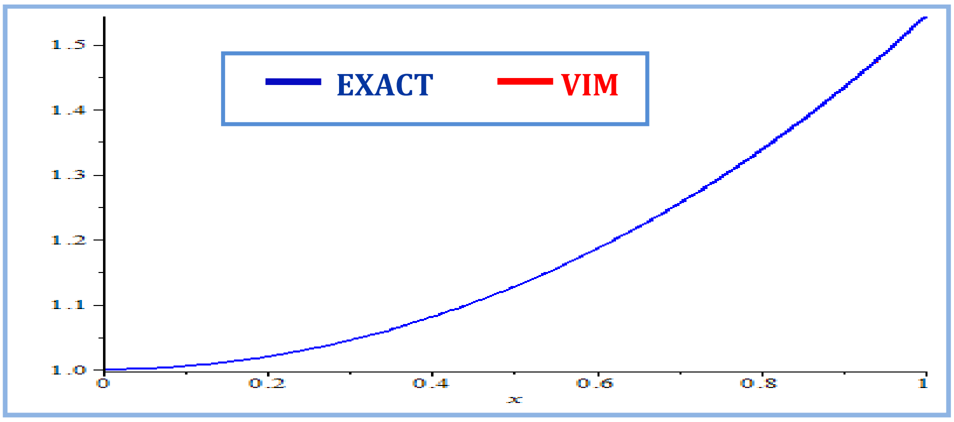

Figure 2.3 Comparison between the exact and VIM solution using two iterations.

2.5.2 Boundary Value Problems

In this section, we apply the variational iteration method for solving boundary value problems.

Consider a general differential equation given in operator form as:

$$Lu + Nu = g(x), \tag{2.186}$$

where L and N are linear and nonlinear operators, respectively, and $g(x)$ is an analytical function. According to VIM, we need to construct a correctional functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\big(Lu_n(s) + N\tilde{u}_n(s) - g(s)\big)ds, \qquad n \geq 0, \tag{2.187}$$

where λ(s) is a general Lagrange multiplier, which can be identified optimally via the variational theory, u_n is the nth approximate solution and $\tilde{u}_n$ is a restricted variation, which means $\delta\tilde{u}_n = 0$. In the following, we will present some examples to illustrate the power of the method in solving certain classes of boundary value problems.

Example 2.10 We will apply the variational iteration method to solve the second order linear differential equation

$$u'' + u + x = 0, \qquad 0 < x < 1, \tag{2.188}$$

subject to the boundary conditions

$$u(0) = u(1) = 0. \tag{2.189}$$

Solution:
From (2.53) we find that $\lambda = \sin(s - x)$. Therefore, the iteration formula is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x \sin(s - x)\,(u''_n(s) + u_n(s) + s)\,ds. \tag{2.190}$$

We can choose $u_0(x) = 0 + Ax$ that can be justified by substituting the given condition in the first two terms of the McLaurin series expansion. Here A is the value of $u'(0)$ which will ne given in the problem, however it will be easily found by applying the second boundary condition, namely $u(1) = 0$. Upon using $u_0(x) = 0 + Ax$ we have

$$\begin{aligned} &u_0(x) = Ax, \\ &u_1(x) = Ax + \int_0^x \sin(s - x)\left[\big(u_0(s)\big)_{ss} + u_0(s) + s\right]ds = A\sin x + \sin x - x. \end{aligned} \tag{2.191}$$

By imposing the second boundary conditions given in (2.189), this yields the value of $A = \frac{1}{\sin 1} - 1$, and so we have $u_1 = \frac{\sin x}{\sin 1} - x$.

Example 2.11 The VIM will be used to solve the fifth order nonlinear boundary value problem

$$u^{(5)} - e^{-x}u^2 = 0, \quad 0 < x < 1, \tag{2.192}$$

with the boundary conditions

$$u(0) = u'(0) = u''(0) = 1, \quad u(1) = u'(1) = e. \tag{2.193}$$

Solution:

From (2.81), we find that $\lambda = (-1)^5 \frac{1}{4!}(s-x)^4$. Therefore, the iteration formula is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \frac{1}{4!}(s-x)^4 \left(u^{(5)}{}_n(s) - e^{-x}u_n^2\right) ds. \tag{2.194}$$

As explained earlier, using the boundary conditions and McLaurin series expansion, one can select the start function to be $u_0(x) = 1 + x + \frac{1}{2}x^2 + \frac{A}{3!}x^3 + \frac{B}{4!}x^4$. Note that A, B are the values of $u'''(0)$ and $u^{(4)}(0)$, respectively which will be determined later from the resulting solution and the other unemployed boundary conditions.

Using $u_0(x) = 1 + x + \frac{1}{2}x^2 + \frac{A}{3!}x^3 + \frac{B}{4!}x^4$, we have

$$\begin{aligned} u_0(x) &= 1 + x + \frac{1}{2}x^2 + \frac{A}{3!}x^3 + \frac{B}{4!}x^4, \\ u_1(x) &= 1 + x + \frac{1}{2}x^2 + \frac{A}{3!}x^3 + \frac{B}{4!}x^4 - \int_0^x \frac{1}{4!}(s-x)^4 \left[\left(u_0(s)\right)_{sssss} - e^{-x}u_0^2\right] ds. \end{aligned} \tag{2.195}$$

By imposing the second set of boundary conditions (2.193), namely $u(1) = u'(1) = e,$ yields

$$A = 0.8582214341 \text{ and } B = 2.044641409. \tag{2.196}$$

The exact solution for this problem is given by

$$u(x) = e^x. \tag{2.197}$$

Table 2.5 and Figure 2.4 show the numerical solution that resulted from the VIM using only one iteration. The error is relatively very small which can be improved by adding more iterates. Note further that the absolute error is almost uniformly distributed within the domain.

x	EXACT	ABSOLUTE ERROR
0	1.0	0.0
0.1	1.105170918	7.1×10^{-5}
0.2	1.221402758	2.8×10^{-6}
0.3	1.349858808	1.6×10^{-4}
0.4	1.491824698	5.2×10^{-4}
0.5	1.648721271	8.2×10^{-4}
0.6	1.822118800	8.2×10^{-4}
0.7	2.013752707	9.5×10^{-4}
0.8	2.225540928	6.2×10^{-4}
0.9	2.459603111	1.9×10^{-4}
1.0	1.105170918	8.2×10^{-6}

Table 2.5 Error obtained from VIM using one iterate.

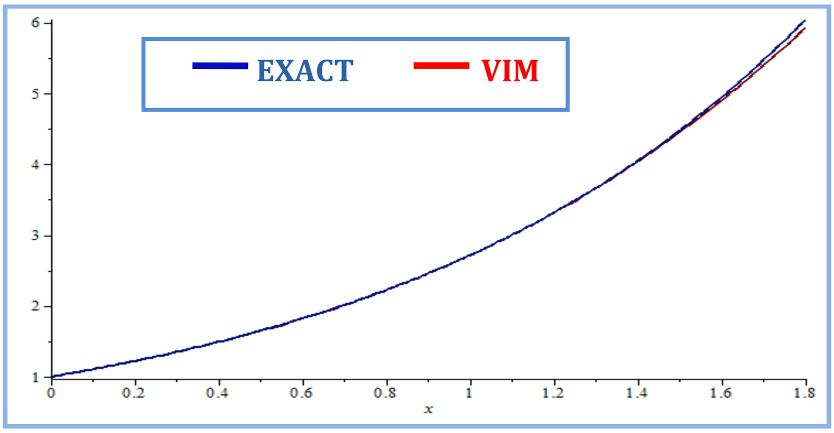

Figure 2.4 Comparison of the exact solution with VIM using the first iteration.

Example 2.12 The thin film flow of a third grade fluid down that includes a plane of inclination $\alpha \neq 0$, is governed by the following nonlinear boundary value problem

$$\frac{d^2u}{dy^2} + \frac{6(\beta_2 + \beta_3)}{\mu}\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} + \frac{\rho g \sin\alpha}{\mu} = 0, \tag{2.198}$$

with the boundary conditions

$$u(0) = 0, \qquad \frac{du}{dy} = 0 \quad at\ y = \delta. \tag{2.199}$$

Introduce the following parameters

$$y = \delta y^*, \qquad u = \frac{\delta^2 \rho g \sin\alpha}{\mu} u^*,$$

$$\beta^* = \frac{6\delta^2\rho^2 g^2 \sin^2\alpha}{\mu^3}(\beta_2 + \beta_3). \tag{2.200}$$

The exact solution for $\beta = 0$ is given by

$$u(y) = -\frac{1}{2}[(y-1)^2 - 1]. \tag{2.201}$$

Solution:

Using the parameters (2.200) we get

$$\frac{du}{dy} = \frac{du}{dy^*}\frac{dy^*}{dy} = \frac{du}{dy^*}\frac{1}{\delta} = \frac{d\left(\frac{\delta^2\rho g \sin\alpha}{\mu}u^*\right)}{dy^*}\frac{1}{\delta} = \frac{\delta\rho g \sin\alpha}{\mu}\frac{du^*}{dy^*} \tag{2.202}$$

and

$$\frac{d^2u}{dy^2} = \frac{d\left(\frac{du}{dy}\right)}{dy} = \frac{\delta\rho g \sin\alpha}{\mu}\frac{d^2u^*}{dy^{*2}}\frac{dy^*}{dy} = \frac{\rho g \sin\alpha}{\mu}\frac{d^2u^*}{dy^{*2}}. \tag{2.203}$$

Substituting (2.202) and (2.203) into equation (2.198), we get

$$\frac{\rho g \sin\alpha}{\mu}\frac{d^2u^*}{dy^{*2}} + \left[\frac{6(\beta_2+\beta_3)}{\mu}\frac{\delta^2\rho^2 g^2 \sin^2\alpha}{\mu^2}\left(\frac{du^*}{dy^*}\right)^2\right] + \frac{\rho g \sin\alpha}{\mu} = 0,$$

$$\frac{\rho g \sin\alpha}{\mu}\frac{d^2u^*}{dy^{*2}} + \frac{6(\beta_2+\beta_3)}{\mu^4}\delta^2\rho^3 g^3 \sin^3\alpha\left(\frac{du^*}{dy^*}\right)^2\frac{d^2u^*}{dy^{*2}} + \frac{\rho g \sin\alpha}{\mu} = 0,$$

$$\frac{d^2u^*}{dy^{*2}} + 6(\beta_2+\beta_3)\frac{\delta^2\rho^2 g^2 \sin^2\alpha}{\mu^3}\left(\frac{du^*}{dy^*}\right)^2\frac{d^2u^*}{dy^{*2}} + 1 = 0. \tag{2.204}$$

Using the parameters (2.200) we get

$$\frac{d^2u^*}{dy^{*2}} + 6\beta^*\left(\frac{du^*}{dy^*}\right)^2\frac{d^2u^*}{dy^{*2}} + 1 = 0, \tag{2.205}$$

with the boundary conditions

$$u(0) = 0, \qquad \frac{du}{dy} = 0 \quad at\ y = 1. \tag{2.206}$$

By integrating both sides of the Equation (2.205) and letting $u = u^*$, we get

$$\frac{du}{dy} + 2\beta\left(\frac{du}{dy}\right)^3 + y = C, \tag{2.207}$$

where C is a constant. Using the second condition of (2.206) in Equation (2.207) we get that $C = 1$. Hence, the system (2.207) can be written as

$$\frac{du}{dy} + 2\beta\left(\frac{du}{dy}\right)^3 + y - 1 = 0, \quad u(0) = 0. \tag{2.208}$$

Now, apply the VIM on Equation (2.208) and note that from (2.22), we have that $\lambda = -1$. Therefore, the iteration formula is given by

$$u_{n+1}(y) = u_n(y) - \int_0^y \left(\frac{du_n}{ds} + 2\beta\left(\frac{du_n}{ds}\right)^3 + s - 1\right) ds. \tag{2.209}$$

We can choose $u_0(x) = 0$ from the given initial condition. Using $u_0(x) = 0$ we have

$$\begin{aligned}
&u_0(x) = 0,\\
&u_1(y) = u_0(y) - \int_0^y \left(\frac{du_0}{ds} + 2\beta\left(\frac{du_0}{ds}\right)^3 + s - 1\right) ds = -\frac{1}{2}[(y-1)^2 - 1],\\
&u_2(y) = u_1(y) - \int_0^y \left(\frac{du_1}{ds} + 2\beta\left(\frac{du_1}{ds}\right)^3 + s - 1\right) ds\\
&\qquad = -\frac{1}{2}[(y-1)^2 - 1] + \frac{\beta}{2}[(y-1)^4 - 1],\\
&u_3(y) = u_2(y) - \int_0^y \left(\frac{du_2}{ds} + 2\beta\left(\frac{du_2}{ds}\right)^3 + s - 1\right) ds\\
&\qquad = -\frac{1}{2}[(y-1)^2 - 1] + \frac{\beta}{2}[(y-1)^4 - 1] - 2\beta^2[(y-1)^6 - 1]\\
&\qquad + 3\beta^3[(y-1)^8 - 1] - \frac{8\beta^4}{5}[(y-1)^{10} - 1],\\
&\vdots
\end{aligned} \tag{2.210}$$

It is clear that by setting $\beta = 0$ in the scheme (2.210), we recover the exact solution for the case of Newtonian fluid. Hence, the first approximation of the nonlinear system solved by the VIM gives the exact solution of this linear equation. Therefore, we can say that the VIM can be

2.5.3 Singular Boundary Value Problems

In this section, we will apply the variational iteration method for the numerical solution of the following class of singular boundary value problems of the form

$$u'' + \frac{\alpha}{x}u' = f(x, u), \tag{2.211}$$

with boundary conditions

$$u(0) = A(or\ u'(0) = B), \quad u(1) = C\ (or\ u'(1) = D). \tag{2.212}$$

The VIM employs the correction functional

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left[(u_n)_{ss} + \frac{\alpha}{s}(u_n)_s - \tilde{f}_n(u, u', u'')\right] ds, \qquad n \geq 0, \tag{2.213}$$

where $\tilde{f}_n$ is a restricted variation, $\left(\delta \tilde{f}_n = 0\right)$.

To find the value of $\lambda(s)$, start with taking the variation with respect to $u_n(x)$, which yields

$$\frac{\delta u_{n+1}}{\delta u_n} = 1 + \frac{\delta}{\delta u_n}\left(\int_0^x \lambda(s)\left[(u_n)_{ss} + \frac{\alpha}{s}(u_n)_s - \tilde{f}_n(u, u', u'')\right] ds\right), \tag{2.214}$$

which is equivalent to

$$\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda(s)\left[(u_n)_{ss} + \frac{\alpha}{s}(u_n)_s - \tilde{f}_n(u, u', u'')\right] ds\right). \tag{2.215}$$

Applying the variation to Eq. (2.215) gives

$$\delta u_{n+1} = \delta u_n + \delta\left(\int_0^x \lambda(s)\left((u_n)_{ss} + \frac{\alpha}{s}(u_n)_s\right) ds\right). \tag{2.216}$$

Evaluating the integral in Eq. (2.216) by parts we have

$$\begin{aligned}\int_0^x \lambda(s)(u_n)_{ss}(s)ds &= [\lambda(x)(u_n)_s(x) - \lambda(0)(u_n)_s(0)] \\ &-[\lambda'(x)(u_n)(x) - \lambda'(0)(u_n)(0)] + \int_0^x \lambda''(s)u_n(s)ds\end{aligned} \tag{2.217}$$

and

$$\int_0^x \lambda(s)\frac{\alpha}{s}(u_n)_s(s)ds = \frac{\alpha}{x}\lambda(x)u_n(x) - \int_0^x \frac{\alpha}{s}\lambda'(s)u_n(s)ds + \int_0^x \frac{\alpha}{s^2}\lambda(s)u_n(s)ds. \tag{2.218}$$

Substituting the integral into Eq.(2.216) by the value of the integral (2.218) we obtain

$$\delta u_{n+1} = \delta u_n + \delta[\lambda(x)(u_n)_s(x)] - \delta[\lambda'(x)(u_n)_s(x)] + \delta\left[\frac{\alpha}{x}\lambda(x)u_n(x)\right]$$
$$+ \delta\left(\int_0^x \lambda''(s)u_n(s)ds\right) - \delta\left(\int_0^x \frac{\alpha}{s}\lambda'(s)(s)u_n(s)ds\right)$$
$$+ \delta\left(\int_0^x \frac{\alpha}{s^2}\lambda(s)u_n(s)ds\right) = 0. \tag{2.219}$$

By simplifying Eq. (2.219) we get

$$\delta u_{n+1} = \left[1 - \lambda'(x) + \frac{\alpha}{x}\lambda(x)\right]\delta u_n + \delta[\lambda(x)(u_n)_s(x)] + \delta\left(\int_0^x \lambda''(s)u_n(s)ds\right)$$
$$- \delta\left(\int_0^x \left(\frac{\alpha}{s}\lambda'(s) - \frac{\alpha}{s^2}\lambda(s)\right)(s)ds\right) = 0. \tag{2.220}$$

So, the following stationary conditions are obtained

$$\lambda''(s) - \left(\frac{\alpha}{s}\lambda'(s) - \frac{\alpha}{s^2}\lambda(s)\right) = 0,$$
$$1 - \lambda'(s)|_{s=x} = 0, \tag{2.221}$$
$$\lambda(s)|_{s=x} = 0.$$

By solving (2.221) for $\lambda(s)$ we have

Case 1: For $\alpha = 0$

$$\lambda(s) = (s - x).$$

Case 2: For $\alpha = 1$, it becomes a cylindrical problem

$$\lambda(s) = \sin\left(\frac{s}{x}\right). \tag{2.223}$$

Case 3: For $\alpha = 2$, it becomes a spherical problem

$$\lambda(s) = \frac{s(s - x)}{x}. \tag{2.224}$$

Case 4: For general case $\alpha \geq 2$,

$$\lambda(s) = \frac{s(s^{\alpha-1} - x^{\alpha-1})}{(\alpha - 1)x^{\alpha-1}}. \tag{2.225}$$

Example 2.13 Consider the linear singular boundary value problem

$$u'' + \frac{1}{x}u' + u + \frac{5}{4} - \frac{x^2}{16} = 0, \tag{2.226}$$

with boundary conditions

$$u'(0) = 0, \quad u(1) = \frac{17}{16}. \tag{2.227}$$

The exact solution is $u(x) = 1 + \frac{x^2}{16}$.

Solution:

Since $\alpha = 1$, and according to (2.222) the iteration formula is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x \sin\left(\frac{s}{x}\right)\left[(u_n)_{ss}(s) + \frac{1}{s}(u_n)_s(s) + u_n + \frac{5}{4} - \frac{s^2}{16}\right] ds. \tag{2.228}$$

We can choose $u_0(x) = u(0) + u'(0)x = u(0) = A$, by using Taylor's expansion and the given conditions, taking into consideration that the value of A will be determined later using the boundary condition at $x = 1$. Using $u_0(x) = A$ we have

$$\begin{aligned} u_0(x) &= A, \\ u_1(x) &= Ax + \int_0^x s \ln\left(\frac{s}{x}\right)\left[(u_0)_{ss}(s) + \frac{1}{s}(u_0)_s(s) + u_0 + \frac{5}{4} - \frac{s^2}{16}\right] ds \\ &= A - \frac{1}{4}Ax^2 + \frac{5}{16}x^2 + \frac{1}{256}x^4. \end{aligned} \tag{2.229}$$

By imposing the second boundary condition in (2.227) yields $A = 0.9947916667.$ Thus we have

$$u_1 = 0.9947916667 + 0.0638020833x^2 + 0.003906250000x^4. \tag{2.230}$$

The error resulting from the VIM using one iteration is listed in Table 2.6. The error is uniformly distributed and can be improved by taking more iterates. This result shows the fast convergence of the VIM for this case. Figure 2.5 shows the numerical and exact solutions and they are almost compatible from the first iterate.

x	EXACT	ERROR
0	1.0	5.2×10^{-3}
0.1	1.000625000	5.2×10^{-3}
0.2	1.002500000	5.2×10^{-3}
0.3	1.005625000	5.1×10^{-3}
0.4	1.010000000	4.9×10^{-3}
0.5	1.015625000	4.6×10^{-3}
0.6	1.022500000	4.2×10^{-3}
0.7	1.030625000	3.6×10^{-3}
0.8	1.040000000	2.7×10^{-3}
0.9	1.050625000	1.6×10^{-3}
1.0	1.000625000	0.0

Table 2.6 Error obtained using VIM with one iteration.

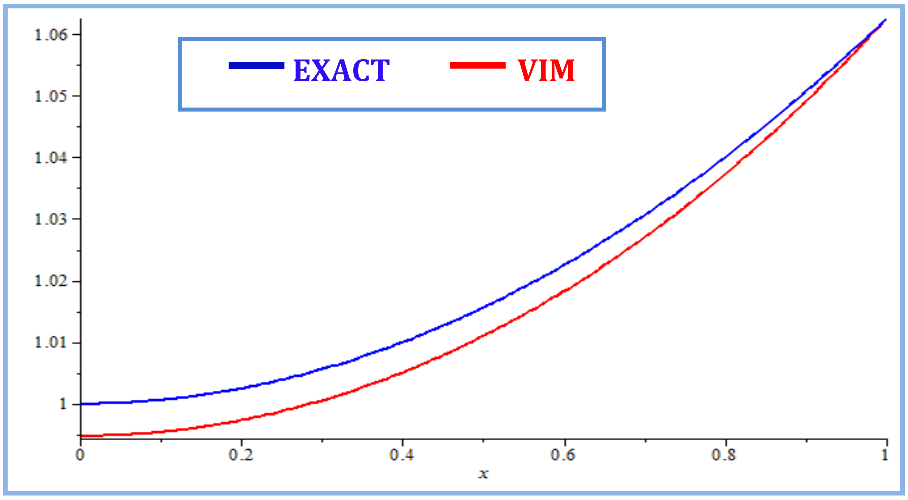

Figure 2.5 The exact solution versus the VIM solution using one iteration.

Example 2.14 Consider the linear singular boundary value problem

$$u'' + \frac{2}{x}u' + u^5 = 0, \tag{2.231}$$

with boundary conditions

$$u'(0) = 0, \quad u(1) = \frac{\sqrt{3}}{2}. \tag{2.232}$$

The exact solution is $u(x) = \frac{1}{\sqrt{1+\frac{x^2}{3}}}$.

Solution:

Since $\alpha = 2$, and according to (2.224) the iteration formula is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{s(s-x)}{x}\left[(u_n)_{ss}(s) + \frac{2}{s}(u_n)_s(s) + u^5{}_n\right] ds. \tag{2.233}$$

We can choose $u_0(x) = A$ from the given conditions. Using $u_0(x) = A$ we have

$$\begin{aligned}
u_0(x) &= A,\\
u_1(x) &= A + \int_0^x \frac{s(s-x)}{x}\left[(u_0)_{ss}(s) + \frac{2}{s}(u_0)_s(s) + u^5{}_0\right] ds\\
&= A - \frac{1}{6}A^5x^2,\\
u_2(x) &= Ax - \frac{1}{6}A^5x^2 + \int_0^x \frac{s(s-x)}{x}\left[(u_1)_{ss}(s) + \frac{2}{s}(u_1)_s(s) + u^5{}_1\right] ds.
\end{aligned} \tag{2.234}$$

By imposing the second boundary condition in (2.232) that is specified at $x = 1$ yields $A = 0.9936779905$. Thus, we have

$$\begin{aligned}
u_2 = 0.9936779905 &- 0.1614645186x^2 + 0.03935498878x^4\\
&- 0.006090345429x^6 + 0.0005772848378x^8\\
&- 0.00003069950679x^{10} + 7.034948337 \cdot 10^{-7}x^{12}.
\end{aligned} \tag{2.235}$$

The numerical results using two iterates of the VIM are given in Table 2.7 and illustrated in Figure 2.6. Obviously, the error is acceptable since we used only two steps of the method.

x	EXACT	ABSOLUTE ERROR
0	1.0	6.3×10^{-3}
0.1	0.9983374885	6.3×10^{-3}
0.2	0.9933992682	6.1×10^{-3}
0.3	0.9853292777	5.9×10^{-3}
0.4	0.9743547036	5.5×10^{-3}
0.5	0.9607689226	5.1×10^{-3}
0.6	0.9449111829	4.5×10^{-3}
0.7	0.9271455412	3.8×10^{-3}
0.8	0.9078412994	2.9×10^{-3}
0.9	0.8873565094	1.6×10^{-3}
1.0	0.8660254041	0

Table 2.7 Error obtained using variational method with two iterations.

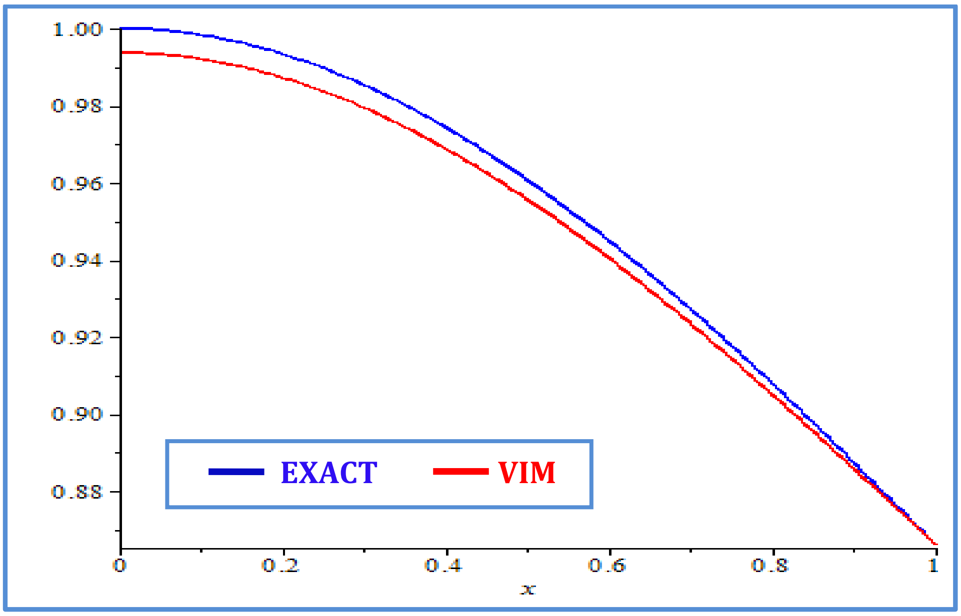

Figure 2.6 Comparison between the exact solution and second iterate of VIM.

2.5.4 System of Equations

In this section, we apply the variational iteration method (VIM) to solve a system of differential equations of first order. Since we can convert every ordinary differential equations of higher order into a system of differential equations of the first order, so this method can be used for solving higher systems as well. The method will be illustrated by discussing some examples.

Consider a system of ordinary differential equations of the first order with initial conditions of the form:

$$\begin{aligned} u'_1 &= f_1(x, u_1, u_2, \dots, u_n), \quad u_1(x_0) = u_1, \\ u'_2 &= f_2(x, u_1, u_2, \dots, u_n), \quad u_2(x_0) = u_2, \\ &\vdots \\ u'_n &= f_n(x, u_1, u_2, \dots, u_n), \quad u_n(x_0) = u_n, \end{aligned} \tag{2.236}$$

where each equation represents the first derivative of one of the unknown functions that depend on the independent variable x, and n unknown functions $f_1, f_2, \dots, f_n$.

Every ordinary differential equation of order n can be written as a system of n ordinary differential equation of order one. For example, consider an equation of the form

$$u^{(n)} = f\left(x, u, u', u'', \dots, u^{(n-1)}\right), \tag{2.237}$$

with initial conditions $u(x_0) = a, u'(x_0) = b, u''(x_0) = c, \dots, u^{(n-1)}(x_0) = d$.

Let $u(x) = u_1(x),\ u'(x) = u_2(x), \ldots, u^{(n-1)}(x) = u_n(x)$. Thus, we can rewrite Eq.(2.236) as follows:

$$\begin{aligned}
&u'_1 = u_2(x), \quad u_1(x_0) = u(x_0),\\
&u'_2 = u_3(x), \quad u_2(x_0) = u'(x_0),\\
&\vdots\\
&u'_n = f_n(x, u_1, u_2, \ldots, u_n), \quad u_n(x_0) = u^{(n-1)}(x_0),
\end{aligned} \tag{2.238}$$

where the system (2.238) is a system of differential equations of the first order.

Example 2.15 We will use the variational iteration method to solve the system of non-homogeneous differential equations:

$$\begin{aligned}
&u'_1 = u_3 - \cos x, \quad u_1(0) = 1,\\
&u'_2 = u_3 - e^x, \quad u_2(0) = 0,\\
&u'_3 = u_1 - u_2, \quad u_3(0) = 2.
\end{aligned} \tag{2.239}$$

Solution:

From (2.22) we find that $\lambda_1(s) = \lambda_2(s) = \lambda_3(s) = -1$. Therefore, the iteration formula is given by

$$\begin{aligned}
u_{1_{n+1}} &= u_{1_n} - \left(\int_0^x \left(u'_{1_n} - u_{3_n} + \cos s\right) ds\right),\\
u_{2_{n+1}} &= u_{2_n} - \left(\int_0^x \left(u'_{2_n} - u_{3_n} + e^s\right) ds\right),\\
u_{3_{n+1}} &= u_{3_n} - \left(\int_0^x \left(u'_{3_n} - u_{1_n} + u_{2_n}\right) ds\right).
\end{aligned} \tag{2.240}$$

We can choose $u_{1_0}(x) = 1, u_{2_0}(x) = 0, u_{3_0}(x) = 2$, from the given conditions. Using $u_{1_0}(x) = 1, u_{2_0}(x) = 0, u_{3_0}(x) = 2$ we have

$$\begin{aligned}
&u_{1_1}(x) = 1 + 2x - \sin x,\\
&u_{2_1}(x) = 1 + 2x - e^x,\\
&u_{3_1}(x) = 2 + x,\\
&u_{1_2}(x) = 1 + 2x - \sin x + \frac{1}{2}x^2,\\
&u_{2_2}(x) = 1 + 2x - e^x + \frac{1}{2}x^2,\\
&u_{3_2}(x) = \cos x + e^x.
\end{aligned} \tag{2.241}$$

If we do more iterates, we will get infinite series solutions which are namely either the exact solution or the McLaurin expansion of the exact solutions that will converge to it. It is worth noting that noise terms might errors as we do more iterates but eventually they will disappear as we pass to the limit. The iterates will converge to the exact solutions which are given by

$$\begin{aligned} u_1(x) &= e^x, \\ u_2(x) &= \sin x, \\ u_3(x) &= e^x + \cos x. \end{aligned} \tag{2.242}$$

Example 2.16 We now use the VIM to solve the following Euler–Lagrange equation of order three:

$$u''' + \frac{1}{x^2}u' - \frac{1}{x^3}u = 0, \tag{2.243}$$

with the initial conditions

$$u(1) = 1, \qquad u'(1) = 2, \quad u''(1) = 3. \tag{2.244}$$

Solution:

Let $u_1(x) = u(x)$, $u_2(x) = u'(x)$, $u_3(x) = u''(x)$. Thus, upon converting Eq.(2.243) into system of three differential equations of order one, we have

$$\begin{aligned} u'_1 &= u_2, & u_1(1) &= 1, \\ u'_2 &= u_3, & u_2(1) &= 2, \\ u'_3 &= -\frac{1}{x^2}u_2 + \frac{1}{x^3}u_1, & u_3(1) &= 3. \end{aligned} \tag{2.245}$$

From (2.22) we find that $\lambda_1(s) = \lambda_2(s) = \lambda_3(s) = -1$. Therefore, the iteration formula is given by

$$\begin{aligned} u_{1_{n+1}} &= u_{1_n} - \left(\int_1^x (u'_{1_n} - u_{2_n}) ds\right), \\ u_{2_{n+1}} &= u_{2_n} - \left(\int_1^x (u'_{2_n} - u_{3_n}) ds\right), \\ u_{3_{n+1}} &= u_{3_n} - \left(\int_1^x \left(u'_{3_n} + \frac{1}{s^2}u_{2_n} - \frac{1}{s^3}u_{1_n}\right) ds\right). \end{aligned} \tag{2.246}$$

We can choose $u_{1_0}(x) = 1, u_{2_0}(x) = 2, u_{3_0}(x) = 3$, from the given conditions. Using $u_{1_0}(x) = 1, u_{2_0}(x) = 2, u_{3_0}(x) = 3$, we have

$$\begin{aligned} u_{1_1}(x) &= -1 + 2x, \\ u_{2_1}(x) &= -1 + 3x, \\ u_{3_1}(x) &= \frac{3}{2} + \frac{2}{x} - \frac{1}{2x^2}. \end{aligned} \tag{2.247}$$

The higher iterates can be found in a similar fashion.

2.5.5 Domain Decomposition Method

In this section, we will use the domain decomposition approach to compliment the VIM, which will help to overcome the deterioration of the error for larger values of the independent variable. The VIM produces accurate error but locally, but the error worsens as we move away from the initial point. The domain decomposition will improve the accuracy for larger values and can make the error uniform across the domain. In order to implement this method, we subdivide our computational domain, as a union of sub-domains, and then solve the BVP or IVP on each of these sub-domains separately. The initial condition on the nth sub-domain can be obtain and approximates from the VIM solution obtained on the $(n-1)$th sub-domain. We will do an example to show the efficiency of the proposed method.

Example 2.17 We will apply the domain decomposition (DD) combined with the VIM on Example (2.6) which is given by:

$$u'(x) - u^2(x) = 1, \qquad u(0) = 0. \tag{2.248}$$

Solution:
To solve our example, and illustrate the DD approach, it suffices to subdivide the domain into two sub-domains, $[0, 0.5]$ and the second is $[0.5, 1]$. Applying the VIM on $[0, 0.5]$ first, then from equation (2.164) we can get the solution in series form and thus use it to estimate the value of the solution at $x = 0.5$, in particular, we get the following value:

$$u(0.5) = 0.5459573413. \tag{2.249}$$

This value is now used as the initial condition when applying the VIM on the sub-interval $[0.5, 1]$. From (2.22) we find that $\lambda = -1$. Therefore, the iteration formula is given by

$$u_{n+1}(x) = u_n(x) - \int_{0.5}^{x} [(u_n(s))_s - u_n^2(s) - 1]\, ds. \tag{2.250}$$

We now choose $u_0(x) = u(0.5) = 0.5459573413$. Using this value we get the following iterates:

$$u_0(x) = 0.5459573413,$$
$$u_1(x) = 0.5459573413 - \int_{0.5}^{x} \left[\left(u_0(s)\right)_s - u_0^2(s) - 1\right] ds =$$
$$-0.1030773677 + 1.298069418x,$$
$$u_2(x) = -0.1030773677 + 1.298069418x - \int_{0}^{x} \left[\left(u_1(s)\right)_s - u_1^2(s) - 1\right] ds$$
$$= 0.0038875885 + 1.010624944x - 0.1338015787x^2$$
$$+ 0.5616614046x^3,$$

$$
\begin{aligned}
u_3(x) &= 0.5462824390 + 1.298424503x - \int_0^x \left[(u_2(s))_s - u_2^2(s) - 1 \right] ds \\
&= -0.000556306464 + 1.000015113x + 0.0039288939x^2 \\
&+ 0.3401074828x^3 - 0.06651985228x^4 + 0.2306321827x^5 \\
&- 0.02505039421x^6 + 0.04506621906\,x^7.
\end{aligned}
\tag{2.251}
$$

The infinite series solution will clearly converge to the exact solution which is given by

$$
u(x) = \tan x. \tag{2.252}
$$

In Table 2.8 we compare the solution arising from VIM alone with that using VIM and DD combined. Clearly the accuracy will improve for larger values of x when we modify the VIM via using DD. Though the accuracy is slightly improved but that is because we subdivided the domain only into two subintervals. One has to subdivide the domain in a larger number of sub-domains in order to achieve uniform convergence.

x	EXACT	Error using VIM	Error using VIM and DD
0	0.0	0.0	0.0
0.1	0.100334672	3.9×10^{-9}	3.9×10^{-9}
0.2	0.202710036	5.0×10^{-7}	5.0×10^{-7}
0.3	0.309336250	8.8×10^{-6}	8.8×10^{-6}
0.4	0.422793219	6.9×10^{-5}	6.9×10^{-5}
0.5	0.546302490	3.5×10^{-4}	3.5×10^{-4}
0.6	0.684136808	1.3×10^{-3}	4.0×10^{-4}
0.7	0.842288380	4.2×10^{-3}	7.0×10^{-4}
0.8	1.029638557	1.2×10^{-2}	2.3×10^{-3}
0.9	1.260158218	3.1×10^{-2}	8.8×10^{-3}
1.0	1.557407725	7.5×10^{-2}	3.0×10^{-2}

Table 2.8 Comparison of the absolute errors obtained by VIM and those by DD and VIM using four iterations for both methods.

2.6 Partial Differential Equations

2.6.1 Initial Value Problems

In this section, we will apply the VIM method as presented before to some examples involving linear and nonlinear PDEs.

Example 2.18 We use the variational iteration method to solve the following homogeneous partial differential equation

$$u_x - u_t = 0, \quad u(0,t) = t, \quad u(x,0) = x. \tag{2.253}$$

Solution:
From (2.22) we find that $\lambda = -1$. Therefore, the iteration formula is given by

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^x \left[\frac{\partial u_n(s,t)}{\partial s} - \frac{\partial u_n(s,t)}{\partial t}\right] ds. \tag{2.254}$$

We can choose $u_0(x,t) = u(0,t) = t$ from the specified conditions. Using $u_0(x,t) = t$ we have

$$\begin{aligned}
&u_0(x,t) = t,\\
&u_1(x,t) = t - \int_0^x \left[\frac{\partial u_0(s,t)}{\partial s} - \frac{\partial u_0(s,t)}{\partial t}\right] ds = t + x,\\
&u_2(x,t) = t + x - \int_0^x \left[\frac{\partial u_1(s,t)}{\partial s} - \frac{\partial u_1(s,t)}{\partial t}\right] d = t + x,\\
&\vdots\\
&u_n(x,t) = t + x.
\end{aligned} \tag{2.255}$$

This gives the exact solution which is given by

$$u(x,t) = x + t. \tag{2.256}$$

Example 2.19 Use the variational iteration method to solve nonhomogeneous partial differential equation

$$u_x + u_t = x + t, \quad u(0,t) = 0, \quad u(x,0) = 0. \tag{2.257}$$

Solution:
From (2.22) we find that $\lambda = -1$. Therefore, the iteration formula is given by

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^x \left[\frac{\partial u_n(s,t)}{\partial s} + \frac{\partial u_n(s,t)}{\partial t} - s - t\right] ds. \tag{2.258}$$

We can choose $u_0(x,t) = u(0,t) = 0$ from the given conditions. Using $u_0(x,t) = 0$ we have

$$
\begin{aligned}
&u_0(x,t) = 0,\\
&u_1(x,t) = 0 - \int_0^x \left[\frac{\partial u_0(s,t)}{\partial s} + \frac{\partial u_0(s,t)}{\partial t} - s - t\right] ds = \frac{1}{2}x^2 + xt,\\
&u_2(x,t) = \frac{1}{2}x^2 + xt - \int_0^x \left[\frac{\partial u_1(s,t)}{\partial s} + \frac{\partial u_1(s,t)}{\partial t} - s - t\right] ds = xt,\\
&\vdots\\
&u_n(x,t) = xt.
\end{aligned}
\tag{2.259}
$$

Note that in the first iterate $u_1(x,t)$ we got the term $\frac{1}{2}x^2$ which we refer to as noise term. This term will disappear or cancels as we take higher iterates. In the limit, the iterates converge to the exact solution which is given by

$$u(x,t) = xt. \tag{2.260}$$

2.6.2 Boundary Value Problems

We will now apply the VIM method as presented before to some examples involving linear and nonlinear PDEs which are complimented with boundary conditions.

Example 2.20 Use the variational iteration method to solve the boundary value problem

$$u_{xx} + u_{tt} = 0, \qquad 0 < x,t < \pi, \tag{2.261}$$

with the boundary conditions

$$
\begin{aligned}
&u(0,t) = 0, \quad u(\pi,t) = \sinh\pi \sin t\\
&u(x,0) = 0, \quad u(x,\pi) = 0.
\end{aligned}
\tag{2.262}
$$

Solution:
From (2.43) we find that $\lambda = s - x$. Therefore, the iteration formula is given by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x (s-x)\left[\frac{\partial^2 u_n(s,t)}{\partial s^2} + \frac{\partial^2 u_n(s,t)}{\partial t^2}\right] ds. \tag{2.263}$$

From the boundary conditions we can see that the solution contains sin t with a function that depends on x. So, we can choose $u_0(x,t) = (0 + x)\sin t$ from the given condition. Using $u_0(x,t) = x\sin t$ we have

$$u_0(x,t) = x\sin t,$$

$$u_1(x,t) = x\sin t + \int_0^x (s-x)\left[\frac{\partial^2 u_0(s,t)}{\partial s^2} + \frac{\partial^2 u_0(s,t)}{\partial t^2}\right] ds$$
$$= x\sin t + \frac{1}{3!}x^3 \sin t,$$

$$u_2(x,t) = x\sin t + \frac{1}{3!}x^3\sin t + \int_0^x (s-x)\left[\frac{\partial^2 u_0(s,t)}{\partial s^2} + \frac{\partial^2 u_0(s,t)}{\partial t^2}\right] ds$$
$$= x\sin t + \frac{1}{3!}x^3\sin t + \frac{1}{5!}x^5 \sin t,$$

$$\vdots \tag{2.264}$$
$$u_n(x,t) = \sin t\left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right).$$

Clearly the iterates will converge to the exact solution which is given by

$$u(x,t) = \sin t \sinh x. \tag{2.265}$$

Example 2.21 We use the variational iteration method to solve the boundary value problem

$$u_{xx} + u_{tt} = 0, \qquad 0 < x, t < \pi \tag{2.266}$$

with the boundary conditions

$$u(0,t) = 0, \quad u(\pi,t) = 0,$$
$$u(x,0) = \cos x, \quad u(x,\pi) = \cosh\pi\ \cos x. \tag{2.267}$$

Solution:
From (2.43) we find that $\lambda = s - t$. Therefore, the iteration formula is given by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\left[\frac{\partial^2 u_n(x,s)}{\partial s^2} + \frac{\partial^2 u_n(x,s)}{\partial t^2}\right] ds. \tag{2.268}$$

From the boundary conditions we can see that the solution includes $\cos x$ with a function that depends on t. So, we can choose $u_0(x,t) = \left(1 + \frac{t^2}{2}\right)\cos x$ from the given condition. Using $u_0(x,t) = \cos x + \frac{t^2}{2}\cos x$, we have

$$u_0(x,t) = \cos x + \frac{t^2}{2}\cos x,$$
$$u_1(x,t) = \cos x + \frac{t^2}{2}\cos x + \int_0^t (s-t)\left[\frac{\partial^2 u_0(x,s)}{\partial s^2} + \frac{\partial^2 u_0(x,s)}{\partial x^2}\right] ds$$
$$= \cos x + \frac{t^2}{2}\cos x + \frac{t^4}{4!}\cos x + \frac{t^6}{6!}\cos x,$$

$$u_2(x,t) = \cos x + \frac{t^2}{2}\cos x + \frac{t^4}{4!}\cos x + \frac{t^6}{6!}\cos x$$
$$+ \int_0^t (s-t)\left[\frac{\partial^2 u_0(t,s)}{\partial s^2} + \frac{\partial^2 u_0(t,s)}{\partial x^2}\right] ds$$
$$= \cos x + \frac{t^2}{2}\cos x + \frac{t^4}{4!}\cos x + \frac{t^6}{6!}\cos x + \frac{1}{8!}t^8 \cos x$$
$$+ \frac{1}{10!}t^{10}\cos x,$$

$$\vdots$$
$$u_n(x,t) = \cos x\left(1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \frac{1}{6!}t^6 + \frac{1}{8!}t^8 + \frac{1}{10!}t^{10} \ldots\right). \quad (2.269)$$

This gives the exact solution by

$$u(x,t) = \cos x \cosh t. \quad (2.270)$$

2.6.3 System of Equations

We will apply the variational iteration method to solve systems of partial differential equations. The method will be illustrated by discussing some examples.

Consider a system of differential equations written in an operator form as

$$\begin{aligned} L_t u + R_1(u,v,w) + N_1(u,v,w) &= g_1, \\ L_t u + R_2(u,v,w) + N_2(u,v,w) &= g_2, \\ L_t u + R_3(u,v,w) + N_3(u,v,w) &= g_3, \end{aligned} \quad (2.271)$$

with initial conditions

$$\begin{aligned} u(x,0) &= f_1(x), \\ v(x,0) &= f_2(x), \\ w(x,0) &= f_3(x), \end{aligned} \quad (2.272)$$

where L_t is a first order partial differential operator, R_1, R_2 and R_3 are linear operators, N_1, N_2 and N_3 are nonlinear operators and g_1, g_2 and g_3 are source terms.
The VIM employs the correction functional as follows:

$$u_{n+1}(x,t) = u_n(x,t)$$
$$+ \int_0^t \lambda_1 \left(Lu_n(x,s) + R_1(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n) + N_1(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n) - g_1(s)\right) ds,$$
$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2 \left(Lv_n(x,s) + R_2(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n) + N_2(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n) - g_2(s)\right) ds,$$

$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \lambda_3\,(Lw_n(x,s) + R_3(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n) + N_3(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n) - g_3(s))ds, \tag{2.273}$$

where $\lambda_1, \lambda_2, \lambda_3$ are Lagrange's multipliers, $\tilde{u}_n, \tilde{v}_n, \tilde{w}_n$ as restricted variations $(\delta\tilde{u}_n = \delta\tilde{v}_n = \delta\tilde{w}_n = 0)$.
The solutions are given by

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t)\,, \\ v(x,t) &= \lim_{n\to\infty} v_n(x,t), \\ w(x,t) &= \lim_{n\to\infty} w_n(x,t). \end{aligned} \tag{2.274}$$

Example 2.22 We use the variational iteration method to solve the inhomogeneous nonlinear system

$$\begin{aligned} u_t + vu_x + u &= 1, \\ u_t - uv_x - v &= 1, \end{aligned} \tag{2.275}$$

with initial conditions

$$u(x,0) = e^x, \;\; v(x,0) = e^{-x}. \tag{2.276}$$

Solution:
From (2.22) we find that $\lambda_1 = \lambda_2 = -1$. Therefore, the iteration formula is given by

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) - \int_0^t \left[\frac{\partial u_n(x,s)}{\partial s} + v_n(x,s)\frac{\partial u_n(x,s)}{\partial x} + u_n(x,s) - 1\right] ds, \\ v_{n+1}(x,t) &= v_n(x,t) - \int_0^t \left[\frac{\partial v_n(x,s)}{\partial s} - u_n(x,s)\frac{\partial v_n(x,s)}{\partial x} - v_n(x,s) - 1\right] ds. \end{aligned} \tag{2.277}$$

We can choose $u_0(x,t) = e^x$ and $v_0(x,t) = e^{-x}$ from the given condition. Using $u_0(x,t) = e^x$ and $v_0(x,t) = e^{-x}$ we have

$$\begin{aligned} u_0(x,t) &= e^x\,, \qquad\qquad v_0(x,t) = e^{-x} \\ u_1(x,t) &= e^x - \int_0^t \left[\frac{\partial u_0(x,s)}{\partial s} + v_0(x,s)\frac{\partial u_0(x,s)}{\partial x} + u_0(x,s) - 1\right] ds \\ &= e^x - te^x = e^x(1-t), \\ v_1(x,t) &= e^{-x} - \int_0^t \left[\frac{\partial v_0(x,s)}{\partial s} - u_0(x,s)\frac{\partial v_0(x,s)}{\partial x} - v_0(x,s) - 1\right] \\ &= e^{-x} + te^{-x} = e^{-x}(1+t), \\ &\vdots \end{aligned} \tag{2.278}$$

Upon taking more iterates we will easily observe that the series solutions are converging to the exact solutions given by

$$u(x,t) = e^{x-t},$$
$$v(x,t) = e^{-x+t}. \tag{2.279}$$

Example 2.23 The variational iteration method is now applied to solve the nonlinear system

$$\begin{aligned} u_t - v_x w_y &= 1, \\ u_t - w_x v_y &= 5, \\ u_t - u_x v_y &= 5, \end{aligned} \tag{2.280}$$

with initial conditions

$$u(x,y,0) = x + 2y, \quad v(x,y,0) = x - 2y, \quad w(x,y,0) = -x + 2y. \tag{2.281}$$

Solution:

From (2.22) we find that $\lambda_1 = \lambda_2 = \lambda_3 = -1$. Therefore, the iteration formula is given by

$$\begin{aligned} u_{n+1}(x,y,t) &= u_n(x,y,t) \\ &\quad - \int_0^t \left[\frac{\partial u_n(x,y,s)}{\partial s} - \frac{\partial v_n(x,y,s)}{\partial x} \frac{\partial w_n(x,y,s)}{\partial y} - 1 \right] ds, \\ v_{n+1}(x,y,t) &= v_n(x,y,t) \\ &\quad - \int_0^t \left[\frac{\partial v_n(x,y,s)}{\partial s} - \frac{\partial w_n(x,y,s)}{\partial x} \frac{\partial u_n(x,y,s)}{\partial y} - 5 \right] ds, \\ w_{n+1}(x,y,t) &= w_n(x,y,t) \\ &\quad - \int_0^t \left[\frac{\partial w_n(x,y,s)}{\partial s} - \frac{\partial u_n(x,y,s)}{\partial x} \frac{\partial v_n(x,y,s)}{\partial y} - 5 \right] ds. \end{aligned} \tag{2.282}$$

We can choose $u_0(x,y,t) = x + 2y$, $v_0(x,y,t) = x - 2y$ and $w_0(x,y,t) = -x + 2y$ from the given conditions. Using these choices we get the higher iterates:

$$u_0(x,t) = x + 2y\,, \qquad v_0(x,t) = x - 2y, \qquad w_0(x,y,t) = -x + 2y,$$

$$\begin{aligned} u_1(x,t) &= x + 2y - \int_0^t \left[\frac{\partial u_0(x,y,s)}{\partial s} - \frac{\partial v_0(x,y,s)}{\partial x} \frac{\partial w_0(x,y,s)}{\partial y} - 1 \right] \\ &= x + 2y + 3t, \\ v_1(x,t) &= x - 2y - \int_0^t \left[\frac{\partial v_0(x,y,s)}{\partial s} - \frac{\partial w_0(x,y,s)}{\partial x} \frac{\partial u_0(x,y,s)}{\partial y} - 5 \right] ds \\ &= x - 2y + 3t, \end{aligned}$$

$$w_1(x,y,t) = -x + 2y - \int_0^t \left[\frac{\partial w_0(x,y,s)}{\partial s} - \frac{\partial u_0(x,y,s)}{\partial x}\frac{\partial v_0(x,y,s)}{\partial y} - 5\right] ds$$
$$= -x + 2y + 3t \quad (2.283)$$

$\vdots$

These iterate will converge to the exact solutions given by

$$u(x,t) = x + 2y + 3t,$$
$$v(x,t) = x - 2y + 3t,$$
$$w(x,t) = -x + 2y + 3t. \quad (2.284)$$

2.7 Integro-Differential Equations

In this section, we will handle integro-differential equations. Recall that an integro-differential equation is an equation that contains $u^{(i)}(x)$, which is the ith derivative of $u(x)$, and an unknown function $u(x)$ that appears under an integral sign. A standard integro-differential equation is of the form:

$$u^{(i)}(x) = f(x) + \int_{g(x)}^{h(x)} K(x,t) F\big(u(t)\big) dt, \quad (2.285)$$

where $F\big(u(x)\big)$is a nonlinear function of $u(x)$, $g(x)$ and $h(x)$ are the limits of the integral, $K(x,t)$ is a function of two variables x and t called the kernel or the nucleus of the equation. We have to mention that the limits of integration $g(x)$ and $h(x)$ can be variables, constants, or mixed.

The correction functional for the nonlinear integro-differential equation (2.285) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left(u_n{}^{(i)}(s) - f(s) - \int_{g(s)}^{h(s)} K(s,t) F\big(u_n(t)\big) dt\right) ds, \quad (2.286)$$

where λ is Lagrange's multiplier, $\tilde{u}_n$ as restricted variation ($\delta\tilde{u}_n = 0$). The zeroth approximation u_n can be any selective function and we showed before how to find it.

Example 2.23 We use the variational iteration method to solve the nonlinear Volterra integro-differential equation

$$u'(x) = 1 + e^x - 2xe^x - e^{2x} + \int_0^x e^{x-t} u^2(t) dt, \qquad u(0) = 2. \quad (2.287)$$

Solution:

From (2.22) we find that $\lambda = -1$. Therefore, the iteration formula is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u'_n(s) - 1 - e^s + 2se^s + e^{2s} - \int_0^s e^{s-r} u_n^2(r)dr \right) ds. \tag{2.288}$$

We can choose $u_0(x) = 2$ from the given initial condition. Using this value we have

$$\begin{aligned}
u_0(x) &= 2,\\
u_1(x) &= 2 - \int_0^x \left(u'_0(s) - 1 - e^s + 2se^s + e^{2s} - \int_0^s e^{s-r} u_0^2(r)dr \right) ds\\
&= 2 + x + \frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{3}{8}x^4 - \frac{19}{120}x^5,\\
u_2(x) &= 2 + x + \frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{3}{8}x^4 - \frac{19}{120}x^5\\
&\quad - \int_0^x \left(u'_1(s) - 1 - e^s + 2se^s + e^{2s} - \int_0^s e^{s-r} u_1^2(r)dr \right) ds\\
&= 2 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{8}x^5.
\end{aligned} \tag{2.289}$$

Clearly the series solution is the Taylor's series expansion of the exact solution which is given by

$$u(x) = 1 + e^x. \tag{2.290}$$

Example 2.24 The variational iteration method will be used to solve the nonlinear Fredholm integro-differential equation

$$u'(x) = \cos x - \frac{\pi}{48}x + \frac{1}{24}\int_0^\pi xu^2(t)dt, \quad u(0) = 0. \tag{2.291}$$

Solution:
From (2.22) we find that $\lambda = -1$. Therefore, the iteration formula is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u'_n(s) - \cos s + \frac{\pi}{48}s - \frac{1}{24}\int_0^\pi s u_n^2(r)dr \right) ds. \tag{2.292}$$

We can choose $u_0(x) = 0$ from the given initial condition. Using $u_0(x) = 0$ we have

$$\begin{aligned}
u_0(x) &= 0,\\
u_1(x) &= 0 - \int_0^x \left(u'_0(s) - cos\, s + \frac{\pi}{48}s - \frac{1}{24}\int_0^\pi s u_0^2(r)dr \right) ds\\
&= sin\, x - 0.03272x^2,
\end{aligned}$$

$$u_2(x) = sin\,x - 0.03272x^2 - \int_0^x \left(u'_1(s) - cos\,s + \frac{\pi}{48}s - \frac{1}{24}\int_0^{\pi} su_1^2(r)dr\right)ds$$
$$= sin\,x - 0.00664x^2,$$
$$u_3(x) = sin\,x - 0.00664x^2 - \int_0^x \left(u'_2(s) - cos\,s + \frac{\pi}{48}s - \frac{1}{24}\int_0^{\pi} su_2^2(r)dr\right)ds \qquad (2.293)$$
$$= sin\,x - 0.001567x^2.$$

Consequently, the solution is given by

$$\lim_{n\to\infty} u_n(x). \qquad (2.294)$$

There are some noise term appearing in the iterates and in the limit they will converge to zero and hence we obtain the exact solution which is clearly

$$u(x) = \sin x. \qquad (2.295)$$

2.8 Integral Equations

2.8.1 Volterra Integral Equations

In this section, we apply the variational iteration method for Volterra type of Integral equations. We can solve the Volterra integral equations in two ways: the first one is by converting the Volterra integral equation to an equivalent integro-differential equation by differentiating both sides of the equation and solve it as in section 2.7, and the second is by converting the Volterra integral equation to an initial value problem and then solve it easily.

A standard Volterra integral equation in $u(x)$ is of the form:

$$u(x) = f(x) + \int_0^x K(x,t)F\big(u(t)\big)dt, \qquad (2.296)$$

Where $F\big(u(x)\big)$ is a nonlinear function of $u(x)$, 0 and x are the limits of the integral, and $K(x,t)$ is a function of two variables x and t called the kernel or the nucleus of the integral equation.

Example 2.25 We will solve the Volterra integral equation by using the variational iteration method

$$u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt. \tag{2.297}$$

Solution:

We can solve this problem by converting this Volterra integral equation to an integro-differntial equation or by converting it to an initial value problem. To do that, differentiate both sides of (2.297) three times with respect to x gives the following two integro-differential equations:

$$u'(x) = 1 + x + \int_0^x (x-t)u(t)dt, \quad u(0) = 1,$$
$$u''(x) = 1 + \int_0^x u(t)dt, \quad u(0) = 1, \; u'(0) = 1, \tag{2.298}$$

and an initial value problem given by

$$u'''(x) = u(x), \qquad u(0) = 1, \qquad u'(0) = 1, \qquad u''(0) = 1. \tag{2.299}$$

Then we can easily solve each equation using the variational iteration method as we mentioned before in the previous sections.

2.8.2 Fredholm Integral Equations

Now, we will apply the variational iteration method to handle Fredholm integral equations.

Consider the standard Fredholm integral equation given by

$$u(x) = f(x) + \int_a^b K(x,t)F\big(u(t)\big)dt, \tag{2.300}$$

where $F\big(u(x)\big)$ is a nonlinear function of $u(x)$, a and b are constants and are the limits of the integral, λ is a constant parameter, and $K(x,t)$ is a function of two variables x and t called the kernel or the nucleus of the integral equation.

Note that $K(x,t)$ is separable and can be written in the form $K(x,t) = g(x)h(t)$. Thus, equation (2.300) can be written as

$$u(x) = f(x) + g(x)\int_a^b h(t)F\big(u(t)\big)dt. \tag{2.301}$$

To solve the Fredholm integral equations we should convert the equation to an equivalent integro-differential equation by differentiating both sides of the equation. In the following we will study the case where $g(x) = x^n$.

Example 2.25 We solve the following Fredholm integral equation by using the variational iteration method

$$u(x) = e^x - x + x\int_0^1 tu(t)dt. \tag{2.302}$$

Solution:

First, we have to convert the Fredholm integral equation to an integro-differential equation by differentiating both sides of the equation (2.302) with respect to x. We have

$$u'(x) = e^x - 1 + \int_0^1 tu(t)dt, \quad (2.303$$

with initial condition $u(0) = 1$.

Recall that the integral at the right side represents a constant value. Now, we can easily solve this integro-differential equation. From (2.22) we find that $\lambda = -1$. Therefore, the iteration formula is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u'_n(s) - e^s + 1 - \int_0^1 tu_n(t)dt\right)ds. \tag{2.304}$$

We can choose $u_0(x) = 1$ from the resulting initial condition. Using $u_0(x) = 1$ we have

$$\begin{aligned}
&u_0(x) = 1,\\
&u_1(x) = 1 - \int_0^x \left(u'_0(s) - e^s + 1 - \int_0^1 tu_0(t)dt\right)ds = e^x - \frac{1}{2}x,\\
&u_2(x) = e^x - \frac{1}{2}x - \int_0^x \left(u'_1(s) - e^s + 1 - \int_0^1 tu_1(t)dt\right)ds = e^x - \frac{1}{6}x,\\
&u_3(x) = e^x - \frac{1}{6}x - \int_0^x \left(u'_2(s) - e^s + 1 - \int_0^1 tu_2(t)dt\right)ds = e^x - \frac{1}{18}x,\\
&\vdots
\end{aligned} \tag{2.305}$$

Note that there are noise terms in the iterates which will disappear as we pass to the limit. Thus, The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x) = e^x, \tag{2.306}$$

which is the exact solution of the problem.

2.9 Calculus of Variations

Finally, in this last section we will handle some problems in calculus of variations. We will apply the VIM for solving Euler-Lagrange equations which arises in calculus of variations problems, more precisely when dealing with maximizing or minimizing a given functional.

Consider the general form of the variational problem:

$$v[u_1, u_2, \dots, u_n] = \int_{x_0}^{x_1} F(x, u_1, u_2, \dots, u_n, u'_1, u'_2, \dots, u'_n) dx, \tag{2.307}$$

with the boundary conditions

$$\begin{aligned} u_1(x_0) &= a_1, \quad u_2(x_0) = a_2, \quad \dots, \quad u_n(x_0) = a_n, \\ u_1(x_1) &= b_1, \quad u_2(x_1) = b_2, \quad \dots, \quad u_n(x_1) = b_n. \end{aligned} \tag{2.308}$$

We indent to maximize the functional to equation (2.306): the solution satisfies the Euler-Lagrange equation of the form

$$F_{u_k} - \frac{d}{dx} F_{u'_k} = 0, \quad k = 1,2, \dots, n, \tag{2.309}$$

with the same boundary conditions (2.308).

In the following examples, the variational iteration method for solving such kinds of problems will be studied.

Example 2.26 We will use the variational iteration method to solve the calculus of variation problem

$$\min v = \int_0^1 \frac{1 + u^2(x)}{u'^2(x)} dx, \tag{2.310}$$

with the boundary conditions

$$u(0) = 0, u(1) = 0.5. \tag{2.311}$$

The exact solution is given by

$$u(x) = \sinh(0.481218250x). \tag{2.312}$$

Solution:

The function which minimizes the integral satisfies the Euler-Lagrange equation which is given by

$$u'' + u''u^2 - uu'^2 = 0, \tag{2.313}$$

with boundary conditions

$$u(0) = 0, \qquad u(1) = 0.5. \tag{2.314}$$

From (2.43) we find that $\lambda = s - x$. Therefore, the iteration formula is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\left(u''_n(s) + u_n^2(s)u''_n(s) - u'^2_n(s)u_n(s)\right) ds. \tag{2.315}$$

We can choose $u_0(x) = 0 + Ax$ from the given initial condition at $x = 0$. Using $u_0(x) = Ax$ we have

$$\begin{aligned}
u_0(x) &= Ax, \\
u_1(x) &= Ax + \int_0^x (s-x)\left(u''_0(s) + u_0^2(s)u''_0(s) - u'^2_0(s)u_0(s)\right) ds \\
&= Ax - \frac{1}{2}A^3x^3, \\
u_2(x) &= Ax - \frac{1}{2}A^3x^3 + \int_0^x (s-x)\left(u''_1(s) + u_1^2(s)u''_1(s) - u'^2_1(s)u_1(s)\right) ds \\
&= Ax + \frac{1}{6}A^3x^3 - \frac{1}{192}A^9x^9 - \frac{1}{40}A^5x^5 + \frac{1}{56}A^7x^7.
\end{aligned} \tag{2.316}$$

By imposing the boundary conditions in u_2 at $x = 1$, we get

$$A = 0.4818977464. \tag{2.317}$$

The error resulting from the VIM using two iterations is listed in Table 2.9. The error is uniformly distributed and can be improved by taking more iterates. This result shows the fast convergence of the VIM for this case. Figure 2.7 shows the numerical and exact solutions and they are almost well-matched starting from the second iterate.

x	EXACT	ERROR
0	0.0	0.0
0.1	0.04813975661	6.9×10^{-5}
0.2	0.09639100946	1.4×10^{-4}
0.3	0.1448655131	2.1×10^{-4}
0.4	0.1936755390	2.7×10^{-4}
0.5	0.2429341358	3.3×10^{-4}
0.6	0.2927553913	3.7×10^{-4}
0.7	0.3432546960	3.7×10^{-4}
0.8	0.3945490113	3.3×10^{-4}
0.9	0.4467571396	2.2×10^{-4}
1.0	0.5	1.0×10^{-10}

Table 2.9 Error obtained using variational method with two iterations.

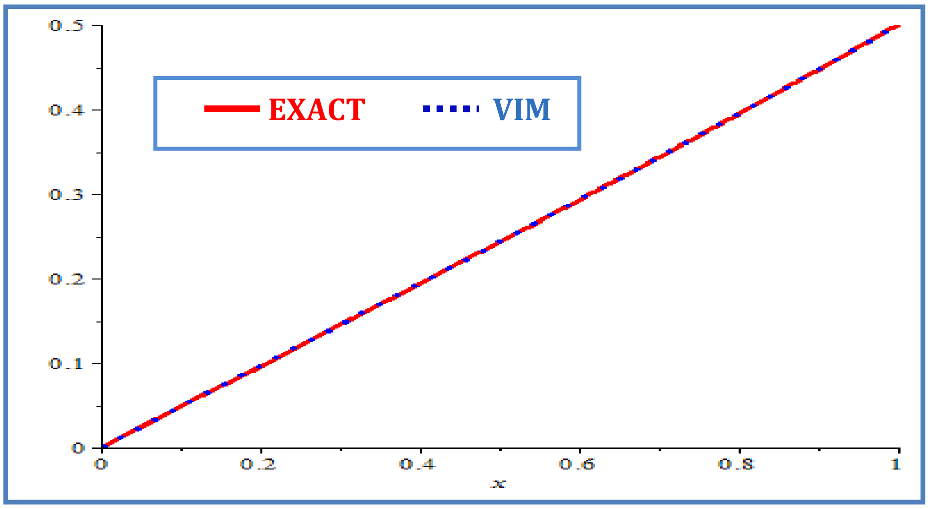

Figure 2.7 Error obtained using variational method with two iterations.

Example 2.27 We use the variational iteration method to solve the calculus of variation problem

$$v[u(x), z(x)] = \int_0^{\pi/2} \left[u'^2(x) + z'^2(x) + 2u(x)z(x)\right] dx, \tag{2.318}$$

with the boundary conditions

$$\begin{aligned} u(0) &= 0, u\left(\frac{\pi}{2}\right) = 1, \\ z(0) &= 0, z\left(\frac{\pi}{2}\right) = -1. \end{aligned} \tag{2.319}$$

Solution:

The Euler-Lagrange equations of this problem are given by

$$\begin{aligned} u'' - z &= 0, \\ z'' - u &= 0, \end{aligned} \tag{2.320}$$

with boundary conditions

$$\begin{aligned} u(0) &= 0, \quad u\left(\frac{\pi}{2}\right) = 1, \\ z(0) &= 0, \quad z\left(\frac{\pi}{2}\right) = -1. \end{aligned} \tag{2.321}$$

Similar to Example 2.26 we find that $\lambda_1 = \lambda_2 = s - x$. Therefore, the iteration formula is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\big(u''_n(s) - z_n(s)\big)\, ds,$$
$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\big(z''_n(s) - u_n(s)\big)\, ds. \tag{2.322}$$

We can choose $u_0(x) = 0 + Ax$ and $z_0(x) = 0 + Bx$ from the given conditions. Using $u_0(x) = Ax$ and $z_0(x) = Bx$ we have

$$u_0(x) = Ax, \qquad z_0(x) = Bx,$$
$$u_1(x) = Ax + \int_0^x (s-x)\big(u''_0(s) - z_0(s)\big)\, ds = Ax + \frac{1}{6}Bx^3,$$
$$z_1(x) = Bx + \int_0^x (s-x)\big(z''_0(s) - u_0(s)\big)\, ds = Bx + \frac{1}{6}Ax^3. \tag{2.323}$$

By imposing the boundary conditions on u_1, z_1 at $x = 1$, we have

$$A = 1.081277196, \quad B = 1.081277196, \tag{2.324}$$

and the series solution of the problem will be found.

CHAPTER 3: A GREEN'S FUNCTION-FIXED POINT ITERATIVE SCHEME

3.1 Green's Functions

In Chapters 1 and 2, we gave a review of two well-known iterative strategies to obtain numerical solution for various problems. However, we noted deficiencies in both of them, particularly the accuracy deteriorates as the applicable domain increases and the convergence is local. In this chapter, we propose an alternate approach based on embedding Green's functions into fixed point iterative schemes and then applying the scheme to a carefully selected integral operator. The prominent characteristic and the main rationale behind this novel technique are to surmount the deterioration of the numerical solution obtained by ADM and VIM as the domain grows.

First, we will introduce Green's functions and investigate how they may be used to derive a general solution to an inhomogeneous boundary value problem. The history of the Green's function dates back to 1828, when George Green published article aimed at seeking solution for the Poisson's equation $\Delta u = f$ for the electric potential u defined inside a bounded volume with given boundary conditions on the surface of the volume. He introduced a function that was later referred to by Riemann as the "Green's function." We will restrict our discussion to Green's functions for ordinary differential equations.

3.1.1 First Order Equations

Consider the first order equation

$$L[u] \equiv u'(t) + p(t)u(t) = f(t), \qquad \text{for } x > a, \tag{3.1}$$

subject to an initial condition,

$$B[u] \equiv u(a) = \alpha. \tag{3.2}$$

The general solution is given by

$$u = u_h + u_p, \tag{3.3}$$

where u_h is a homogeneous solution which is the solution of $L[u] = 0$ subject to the initial condition (3.2) and u_p is a particular solution which satisfies $L[u] = f(t)$ with homogeneous initial condition $u(a) = 0$. We represent the inhomogeneous solution (the particular solution) as an integral of the Green's function $G(t|s)$ given by

$$u = \int_a^\infty G(t|s)f(s)\,ds, \tag{3.4}$$

where the Green function $G(t|s)$ is defined as the solution to

$$L[G(t|s)] = \delta(t-s) \text{ subject to } G(a|s) = 0, \tag{3.5}$$

where $\delta(t-s)$ is the Dirac delta function defined as

$$\delta(t-s) = \begin{cases} 0, & t \neq s \\ \infty, & t = s \end{cases}. \tag{3.6}$$

Another way to define the delta function is as one that satisfies the following properties:

$$\text{i. } \int_{-\infty}^{\infty} \delta(t-s)dt = 1,$$
$$\text{ii. } \int_{-\infty}^{\infty} \delta(t-s)f(s)dt = f(t), \tag{3.7}$$

where the integral can be taken over any interval that includes $t = s$. To show that (3.3) is the solution, we can use the definition of Green's function (3.5) and the properties of the Dirac delta function (3.7). Applying the linear operator L to the solution u

$$\begin{aligned} L[u_h + u_p] &= L\left[\int_a^{\infty} G(t|s)f(s)\,ds\right] \\ &= \int_a^{\infty} L[G(t|s)]f(s)\,ds \\ &= \int_a^{\infty} \delta(t-s)f(s)\,ds \\ &= f(t). \end{aligned} \tag{3.8}$$

Applying the initial condition to u yields

$$\begin{aligned} B[u_h + u_p] &= B\left[u_h + \int_a^{\infty} G(t|s)f(s)\,ds\right] \\ &= \alpha + \int_a^{\infty} B[G(t|s)]f(s)\,ds \\ &= \alpha + \int_a^{\infty} 0 \cdot f(s)\,ds = \alpha. \end{aligned} \tag{3.9}$$

The Green's function satisfies the equation

$$G'(t|s) + p(t)G(t|s) = \delta(t-s). \tag{3.10}$$

The solution to the corresponding homogeneous equation is

$$u_h = e^{-\int_s^t p(x)\,dx}. \tag{3.11}$$

For $t \neq s$, Green's function is a homogeneous solution to the differential equation $L[u] = 0$, However, at $t = s$ Green's function has a singular behavior. Therefore, Green's function is given by

$$G(t|s) = \begin{cases} c_1 e^{-\int_s^t p(x)\,dx}, & a < t < s \\ c_2 e^{-\int_s^t p(x)\,dx}, & t > s \end{cases}, \tag{3.12}$$

where c_1 and c_2 are constants. We can find these two constants using the following properties of Green's function:

1. The Green function satisfies the homogeneous initial condition

$$u = \int_a^{\infty} G(t|s) f(s)\, ds, \tag{3.13}$$

thus, we get $c_1 = 0$. Green's function becomes

$$G(t|s) = \begin{cases} 0, & a < t < s \\ c_2 e^{-\int_s^t p(x)\,dx}, & t > s \end{cases}. \tag{3.14}$$

2. Integrating equation (3.10), we get

$$\begin{gathered} \int_{s^-}^{s^+} [G'(t|s) + p(t)G(t|s)]dx = \int_{s^-}^{s^+} [\delta(t-s)]dx, \\ G(s^+|s) - G(s^-|s) + \int_{s^-}^{s^+} p(t)G(t|s) = 1, \\ G(s^+|s) - G(s^-|s) = 1. \end{gathered} \tag{3.15}$$

Since $G'(t|s)$ has a Dirac delta function type of singularity, thus $G(t|s)$ has a jump discontinuity at $t = s$. Since at $t = s$, $\int_s^t p(x)\, dx = 0$ it follows from (3.15) that

$$\begin{gathered} c_2 e^{-\int_s^t p(x)\,dx} - 0 = 1, \\ c_2 = 1. \end{gathered} \tag{3.16}$$

The Green's function becomes

$$G(t|s) = \begin{cases} 0, & a < t < s \\ e^{-\int_s^t p(x)\,dx}, & t > s \end{cases}. \tag{3.17}$$

We can use the Heaviside function to rewrite the equation (3.17). The Heaviside function is defined as

$$H(t-s) = \begin{cases} 0, & t < s \\ 1, & t \geq s \end{cases}. \tag{3.18}$$

Thus, equation (3.17) becomes

$$G(t|s) = e^{-\int_s^t p(x)\,dx} H(t-s). \tag{3.19}$$

3.1.2 Initial Value Problems for Second Order Equations

Consider the second order equation

$$L[u] \equiv u'' + p(t)u' + q(t)u = f(t), \qquad \text{for } a < t < b, \tag{3.20}$$

subject to the initial conditions,

$$\begin{aligned} u(a) &= \alpha, \\ u'(a) &= \beta. \end{aligned} \tag{3.21}$$

The general solution is given by

$$u = u_h + u_p, \tag{3.22}$$

where u_h is the homogeneous solution which is the solution of $L[u] = 0$ subject to the initial conditions (3.21) while u_p is a particular solution which satisfies $L[u] = f(t)$ with the initial conditions $u(a) = u'(a) = 0$. We represent the inhomogeneous solution (the particular solution) as an integral of the Green's function $G(t|s)$ as

$$u(t) = \int_a^t G(t|s)f(s)\,ds. \tag{3.23}$$

By applying the linear operator L, we can verify that (3.22) is correct

$$\begin{aligned} L[u_h + u_p] &= L\left[\int_a^t G(t|s)f(s)\,ds\right] \\ &= \int_a^t L[G(t|s)]f(s)\,ds \\ &= \int_a^t \delta(t-s)f(s)\,ds \\ &= f(t), \end{aligned} \tag{3.24}$$

so we do indeed have a solution to (3.20), namely (3.22). Let us consider the general solution of the inhomogeneous problem $u = c_1u_1 + c_2u_2$, then we have

$$u' = c'_1u_1 + c_1u'_1 + c'_2u_2 + c_2u'_2. \tag{3.25}$$

Since $c_1$and c_2 are constants, thus

$$c'_1u_1 + c'_2u_2 = 0. \tag{3.26}$$

Therefore, from (3.26), we get

$$u' = c_1u'_1 + c_2u'_2. \tag{3.27}$$

Substituting expressions for u and u' into equation (3.20), gives

$$\begin{aligned} &c'_1u'_1 + c_1u''_1 + c'_2u'_2 + c_2u''_2 + p(t)(c_1u'_1 + c_2u'_2) + q(t)(c_1u_1 + c_2u_2) \\ &\qquad = f(t), \\ &c'_1u'_1 + c'_2u'_2 + c_1(u''_1 + p(t)u'_1 + q(t)u_1) + c_2(u''_2 + p(t)u'_2 + q(t)u_2) \\ &\qquad = f(t). \end{aligned} \tag{3.28}$$

As we know, u_1 and u_2 satisfy the linear equation. Thus

$$c'_1 u'_1 + c'_2 u'_2 = f(t). \tag{3.29}$$

The Green's function satisfies the equation

$$G''(t|s) + p(t)G'(t|s) + q(t)G(t|s) = \delta(t-s), \tag{3.30}$$

where $G(t|s)$ is defined as the solution to

$$L[G(t|s)] = \delta(t-s) \text{ subject to } G(a|s) = G'(a|s) = 0. \tag{3.31}$$

Now, assume that u_1 and u_2 are two linearly independent solutions to the homogeneous equation $L[u] = 0$. For $t < s$, $G(t|s)$ is a linear combination of these solutions. Therefore, we can write

$$G(t|s) = \begin{cases} 0, & t < s \\ c_1 u_1 + c_2 u_2, & t > s \end{cases}. \tag{3.32}$$

The constants can be found by solving the system

$$\begin{cases} c'_1 u_1 + c'_2 u_2 = 0, \\ c'_1 u'_1 + c'_2 u'_2 = f(t), \end{cases} \tag{3.33}$$

whose solution is

$$\begin{cases} c'_1 = \dfrac{-u_2 f(t)}{W(t)} \to c_1 = \displaystyle\int_a^t \frac{-u_2 f(s)}{W(s)} ds \\ c'_2 = \dfrac{u_1 f(t)}{W(t)} \to c_2 = \displaystyle\int_a^t \frac{u_1 f(s)}{W(s)} ds \end{cases}, \tag{3.34}$$

where $W(t) = u_1 u'_2 + u_2 u'_1$ is usually referred to as the Wronskian of u_1 and u_2. Since the Wronskian is non-vanishing, only the trivial solution satisfies the homogeneous initial conditions. The Green's function must be

$$G(t|s) = \begin{cases} 0, & t < s \\ u_1 \displaystyle\int_a^t \frac{-u_2(s) f(s)}{W(s)} ds + u_2 \int_a^t \frac{u_1(s) f(s)}{W(s)} ds, & t > s \end{cases}. \tag{3.35}$$

Thus, The solution for u is

$$u = u_h + u_1 \int_a^t \frac{-u_2(s) f(s)}{W(s)} ds + u_2 \int_a^t \frac{u_1(s) f(s)}{W(s)} ds. \tag{3.36}$$

simplifying Eq. (3.36) we get

$$\begin{aligned} u &= u_h + \int_a^t \frac{u_1(s)u_2(t) - u_2(s)u_1(t)}{W(s)} f(s) ds \\ &= u_h + \int_a^x \frac{\Delta(s,t)}{W(s)} f(s) ds \end{aligned}$$

$$= u_h + \int_a^t G(t|s)f(s)ds, \tag{3.37}$$

where $\frac{\Delta(s,t)}{W(s)} = G(t|s)$ is the Green's function. Note that solution u will satisfy the initial conditions if u_p satisfies the initial conditions of the inhomogeneous problem:

$$u(a) = u'(a) = 0. \tag{3.38}$$

We can also find Green's function in another way by using the properties of Green's function to determine the constants in (3.32). Green's function for Equation (3.20) has the following properties:

1. $G(t|s)$ satisfies the homogeneous initial condition

$$G(a|s) = G'(a|s) = 0. \tag{3.39}$$

2. $G(t|s)$ is continuous, that is

$$G(t|s)|_{t\mapsto s^-} = G(t|s)|_{t\mapsto s^+}, \tag{3.40}$$

and hence,

$$c_1u_1(s) + c_2u_2(s) = d_1u_1(s) + d_2u_2(s). \tag{3.41}$$

3. Integrating equation (3.30), we get

$$\int_{s^-}^{s^+} [G''(t|s) + p(t)G'(t|s) + q(t)G(t|s)]dt = \int_{s^-}^{s^+} [\delta(t-s)]dt. \tag{3.42}$$

Since $G(t|s)$ is continuous and $G''(t,s)$ has a Dirac delta function type of singularity, thus $G'(t|s)$ has only a jump discontinuity. Then

$$\int_{s^-}^{s^+} p(t)G'(t|s)dt = 0 \quad \text{and} \quad \int_{s^-}^{s^+} q(t)G(t|s)dt = 0. \tag{3.43}$$

Therefore,

$$\int_{s^-}^{s^+} G''(t|s)dx = \int_{s^-}^{s^+} [\delta(t-s)]dx$$
$$[G'(t|s)]_{s^-}^{s^+} = [H(t-s)]_{s^-}^{s^+} \tag{3.44}$$
$$G'(s^+|s) - G'(s^-|s) = 1.$$

As a result, we can easily determine the coefficients using these properties.

Example 3.1 Solve the initial value problem

$$u''(t) + u(t) = 2\cos t, \qquad u(0) = 4, \qquad u'(0) = 0. \tag{3.45}$$

Solution:
We first solve the homogeneous problem with nonhomogeneous initial conditions:

$$u''(t) + u(t) = 0, \quad u(0) = 4, \; u'(0) = 0. \tag{3.46}$$

Thus,

$$u_h = 4\cos t. \tag{3.47}$$

The Green's function satisfies the equation

$$G''(t|s) + G(t|s) = \delta(t - s). \tag{3.48}$$

Next, we construct the Green's function. We need two linearly independent solutions, $u_1(t)$, $u_2(t)$, to the homogeneous differential equation satisfying $u(0) = 0$ and $u'(0) = 0$.
So, Green's function is given by

$$G(t|s) = \begin{cases} a_1 \sin t + b_1 \cos t, & 0 < t < s \\ a_2 \sin t + b_2 \cos t, & t > s \end{cases}. \tag{3.49}$$

Applying the homogeneous initial conditions $G(0|s) = G'(0|s) = 0$, we get

$$G(t|s) = \begin{cases} 0, & 0 < t < s \\ a_2 \sin t + b_2 \cos t, & t > s \end{cases}. \tag{3.50}$$

Now $G(t|s)$ is continuous, so we have

$$G(t|s)|_{t\mapsto s^-} = G(t|s)|_{t\mapsto s^+}, \tag{3.51}$$

therefore,

$$0 = a_2 \sin s + b_2 \cos s. \tag{3.52}$$

Integrating equation (3.49), we get

$$\int_{s^-}^{s^+} [G''(t|s) + G(t|s)]dt = \int_{s^-}^{s^+} [\delta(t - s)]dt. \tag{3.53}$$

Since $G(t|s)$ is continuous and $G''(t|s)$ has a Dirac delta function type of singularity, thus $G'(t|s)$ has only a jump discontinuity. Hence

$$\begin{gathered} \int_{s^-}^{s^+} G''(t|s)dx = \int_{s^-}^{s^+} [\delta(t - s)]dx \\ [G'(t|s)]_{s^-}^{s^+} = [H(t - s)]_{s^-}^{s^+} \\ G'(s^+|s) - G'(s^-|s) = 1, \\ a_2 \cos s - b_2 \sin s = 1. \end{gathered} \tag{3.54}$$

From (3.52) and (3.54), we get

$$a_2 = \cos s\ , \quad b_2 = -\sin s, \tag{3.55}$$

therefore,

$$G(t|s) = \begin{cases} 0, & 0 < t < s \\ \cos s \sin t \ - \sin s \cos t, & t > s \end{cases}. \tag{3.56}$$

Then

$$\begin{aligned} u_p(t) &= \int_0^t G(t|s) f(s)\, ds \\ &= \int_0^t (\cos s \sin t \ - \sin s \cos t) 2 \cos s \ \ ds \\ &= t \sin t. \end{aligned} \tag{3.57}$$

The general solution is given by

$$u = u_h + u_p. \tag{3.58}$$

From (3.56) and (3.57), we get the general solution

$$u(t) = 4 \cos t + t \sin t. \tag{3.59}$$

We can find the solution directly using the Wronskian. We pick $u_1(t) = \sin t$ and $u_2(t) = cos\ t$. The Wronskian is found as

$$W(t) = u_1(t) u'_2(t) - u'_1(t) u_2(t) = -\sin^2 t \ - \cos^2 t = -1. \tag{3.60}$$

Thus, we can write Green's function in the form

$$G(t|s) = \begin{cases} 0, & 0 < t < s \\ \dfrac{u_1(s) u_2(t) - u_1(t) u_2(s)}{W(t)}, & t > s \end{cases}. \tag{3.61}$$

Therefore,

$$G(t|s) = \begin{cases} 0, & 0 < t < s \\ -\cos s \sin t \ + \sin s \cos t, & t > s \end{cases}. \tag{3.62}$$

The particular solution is $u(t) = 4 \cos t + t \sin t$. This is the same solution as the one we have found earlier using Green's function properties.

3.1.3 Boundary Value Problems for Second Order Equations

Consider the second order equation

$$L[u] \equiv u''(t) + p(t) u'(t) + q(t) u(t) = f(t), \qquad \text{for } a < t < b, \tag{3.63}$$

subject to the boundary conditions,

$$B_1[u] \equiv a_1 u(a) + a_2 u'(a) = \alpha,$$
$$B_2[u] \equiv b_1 u(b) + b_2 u'(b) = \beta. \tag{3.64}$$

The general solution is given by

$$u = u_h + u_p, \tag{3.65}$$

where u_h is the homogeneous solution which is the solution of $L[u] = 0$ subject to the boundary conditions (3.64) and u_p is a particular solution which satisfies $L[u] = f(t)$ with the boundary conditions $B_1[u] = B_2[u] = 0$. We represent the inhomogeneous solution (the particular solution) as an integral of the Green's function $G(t|s)$:

$$u_p = \int_a^b G(t|s) f(s)\, ds, \tag{3.66}$$

where the Green's function $G(t|s)$ is defined as the solution to

$$L[G(t|s)] = \delta(t-s) \quad \text{subject to} \quad B_1[G(t|s)] = B_2[G(t|s)] = 0. \tag{3.67}$$

Here $\delta(t-s)$ is the Dirac delta function, which has been mentioned in Section 3.1.1.

To show that (3.65) is the solution, we can use the definition of Green's function (3.5) and the properties of the Dirac delta function (3.7). Applying the linear operator L to the solution

$$\begin{aligned} L[u_h + u_p] &= L\left[\int_a^b G(t|s) f(s)\, ds\right] \\ &= \int_a^b L[G(t|s)] f(s)\, ds \\ &= \int_a^b \delta(t-s) f(s)\, ds \\ &= f(t). \end{aligned} \tag{3.68}$$

Applying the boundary conditions

$$\begin{aligned} B_1[u_h + u_p] &= B_1\left[u_h + \int_a^b G(t|s) f(s)\, ds\right] \\ &= \alpha + \int_a^b B_1[G(t|s)] f(s)\, ds \\ &= \alpha + \int_a^b [0] f(s)\, ds \\ &= \alpha, \end{aligned} \tag{3.69}$$

and

$$\begin{aligned} B_2[u_h + u_p] &= B_2\left[u_h + \int_a^b G(t|s) f(s)\, ds\right] \\ &= \beta + \int_a^b B_2[G(t|s)] f(s)\, ds \end{aligned}$$

$$= \beta + \int_a^b [0] f(s)\, ds \tag{3.70}$$
$$= \beta.$$

The Green's function satisfies the equation

$$G''(t|s) + p(t)G'(t|s) + q(t)G(t|s) = \delta(t-s). \tag{3.71}$$

Let u_1 and u_2 be two linearly independent solutions to $L[u] = 0$, which is the homogeneous equation. For $x \neq s$, Green's function is a homogeneous solution of the differential equation. Therefore, Green's function is given by

$$G(t|s) = \begin{cases} c_1 u_1 + c_2 u_2, & t < s \\ d_1 u_1 + d_2 u_2, & t > s' \end{cases} \tag{3.72}$$

where c_1, c_2 , d_1 and d_2 are constants. We consider the properties of the Green's function:

1. $G(t|s)$ satisfies the homogeneous initial conditions

$$B_1[G(t|s)] = B_2[G(t|s)] = 0. \tag{3.73}$$

2. $G(t|s)$ is continuous, that is

$$G(t|s)|_{t\mapsto s^-} = G(t|s)|_{t\mapsto s^+}, \tag{3.74}$$

and hence,

$$c_1 u_1(s) + c_2 u_2(s) = d_1 u_1(s) + d_2 u_2(s). \tag{3.75}$$

3. Integrating equation (3.71) gives

$$\int_{s^-}^{s^+} [G''(t|s) + p(t)G'(t|s) + q(t)G(t|s)]dt = \int_{s^-}^{s^+} [\delta(t-s)]dt. \tag{3.76}$$

Since $G(t|s)$ is continuous and $G''(t|s)$ has a Dirac delta function type singularity. Thus, $G'(t|s)$ has only a jump discontinuity, then

$$\int_{s^-}^{s^+} p(t)G'(t|s)dt = 0 \qquad \text{and} \qquad \int_{s^-}^{s^+} q(t)G(t|s)dt = 0. \tag{3.77}$$

Therefore,

$$\int_{s^-}^{s^+} G''(t|s)dx = \int_{s^-}^{s^+} [\delta(t-s)]dt$$

$$[G'(t|s)]_{s^-}^{s^+} = [H(t-s)]_{s^-}^{s^+}$$

$$G'(s^+|s) - G'(s^-|s) = 1. \tag{3.78}$$

Hence,

$$d_1u_1(s) + d_2u_2(s) - c_1u_1(s) - c_2u_2(s) = 1. \tag{3.79}$$

Using Green's function properties 1, 2 and 3 we can easily determine the four constants c_1, c_2 , $d_1$and d_2 to find the Green's function for the second order differential equations with the specified boundary conditions.

We can find Green's function using the Wronskian $W(t)$ of u_1 and u_2. Since the homogeneous equation with the homogeneous boundary conditions has only one trivial solution, $W(t)$ is nonzero on the given interval. The Green's function has the form

$$G(t|s) = \begin{cases} c_1u_1(t), & a < t < s \\ c_2u_2(t), & s < t < b \end{cases}. \tag{3.80}$$

From the continuity and jump conditions for the Green's function we get

$$\begin{aligned} c_1u_1(s) - c_2u_2(s) &= 0, \\ c_1u'_1(s) - c_2u'_2 &= -1. \end{aligned} \tag{3.81}$$

Thus, the solution is

$$c_1 = \frac{u_2(s)}{W(s)}, \quad c_2 = \frac{u_1(s)}{W(s)}. \tag{3.82}$$

Therefore, The Green's function is

$$G(t|s) = \begin{cases} \dfrac{u_2(s)u_1(t)}{W(s)}, & a < t < s \\ \dfrac{u_1(s)u_2(t)}{W(s)}, & s < t < b \end{cases}, \tag{3.83}$$

where the Wronskian is given by

$$W(t) = u_1(t)u'_2(t) + u_2(t)u'_1(t). \tag{3.84}$$

Example 3.2 Use a Green function to solve the boundary value problem

$$u''(t) = f(t), \qquad u(0) = u(2\pi) = 0. \tag{3.85}$$

Solution:
First solve the homogeneous equation that satisfies the boundary conditions

$$u''(t) = 0, \quad u(t) = at + b \tag{3.86}$$

The Green function satisfies

$$G''(t|s) = \delta(t-s), \qquad G(0|s) = G(2\pi|s) = 0. \tag{3.87}$$

Thus, the Green function has the form

$$G(t|s) = \begin{cases} a_1 t + b_1, & 0 < t < s \\ a_2 t + b_2, & s < t < 2\pi \end{cases}. \tag{3.88}$$

Applying the two boundary conditions, we get that $b_1 = 0$ and $b_2 = -2\pi a_2$. Hence

$$G(t|s) = \begin{cases} a_1 t, & 0 < t < s \\ a_2(t - 2\pi), & s < t < 2\pi \end{cases}. \tag{3.89}$$

Since Green's function is continuous at $t = s$, then

$$a_1 s = a_2(s - 2\pi) \qquad \rightarrow \quad a_1 = a_2\left(1 - \frac{2\pi}{s}\right). \tag{3.90}$$

From the jump condition, we have

$$[G'(t|s)]_{s^-}^{s^+} = 1,$$
$$G'(s^+|s) - G'(s^-|s) = 1,$$
$$a_2 - a_1 = 1. \tag{3.91}$$

From (3.90) and (3.91), we get

$$a_1 = \frac{s}{2\pi} - 1, \qquad a_2 = \frac{s}{2\pi}. \tag{3.92}$$

Therefore,

$$G(t|s) = \begin{cases} \left(\frac{s}{2\pi} - 1\right) t, & 0 < t < s \\ \frac{s}{2\pi}(t - 2\pi), & s < t < 2\pi \end{cases}. \tag{3.93}$$

and

$$u(t) = \int_0^{2\pi} G(t|s) f(s) ds = \int_0^t \left(\frac{s}{2\pi} - 1\right) t\, f(s) ds + \int_t^{2\pi} \frac{s}{2\pi}(t - 2\pi) f(s) ds. \tag{3.94}$$

Green's function (3.93) is symmetric, i.e. G(t|s) = G(s|t).

Example 3.3 Construct a Green's function for the problem

$$u''(t) + u(t) = f(t), \qquad u(0) = u(1) = 0. \tag{3.95}$$

Solution:

The general solution to the homogeneous equation is

$$u_h(t) = a \sin t + b \cos t. \tag{3.96}$$

Since $G(t|s)$ satisfies

$$G''(t|s) + G(t|s) = \delta(t - s), \qquad G(0|s) = G(1|s) = 0, \tag{3.97}$$

then,

$$G(t|s) = \begin{cases} a_1 \sin t + b_1 \cos t, & 0 < t < s \\ a_2 \sin t + b_2 \cos t, & s < t < 1 \end{cases}. \tag{3.98}$$

The condition $G(0|s) = 0$ for $0 \le t \le s$ implies that $b_1 = 0$, and the condition $G(1|s) = 0$ for $s \le t \le 1$ leads to

$$G(1|s) = a_2 \sin 1 + b_2 \cos 1 = 0. \tag{3.99}$$

Therefore,

$$G(t|s) = \begin{cases} a_1 \sin t, & 0 < t < s \\ a_2 \sin t - a_2 \tan 1 \cos t, & s < t < 1 \end{cases}. \tag{3.100}$$

Notice that

$$\begin{aligned} a_2 \sin t - a_2 \tan 1 \cos t &= \frac{a_2}{\cos 1}(\sin t \cos t - \sin 1 \cos t) \\ &= -\frac{a_2}{\cos 1}\sin(1 - t). \end{aligned} \tag{3.101}$$

Since the coefficient is arbitrary at this point, we can write the result as

$$-\frac{a_2}{\cos 1}\sin(1 - t) = c_1 \sin(1 - t). \tag{3.102}$$

Therefore, the Green's function has the form

$$G(t|s) = \begin{cases} a_1 \sin t, & 0 < t < s \\ c_1 \sin(1 - t), & s < t < 1 \end{cases}. \tag{3.103}$$

The continuity at $t = s$, implies

$$a_1 \sin s = c_1 \sin(1 - s) \qquad \rightarrow \quad a_1 = c_1 \frac{\sin(1 - s)}{\sin s}. \tag{3.104}$$

From the jump condition, we have

$$\begin{gathered} [G'(t|s)]_{s^-}^{s^+} = 1, \\ G'(s^+|s) - G'(s^-|s) = 1, \\ -c_1 \cos(1 - s) - a_1 \cos s = 1 \\ -c_1 \cos(1 - s) - c_1 \frac{\sin(1 - s)}{\sin s}\cos s = 1 \\ -c_1 \sin(s + 1 - s) = 1 \\ c_1 = -\frac{1}{\sin 1}, \qquad a_1 = -\frac{\sin(1 - s)}{\sin 1 \sin s}. \end{gathered} \tag{3.105}$$

Hence,

$$G(t|s) = \begin{cases} -\dfrac{\sin t\ \sin(1-s)}{\sin 1 \sin s}, & 0 < t < s \\ -\dfrac{\sin(1-t)}{\sin 1}, & s < t < 1 \end{cases}. \tag{3.106}$$

3.1.4 Sturm –Liouville Problems

Consider the following problem

$$L[u] \equiv (p(t)u')' + q(t)u(t) = f(t), \tag{3.107}$$

subject to the boundary conditions

$$\begin{aligned} B_1[u] &= a_1 u(a) + b_1 u'(a) = 0, \\ B_2[u] &= a_2 u(b) + b_2 u'(b) = 0. \end{aligned} \tag{3.108}$$

The Green's function $G(t|s)$ is defined as the solution to

$$L[G(t|s)] = \delta(t-s) \text{ subject to } B_1[G] = B_2[G] = 0. \tag{3.109}$$

Let u_1 and u_2 be two linearly independent solutions to $L[u] = 0$, which is the homogeneous equation. For $x \neq s$, the Green's function is a homogeneous solution of the differential equation. Therefore, the Green's function is given by

$$G(t|s) = \begin{cases} c_1 u_1(t), & a < t < s \\ c_2 u_2(t), & s < t < b \end{cases}. \tag{3.110}$$

Green's function satisfies the equation

$$G''(t|s) + \frac{p'(t)}{p(t)} G' + \frac{q(t)}{p(t)} G(t|s) = \frac{\delta(t-s)}{p(t)}. \tag{3.111}$$

The continuity of $G(t|s)$ at $t = s$ implies

$$\begin{gathered} G(t|s)|_{t\mapsto s^-} = G(t|s)|_{t\mapsto s^+} \\ c_1 u_1(s) = c_2 u_2(s). \end{gathered} \tag{3.112}$$

Further, since $G(t|s)$ is continuous and $G''(t,s)$ has a Dirac delta function type of singularity, thus $G'(t|s)$ has only a jump discontinuity. Then

$$\int_{s^-}^{s^+} G''(t|s)dx = \int_{s^-}^{s^+} \left[\frac{\delta(t-s)}{p(t)}\right] dt,$$

$$G'(s^+|s) - G'(s^-|s) = \frac{1}{p(s)}, \tag{3.113}$$
$$c_2 u'_2(s) - c_1 u'_1(s) = \frac{1}{p(s)}.$$

We can determine these two constants c_1 and c_2 by solving the following system:

$$c_1(s)u_1(s) - c_2(s)u_2(s) = 0,$$
$$c_1(s)u'_1(s) - c_2(s)u'_2(s) = -\frac{1}{p(s)}. \tag{3.114}$$

We can solve (3.114) by using Kramer's rule. Hence

$$c_1(s) = \frac{u_2(s)}{p(s)W(s)}, \quad c_2(s) = \frac{u_1(s)}{p(s)W(s)}, \tag{3.115}$$

where W(t) is the Wronskian of $u_1(t)$ and $u_2(t)$. Thus, Green's function is given

$$G(t|s) = \begin{cases} \dfrac{u_2(s)u_1(t)}{p(s)W(s)}, & a < t < s \\ \dfrac{u_1(s)u_2(t)}{p(s)W(s)}, & s < t < b \end{cases}. \tag{3.116}$$

The solution for this problem is given by

$$u(t) = u_h(t) + u_p(t)$$
$$= u_h(t) + \int_a^b G(t|s)f(s)ds. \tag{3.117}$$

Example 3.4 Use a Green's function to solve the boundary value problem

$$u''(t) = t^2, \quad u(0) = u(1) = 0. \tag{3.118}$$

Solution:

The general solution to the homogeneous equation is

$$u_h(t) = at + b. \tag{3.119}$$

The Green's function satisfies

$$G''(t|s) = \delta(t - s), \qquad G(0|s) = G(1|s) = 0. \tag{3.120}$$

Thus, the Green's function has the form

$$G(t|s) = \begin{cases} a_1 t + b_1, & 0 < t < s \\ a_2 t + b_2, & s < t < 1 \end{cases}. \tag{3.121}$$

The condition $G(0|s) = 0$ for $0 \leq t \leq s$ gives $b_1 = 0$, and the condition $G(1|s) = 0$ for $s \leq t \leq 1$ leads to

$$G(1|s) = a_2 + b_2 = 0. \tag{3.122}$$

Thus,

$$G(t|s) = \begin{cases} a_1 t, & 0 < t < s \\ a_2 t + b_2, & s < t < 1 \end{cases}. \tag{3.123}$$

Since $G(t|s)$ is continuous, then

$$a_1 s = a_2 s + b_2. \tag{3.124}$$

By the jump condition, we have

$$[G''(t|s)]_{s^-}^{s^+} = 1,$$
$$G''(s^+|s) - G''(s^-|s) = 1,$$
$$a_2 - a_1 = 1. \tag{3.125}$$

Equations (3.124) and (3.125), give

$$a_1 = s - 1, \quad a_2 = s, \quad b_2 = -s, \tag{3.126}$$

and hence,

$$G(t|s) = \begin{cases} (s-1)t, & 0 < t < s \\ (t-1)s, & s < t < 1 \end{cases}. \tag{3.127}$$

Now, in order to find the particular solution, we insert the Green's function into the integral form of the solution

$$\begin{aligned} u_p(t) = \int_0^1 G(t|s) f(s) ds &= \int_0^t (s-1)t(s^2) ds + \int_t^1 (t-1)s(s^2) ds \\ &= \frac{1}{4}\left(-\frac{1}{3}t^4 + t - 1\right). \end{aligned} \tag{3.128}$$

3.1.5 Boundary Value Problems for Third Order Equations

Third order BVPs arise in many scientific and engineering applications such as the deflection of a curved beam having a constant or varying cross-section, three-layer beam, the motion of rocket, thin film flow, electromagnetic waves, gravity-driven flows, the study of draining and coating flows, boundary layer theory, the study of stellar interiors, control and optimization theory and flow networks in biology.

There are various theorems regarding existence of a unique solution. In particular Lu¨ and Cui [47] provide the existence of a solution for the following case:

$$u'''(x) - f\big(x, u(x), u'(x), u''(x)\big) = 0, \quad 0 \le x \le 1, \tag{3.129}$$
$$u(1) = 0, \quad u'(0) = 0, \quad u'(1) = 0.$$

They proved that under the following assumptions:

(H1) $f(x, y, z, w) \in [0,1] \times R^3$ is completely continuous;

(H2) $f(x, y, z, w), f_x(x, y, z, w),\ f_y(x, y, z, w), f_z(x, y, z, w)$ and $f_w(x, y, z, w)$ are bounded;

(H3) $f(x, y, z, w) > 0$ on $[0,1] \times R^3$,

where $f(x, y, z, w) \in W_1[0,1]$ as $y = y(x) \in W_1[0,1],\ z = z(x) \in W_1[0,1],\ w = w(x) \in W_1[0,1]$, $(0 \le x \le 1, -\infty \le y, z, w \le \infty)$.

Problem (3.129) has a solution in $W_2[0,1]$. For more existence theorems see Lu¨ and Cui, Feng and Yao, and Feng [47, 49-50] and the references therein.

Now, consider the third order nonlinear equation

$$L[u] \equiv u'''(t) + p(t)u''(t) + q(t)u'(t) + r(t)u(t) = f(t), \tag{3.130}$$

where $a < t < b$ and subject to the boundary conditions,

$$B_1[u] \equiv a_1 u(a) + a_2 u'(a) + a_3 u''(a) = \alpha, \tag{3.131}$$
$$B_2[u] \equiv b_1 u(b) + b_2 u'(b) + a_3 u''(b) = \beta.$$

The general solution is given by

$$u = u_h + u_p, \tag{3.132}$$

where u_h is a homogeneous solution subject to boundary conditions (3.131) and u_p is a particular solution which satisfies $L[u] = f(t)$ with boundary conditions $B_1[u] = B_2[u] = 0$. We represent the inhomogeneous solution (the particular solution) as an integral of the Green's function $G(t|s)$

$$u_p = \int_a^b G(t|s) f(s)\, ds, \tag{3.133}$$

where Green's function $G(t|s)$ is defined as the solution to

$$L[G(t|s)] = \delta(t - s) \qquad \text{subject to} \qquad B_1[G(t|s)] = B_2[G(t|s)] = 0, \tag{3.134}$$

where $\delta(t - s)$ is the Dirac delta function. To show that (3.132) is the solution, we can use the definition of Green's function (3.5) and the properties of Dirac delta function (3.7). Applying the linear operator L to the solution

$$
\begin{aligned}
L[u_h + u_p] &= L\left[\int_a^b G(t|s) f(s)\, ds\right] \\
&= \int_a^b L[G(t|s)] f(s)\, ds \\
&= \int_a^b \delta(t-s) f(s)\, ds \\
&= f(t).
\end{aligned} \tag{3.135}
$$

Applying the boundary conditions

$$
\begin{aligned}
B_1[u_h + u_p] &= B_1\left[u_h + \int_a^b G(t|s) f(s)\, ds\right] \\
&= \alpha + \int_a^b B_1[G(t|s)] f(s)\, ds \\
&= \alpha + \int_a^b 0 \cdot f(s)\, ds \\
&= \alpha.
\end{aligned} \tag{3.136}
$$

and

$$
\begin{aligned}
B_2[u_h + u_p] &= B_2\left[u_h + \int_a^b G(t|s) f(s)\, ds\right] \\
&= \beta + \int_a^b B_2[G(t|s)] f(s)\, ds \\
&= \beta + \int_a^b 0 \cdot f(s)\, ds \\
&= \beta.
\end{aligned} \tag{3.137}
$$

The Green's function satisfies the equation

$$
G'''(t|s) + p(t)G''(t|s) + q(t)G'(t|s) + r(t)G(t|s) = \delta(t-s). \tag{3.138}
$$

Let u_1, u_2 and u_3 be three linearly independent solutions to $L[u] = 0$, which is the homogeneous equation. For $t \neq s$, Green's function is a homogeneous solution to the differential equation. Therefore,

$$
G(t|s) = \begin{cases} c_1u_1 + c_2u_2 + c_3u_3, & a < t < s \\ d_1u_1 + d_2u_2 + d_3u_3, & s < t < b' \end{cases} \tag{3.139}
$$

where c_1, c_2, c_3, d_1, d_2 and d_3 are constants. The properties of Green's function are:

1. $G(t|s)$ satisfies the homogeneous boundary conditions

$$
B_1[G(t|s)] = B_2[G(t|s)] = 0. \tag{3.140}
$$

2. $G(t|s)$ is continuous, that is

$$G(t|s)|_{t\mapsto s^-} = G(t|s)|_{t\mapsto s^+} \tag{3.141}$$

Therefore,

$$c_1u_1(s) + c_2u_2(s) + c_3u_3(s) = d_1u_1(s) + d_2u_2(s) + d_3u_3(s). \tag{3.142}$$

3. $G'(t|s)$ is continuous, that is,

$$G'(t|s)|_{t\mapsto s^-} = G'(t|s)|_{t\mapsto s^+}. \tag{3.143}$$

Therefore,

$$c_1u'_1(s) + c_2u'_2(s) + c_3u'_3(s) = d_1u'_1(s) + d_2u'_2(s) + d_3u'_3(s). \tag{3.144}$$

4. Integrating equation (3.138), implies that

$$\int_{s^-}^{s^+} [G'''(t|s) + p(t)G''(t|s) + q(t)G'(t|s) + r(t)G(t|s)]dt = \int_{s^-}^{s^+} [\delta(t-s)]dt. \tag{3.145}$$

Since $G(t|s)$ and $G'(t|s)$ are continuous and $G'''(t|s)$ has a Dirac delta function type of singularity, thus $G''(t|s)$ has only a jump discontinuity. Note that

$$\int_{s^-}^{s^+} p(t)G''(t|s)dt = 0, \qquad \int_{s^-}^{s^+} q(t)G'(t|s)dt = 0 \quad \text{and} \quad \int_{s^-}^{s^+} r(t)G(t|s)dt = 0.$$

Therefore,

$$\begin{aligned} \int_{s^-}^{s^+} G'''(t|s)dx &= \int_{s^-}^{s^+} [\delta(t-s)]dt \\ [G''(t|s)]_{s^-}^{s^+} &= [H(t-s)]_{s^-}^{s^+} \\ G''(s^+|s) - G''(s^-|s) &= 1 \end{aligned} \tag{3.146}$$

Hence,

$$d_1u''_1(s) + d_2u''_2(s) + d_3u''_3(s) - c_1u''_1(s) - c_2u''_2(s) - c_3u''_3(s) = 1. \tag{3.147}$$

Using Green's function properties 1, 2, 3 and 4, we can easily determine the four constants c_1, c_2 , d_1 and d_2 to find Green's function for the third order differential equation.

Example 3.5 Use a Green's function to solve the boundary value problem

$$u'''(t) + u''(t) - (u'(t))^2 + 1 = 0, \quad 0 < t < 1, \tag{3.148}$$

subject to the boundary conditions

$$u(0) = u'(0) = u(1) = 0. \tag{3.149}$$

Solution:

The general solution to the homogeneous equation is

$$u_h(t) = at^2 + bt + c. \tag{3.150}$$

Green's function satisfies

$$\begin{gathered} G'''(t|s) + G''(t|s) - \left(G'(t|s)\right)^2 + 1 = \delta(t-s), \\ G(0|s) = G'(0|s) = G(1|s) = 0. \end{gathered} \tag{3.151}$$

Thus, Green's function has the form

$$G(t|s) = \begin{cases} At^2 + Bt + C, & 0 < t < s \\ Dt^2 + Et + F, & s < t < 1 \end{cases}. \tag{3.152}$$

The condition $G(0|s) = 0$ for $0 \le t \le s$, implies that $C = 0$ and the condition $G'(0|s) = 0$ for $0 \le t \le s$, implies that $B = 0$.

Finally, $G(1|s) = 0$ for $s \le t \le 0$, leads to

$$D + E + F = 0. \tag{3.153}$$

Hence,

$$G(t|s) = \begin{cases} At^2, & 0 < t < s \\ Dt^2 + Et + F, & s < t < 1 \end{cases}. \tag{3.154}$$

Since Green's function $G(t|s)$ must be continuous at $t = s$, then

$$As^2 = Ds^2 + Es + F. \tag{3.155}$$

Also, $G'(t|s)$ is continuous at $t = s$, then

$$2As + B = 2Ds + E. \tag{3.156}$$

From the jump condition, we have

$$\begin{gathered} [G''(t|s)]_{s^-}^{s^+} = 1, \\ G''(s^+|s) - G''(s^-|s) = 1, \\ 2D - 2A = 1. \end{gathered} \tag{3.157}$$

From (3.155), (3.156) and (3.157), we get

$$A = -\frac{1}{2}s^2 + s - \frac{1}{2}, \quad D = -\frac{1}{2}s^2 + s, \quad E = -s, \quad F = \frac{1}{2}s^2. \tag{3.158}$$

Therefore,

$$G(t|s) = \begin{cases} \left(-\frac{1}{2}s^2 + s - \frac{1}{2}\right)t^2, & 0 < t < s \\ \left(-\frac{1}{2}s^2 + s\right)t^2 - st + \frac{s^2}{2}, & s < t < 1 \end{cases}. \tag{3.159}$$

Example 3.6 Use a Green function to solve the boundary value problem

$$u'''(t) - t^2u''(t) = 0, \quad 0 < t < 1, \tag{3.160}$$

subject to the boundary conditions

$$u(0) = u'(1) = u(1) = 0. \tag{3.161}$$

Solution:
The general solution to the homogeneous equation is

$$u_h(t) = at^2 + bt + c. \tag{3.162}$$

Green's function satisfies

$$G'''(t|s) - t^2G''(t|s) = \delta(t - s), \qquad G(0|s) = G'(1|s) = G(1|s) = 0. \tag{3.163}$$

Thus, Green's function has the form

$$G(t|s) = \begin{cases} At^2 + Bt + C, & 0 < t < s \\ Dt^2 + Et + F, & s < t < 1 \end{cases}. \tag{3.164}$$

The condition $G(0|s) = 0$ for $0 \le t \le s$, implies that $C = 0$, and the condition $G'(1|s) = 0$ for $0 \le t \le s$, implies that

$$2Ds + E = 0. \tag{3.165}$$

Finally, $G(1|s) = 0$ for $s \le t \le 0$, leads to

$$D + E + F = 0. \tag{3.166}$$

Hence,

$$G(t|s) = \begin{cases} At^2 + Bt, & 0 < t < s \\ Dt^2 + Et + F, & s < t < 1 \end{cases}. \tag{3.167}$$

Since Green's function $G(t|s)$ must be continuous at $t = s$, then

$$As^2 + Bs = Ds^2 + Es + F. \tag{3.168}$$

Also, $G'(t|s)$ is continuous at $t = s$, thus

$$2As + B = 2Ds + E. \tag{3.169}$$

From the jump condition, we have

$$[G''(t|s)]_{s^-}^{s^+} = 1, \tag{3.170}$$

$$G''(s^+|s) - G''(s^-|s) = 1,$$

$$2D - 2A = 1.$$

From (3.168), (3.169) and (3.170), we get

$$A = \frac{1}{2}\frac{s^2 - 2s + 1}{2s - 1}, \quad B = -\frac{s(s^2 - 2s + 1)}{2s - 1}, \quad D = \frac{1}{2}\frac{s^2}{2s - 1},$$
$$E = -\frac{s^3}{2s - 1}, \quad F = \frac{1}{2}s^2. \tag{3.171}$$

Therefore,

$$G(t|s) = \begin{cases} \left(\frac{1}{2}\frac{s^2 - 2s + 1}{2s - 1}\right)t^2 - \left(\frac{s(s^2 - 2s + 1)}{2s - 1}\right)t, & 0 < t < s \\ \left(\frac{1}{2}\frac{s^2}{2s - 1}\right)t^2 - \frac{s^3}{2s - 1}t + \frac{s^2}{2}, & s < t < 1 \end{cases}. \tag{3.172}$$

3.1.6 Properties of Green's Functions

In this section, we will summarize the properties of Green's functions as a tool for quickly constructing Green's functions for boundary value problems. Here is a list of the properties based upon the third order BVPs.

1. Differential Equation:

$$p(t)u'''(t) + q(t)u''(t) + r(t)u'(t) + h(t)u(t) = f(t). \tag{3.173}$$

The Green's function satisfies

$$p(t)G'''(t|s) + q(t)G''(t|s) + r(t)G'(t|s) + h(t)G(t|s) = \delta(t - s). \tag{3.174}$$

Let u_1, u_2 and u_3 be three linearly independent solutions to $L[u] = 0$, which is the homogeneous equation. For $t \neq s$, Green's function is a homogeneous solution of the differential equation.

2. Boundary Conditions:

$$B_1[G(t|s)] = B_2[G(t|s)] = 0. \tag{3.175}$$

Green's function satisfies the homogeneous boundary conditions

3. Continuity of $\mathbf{G(t|s)}$:

$$G(t|s)|_{t\mapsto s^-} = G(t|s)|_{t\mapsto s^+}, \quad (3.176)$$

where

$$G(t|s)|_{t\mapsto s^-} = \lim_{t\to s} G(t|s), \quad t < s,$$

$$G(t|s)|_{t\mapsto s^+} = \lim_{t\to s} G(t|s), \quad t > s.$$

4. Continuity of $\mathbf{G'(t|s)}$:

$$G'(t|s)|_{t\mapsto s^-} = G'(t|s)|_{t\mapsto s^+}, \quad (3.177)$$

where

$$G'(t|s)|_{t\mapsto s^-} = \lim_{t\to s} G'(t|s), \quad t < s,$$

$$G'(t|s)|_{t\mapsto s^+} = \lim_{t\to s} G'(t|s), \quad t > s.$$

5. Jump Discontinuity of $\boldsymbol{G''(t|s)}$ at $\boldsymbol{t = s}$:

$$\begin{aligned} [G''(t|s)]_{s^-}^{s^+} &= [H(t-s)]_{s^-}^{s^+}, \\ G''(s^+|s) - G''(s^-|s) &= \frac{1}{p(t)}. \end{aligned} \quad (3.178)$$

3.2 Picard's iterative Method

In this section, we will discuss Picard's iterative method for finding approximate solutions of first order nonlinear ordinary differential equation of the form

$$u' = \frac{du}{dx} = f(x,u), \quad (3.179)$$

with initial condition

$$u(x_0) = u_0. \quad (3.180)$$

We integrate both sides of the equation (3.179) over the interval (x_0, x). This gives

$$\begin{aligned} u(x) - u(x_0) &= \int_{x_0}^{x} f(x,u)\,dx \\ &= u_0 + \int_{x_0}^{x} f\big(x,u(x)\big)\,dx. \end{aligned} \quad (3.181)$$

The integral in (3.181) cannot be evaluated. Hence the exact value cannot be obtained. So we try to solve this by iteration. Substituting an initial guess of $u(x) = u_0$ into the right hand side of (3.181), we get

$$u_1(x) = u_0 + \int_{x_0}^{x} f(x, u_0)\, dx, \tag{3.182}$$

where the corresponding $u_1(x)$ is the value of $u(x)$ and is called first approximation. To determine better approximation we replace $u_1(x)$ by $u_2(x)$ as

$$u_2(x) = u_0 + \int_{x_0}^{x} f(x, u_1)\, dx. \tag{3.183}$$

In general, the $n + 1$ approximation is given by

$$u_{n+1}(x) = u_0 + \int_{x_0}^{x} f(x, u_n)\, dx. \tag{3.184}$$

Therefore, we have a sequence of approximate solutions

$$u_1(x), u_2(x), \dots, u_{n+1}(x), \dots. \tag{3.185}$$

3.3 The Krasnonsel'skii-Mann iteration algorithm (K-M)

There are several iteration techniques for approximating fixed points equations of various classes. Some of them are Picard iteration technique, Mann iteration technique, Krasnosel'skii iteration technique, and Newton iteration technique. The Picard's iteration technique, the Mann iteration technique and the Krasnosel'skii iteration technique are the most used of all those methods. In particular, to implement our method, we need to use Krasnonsel'skii-Mann (K-M) iteration algorithm.

The Krasnonsel'skii-Mann (K-M) iteration algorithm is aimed at solving the fixed point equation

$$Tx = x, \tag{3.186}$$

where T is a self-mapping of closed convex subset. The K-M algorithm generates a sequence $\{u_n\}$ according to the recursive formula

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T[u_n], \qquad n \geq 0. \tag{3.187}$$

Obviously, for the special case $\alpha_n = 1$ for each n in the Krasnonsel'skii -Mann iterative scheme, the result is Picard's iteration technique.

3.4 Method Description

In this section, we will discuss and describe the method that we will apply to obtain numerical solutions for a class of third order linear and nonlinear boundary value problems.

First, consider the general third order equation

$$p(t)u'''(t) + q(t)u''(t) + r(t)u'(t) + h(t)u(t) = f(t), \qquad a \le t \le b, \tag{3.188}$$

with boundary conditions

$$\begin{aligned} B_1[u] &\equiv a_1 u(a) + a_2 u'(a) + a_3 u''(a) = \alpha, \\ B_2[u] &\equiv b_1 u(b) + b_2 u'(b) + a_3 u''(a) = \beta. \end{aligned} \tag{3.189}$$

For the implementation of Green’s Function-Picard's fixed point iteration, we first define the following linear integral operator $T[u]$

$$T[u] = u_h + \int_a^b G(t|s)\,[p(s)u'''(s) + q(s)u''(s) + r(s)u'(s) + h(s)u(s)]\,ds. \tag{3.190}$$

Now, adding and subtracting $f(t)$ we get

$$\begin{aligned} T[u] = u_h + \int_a^b G(t|s)\,[p(s)u'''(s) + q(t)u''(s) + r(s)u'(s) + h(s)u(s) \\ - f(s)]\,ds + \int_a^b G(t|s)\,f(s)ds. \end{aligned} \tag{3.191}$$

From (3.132) and (3.133) we get

$$\begin{aligned} T[u] &= u_h + \int_a^b G(t|s)\,[p(s)u'''(s) + q(s)u''(s) + r(s)u'(s) + h(s)u(s) \\ &\quad - f(s)]\,ds + u - u_h, \\ T[u] &= u + \int_a^b G(t|s)\,[p(s)u'''(s) + q(s)u''(s) + r(s)u'(s) + h(s)u(s) \\ &\quad - f(s)]\,ds. \end{aligned} \tag{3.192}$$

Applying Picard's iteration for $n \ge 0$, gives

$$\begin{aligned} u_{n+1} = u_n + \int_a^b G(t|s)\,[p(t){u_n}'''(s) + q(s){u_n}''(s) + r(s){u_n}'(s) + h(s)u_n(s) \\ - f(s)]\,ds, \end{aligned} \tag{3.193}$$

which is equivalent to

$$
\begin{aligned}
u_{n+1} = u_0 + \int_a^b & G(t|s)\,[p(s)u_0{}'''(s) + q(s)u_0{}''(s) + r(s)u_0{}'(s) + h(s)u_0(s) \\
& - f(s'))]\,ds \\
& + \int_a^b G(t|s)\,[p(s)u_1{}'''(s) + q(s)u_1{}''(s) + r(s)u_1{}'(s) \\
& + h(s)u_1(s) - f(s))]\,ds + \cdots \\
& + \int_a^b G(t|s)\,[p(t)u_n{}'''(s) + q(s)u_n{}''(s) + r(s)u_n{}'(s) \\
& + h(s)u_n(s) - f(s)]\,ds.
\end{aligned}
\tag{3.194}
$$

We can choose u_0 by finding the solution for $L[u] = 0$ subject to the specified boundary conditions (3.189). Next, we apply Krasnoselskii–Mann iterative algorithm (K-M) for the approximation of fixed points given by (3.188). This implies

$$
\begin{aligned}
u_{n+1} = (1-\alpha_n)u_n & \\
& + \alpha_n \Bigg[u_n \\
& + \int_a^b G(t|s)\,[p(t)u_n{}'''(s) + q(s)u_n{}''(s) + r(s)u_n{}'(s) + h(s)u_n(s) \\
& - f(s)]\,ds\Bigg],
\end{aligned}
\tag{3.195}
$$

or equivalently

$$
u_{n+1} = u_n + \alpha_n \left[\int_a^b G(t|s)\,[p(t)u_n{}'''(s) + q(s)u_n{}''(s) + r(s)u_n{}'(s) + h(s)u_n(s)\right.
\tag{3.196}
$$

3.5 Numerical Results

In this section, we will apply this method on the class of third order nonlinear and linear boundary value problems and then comparing the numerical results to illustrate the efficiency of this method.

Problem 3.1 Consider the following third order nonlinear (BVP)

$$
u'''(t) + u(t)u''(t) - u'^2(t) + 1 = 0, \quad 0 \le t \le 1,
\tag{3.197}
$$

with boundary conditions

$$u(0) = u(1) = u'(0) = 0. \tag{3.198}$$

This problem has no known exact solution.

First, we find Green's function for $L[u] = u'''$ subject to $u(0) = u(1) = u'(0) = 0$ by applying the properties described before. Using the Computer Algebra System *Maple* we find that

$$G(t|s) = \begin{cases} \left(-\frac{1}{2}s^2 + s\right)t^2 - ts + \frac{s^2}{2}, & 0 < t < s \\ \left(-\frac{1}{2}s^2 + s - \frac{1}{2}\right)t^2, & s < t < 1 \end{cases}. \tag{3.199}$$

Next we apply the method as shown in Section 3.4, where u_0 is the solution of $L[u] \equiv u''' = 0$ subject to $u(0) = u(1) = u'(0) = 0$. Hence the iterative algorithm is

$$\begin{aligned} u_0 &= 0, \\ u_{n+1} &= u_n + \int_0^t \left(\left(-\frac{1}{2}s^2 + s\right)t^2 - ts + \frac{s^2}{2}\right)\left(u_n'''(s) + u_n(s)u_n''(s)\right. \\ &\quad \left. - {u'_n}^2(s) + 1\right)ds \\ &\quad + \int_t^1 \left(\left(-\frac{1}{2}s^2 + s - \frac{1}{2}\right)t^2\right)\left(u_n'''(s) + u_n(s)u_n''(s) - {u'_n}^2(s)\right. \\ &\quad \left. + 1\right)ds. \end{aligned} \tag{3.200}$$

Numerical results are given to illustrate the efficiency of the proposed method and are compared with other numerical methods that exist in the literature. It is clear that the method is highly accurate and reliable because it yields very accurate approximate solutions as is shown by Tables 3.1 (a), (b), (c), (d) and (e) and depicted in Figure 3.1 (a). The error is better when using Picard iterations while the suggested Krasnoselskii–Mann iteration is not implemented in this problem since it did not show any significant improvement. Since this problem does not have a known exact solution, thus we found the error by subtracting the nth iteration from the $n+1$ iteration $|u_{n+1} - u_n|$.

Also, the numerical solution can be substituted in the differential equation to show that it satisfies the equation with high accuracy. It is worth mentioning that when we increase the number of iterations, the maximum error at the mesh points $t = 0.1, 0.2, \dots, 0.9$ is reduced as is shown in Table 3.1 (a). Comparisons of the absolute error values between our proposed method and those methods in [39-41] are shown in Tables 3.1 (c) and (e). It is obvious that our method yields better results.

t	Numerical Solution	Present Method: $\|u_7 - u_6\|$
0.0	0	0
0.1	0.00149606946549957007926034455974	1.0×10^{-16}
0.2	0.00531781872516839635122571694105	4.0×10^{-16}
0.3	0.0104662029209492814015920759592	8.8×10^{-16}
0.4	0.0159432809774814504168200940993	1.5×10^{-15}
0.5	0.0207523246456161601691631635018	2.2×10^{-15}
0.6	0.0238978611427057135484820595047	2.8×10^{-15}
0.7	0.0243858478971903422282362025053	3.2×10^{-15}
0.8	0.0212241768032093085434390272793	3.2×10^{-15}
0.9	0.0134237040224862097880788967389	2.3×10^{-15}
1.0	-2.3×10^{-32}	4.7×10^{-42}

Table 3.1 (a) Numerical solution for Problem 3.1 using 7 iterations of the iterative method.

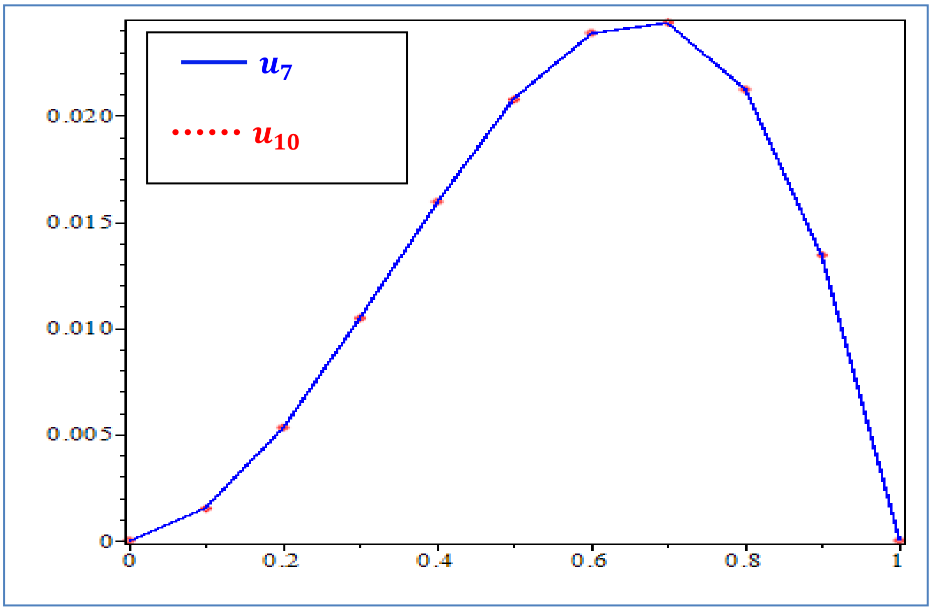

Figure 3.1 (a) Absolute error between approximate solution for 10^{th} and 7^{th} iterations.

Number of iterations	Maximum Error
2	1.2×10^{-4}
3	1.2×10^{-7}
4	7.2×10^{-9}
5	5.5×10^{-11}
6	4.2×10^{-13}
7	3.2×10^{-15}
8	2.5×10^{-17}
9	9.7×10^{-19}
10	9.7×10^{-22}

Table 3.1 (b) Maximum error of our method applied to Problem 3.1 using various iterations.

x	Shooting Method[41]	10th HAM for $h = -0.922$ [39]	Present Method
$\mathbf{0.1}$	0.0006723	3.96×10^{-7}	4.5×10^{-23}
$\mathbf{0.3}$	0.0021682	3.00×10^{-11}	3.9×10^{-22}
$\mathbf{0.5}$	0.0033009	1.19×10^{-8}	9.7×10^{-22}
$\mathbf{0.7}$	0.0036213	6.89×10^{-8}	1.4×10^{-22}
$\mathbf{0.9}$	0.0021849	1.02×10^{-7}	1.0×10^{-22}

Table 3.1 (c) Comparison with other methods, for Problem 3.1 using 10 iterations.

t	**Numerical Solution**	**Present Method:** $\lvert u_{10} - u_9 \rvert$
$\mathbf{0.0}$	0	0
$\mathbf{0.1}$	0.00149606946549956931554415153542419674331216281	4.5×10^{-23}
$\mathbf{0.2}$	0.00149606946549956931554415153542419674331216281	1.8×10^{-22}
$\mathbf{0.3}$	0.0104662029209492747487539290229974468662108890	3.9×10^{-22}
$\mathbf{0.4}$	0.0159432809774814390988468700292006259628868318	6.7×10^{-22}
$\mathbf{0.5}$	0.0207523246456161437085590234114926679241014935	9.7×10^{-22}
$\mathbf{0.6}$	0.0238978611427056923179285459540071998662169585	1.2×10^{-22}
$\mathbf{0.7}$	0.0243858478971903178263927431619211544095578985	1.4×10^{-22}
$\mathbf{0.8}$	0.0212241768032092843770770158736263620581738920	1.4×10^{-22}
$\mathbf{0.9}$	0.0134237040224861921260805616667454183654135393	1.0×10^{-22}
$\mathbf{1.0}$	6.8×10^{-47}	1.4×10^{-59}

Table 3.1 (d) Numerical solutions for Problem 3.1 using 10 iterations of the present method.

t	5^{th} **HAM[40]**	10^{th} **HAM [40]**	15^{th} **HAM [40]**	**Present Method :** u_{10}
$\mathbf{0.1}$	2.7×10^{-3}	1.4×10^{-5}	3.2×10^{-6}	4.5×10^{-23}
$\mathbf{0.2}$	2.6×10^{-3}	1.3×10^{-5}	1.4×10^{-6}	1.8×10^{-22}
$\mathbf{0.3}$	2.6×10^{-3}	1.4×10^{-5}	7.6×10^{-7}	3.9×10^{-22}
$\mathbf{0.4}$	2.8×10^{-3}	1.5×10^{-5}	4.8×10^{-7}	6.7×10^{-22}
$\mathbf{0.5}$	3.2×10^{-3}	1.7×10^{-5}	3.5×10^{-7}	9.7×10^{-22}
$\mathbf{0.6}$	3.7×10^{-3}	1.8×10^{-5}	2.4×10^{-7}	1.2×10^{-22}
$\mathbf{0.7}$	4.6×10^{-3}	2.0×10^{-5}	1.2×10^{-7}	1.4×10^{-22}
$\mathbf{0.8}$	5.9×10^{-3}	2.2×10^{-5}	1.7×10^{-8}	1.4×10^{-22}
$\mathbf{0.9}$	7.7×10^{-3}	2.3×10^{-5}	1.1×10^{-7}	1.0×10^{-22}

Table 3.1 (e) Comparison with the other methods for Problem 3.1 using 10 iterations.

Problem 3.2 Solve the third order nonlinear equation

$$u'''(t) + 2e^{-3u(t)} = \frac{4}{(1+t)^3}, \quad 0 \le t \le 1, \tag{3.201}$$

with boundary conditions

$$u(0) = u'(0) = 1, u(1) = ln(2). \tag{3.202}$$

The exact solution is given by

$$u(t) = \ln(1+t). \tag{3.203}$$

First, we find Green's function for $L[u] = u'''$ subject to $u(0) = u(1) = u'(0) = 0$ by applying the properties described before and using *Maple*, we find that

$$G(t|s) = \begin{cases} \left(-\frac{1}{2}s^2 + s\right)t^2 - ts + \frac{s^2}{2}, & 0 < t < s \\ \left(-\frac{1}{2}s^2 + s - \frac{1}{2}\right)t^2, & s < t < 1 \end{cases}. \tag{3.204}$$

Next, we apply the method as described in Section 3.4, where u_0 is the solution of $L[u] \equiv u''' = 0$ subject to $u(0) = u'(0) = 1, u(1) = ln(2)$. We get the following iterative algorithm:

$$\begin{aligned} u_0 &= t + (\ln(2) - 1)t^2, \\ u_{n+1} &= u_n + \int_0^t \left(\left(-\frac{1}{2}s^2 + s\right)t^2 - ts + \frac{s^2}{2}\right)\left(u_n'''(s) + 2e^{-3u_n(s)} - \frac{4}{(1+s)^3}\right) ds \\ &\quad + \int_t^1 \left(\left(-\frac{1}{2}s^2 + s - \frac{1}{2}\right)t^2\right)\left(u_n'''(s) + 2e^{-3u_n(s)} - \frac{4}{(1+s)^3}\right) ds. \end{aligned} \tag{3.205}$$

The comparison of the absolute error values between the method developed in this section and those in references [42] and [43] are shown in Table 3.2(a). The results show that our present method is much better and gives more accurate results using only 31 iterations.

The absolute errors for our numerical solution are shown in Table 3.2 (c) and depicted in Figures 3.2 (a), (b) and (c). Note that the numerical result is highly accurate. The K-M iteration is not reported since it did not show any improvement of the error as compared with the Picard's iteration.

t	Khan and Aziz [43]	HPM and RKM [42]	Present Method
0.0	0	0	0
0.1	5.6×10^{-6}	3.1×10^{-7}	1.4×10^{-53}
0.2	9.5×10^{-6}	1.6×10^{-7}	1.0×10^{-52}
0.3	3.2×10^{-6}	1.3×10^{-7}	3.3×10^{-52}
0.4	1.6×10^{-5}	3.7×10^{-7}	6.4×10^{-52}
0.5	2.9×10^{-6}	4.8×10^{-7}	9.4×10^{-52}
0.6	2.9×10^{-5}	4.5×10^{-7}	1.2×10^{-51}
0.7	1.3×10^{-5}	3.3×10^{-7}	1.3×10^{-51}
0.8	5.1×10^{-6}	1.8×10^{-7}	1.4×10^{-51}
0.9	–	6.9×10^{-7}	7.3×10^{-51}
1.0	0	0	4.5×10^{-78}

Table 3.2 (a) Comparison with other methods for Problem 3.1 using 31 iterations.

Number of Iterations	Maximum Error
2	1.6×10^{-5}
4	1.1×10^{-8}
6	7.3×10^{-12}
8	5.0×10^{-15}
10	3.3×10^{-18}
12	2.2×10^{-21}
16	9.6×10^{-28}
21	4.9×10^{-36}
26	1.8×10^{-40}
31	1.2×10^{-51}

Table 3.2 (b) Maximum error of the present method for various iterations

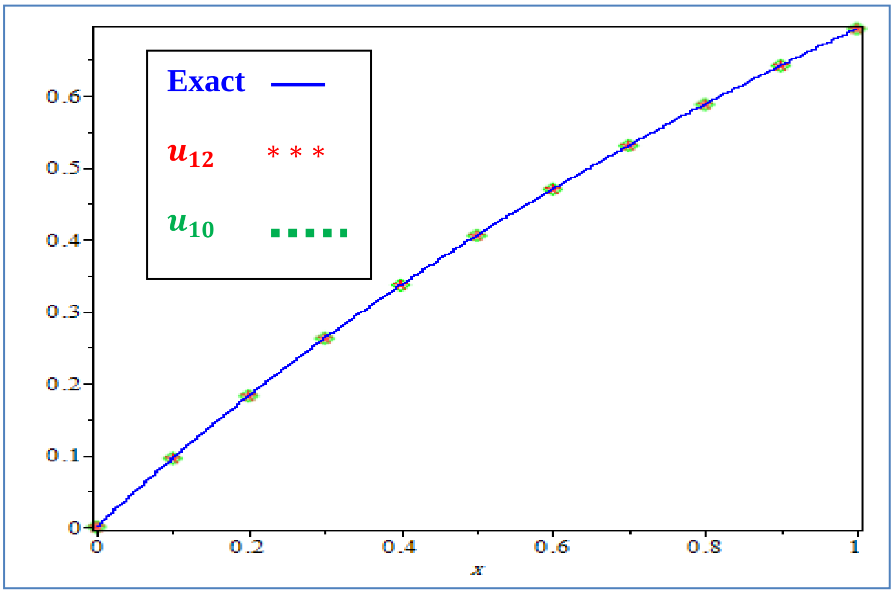

Figure 3.2 (a) Exact solution versus numerical solutions using 10 and 12 iterations

t	Exact Solution	Numerical Solution	Error
0.0	0	0	0
0.1	0.0953101798043248600439521232808	0.0953101798043248600440463833557	9.4×10^{-23}
0.2	0.182321556793954626211718025155	0.182321556793954626212081386422	3.6×10^{-22}
0.3	0.262364264467491052035495986881	0.262364264467491052036270847385	7.7×10^{-22}
0.4	0.336472236621212930504593410217	0.336472236621212930505857497638	1.3×10^{-21}
0.5	0.405465108108164381978013115464	0.405465108108164381979750373379	1.7×10^{-21}
0.6	0.470003629245735553650937031148	0.470003629245735553653018696893	2.1×10^{-21}
0.7	0.530628251062170396231543163189	0.530628251062170396233722122096	2.2×10^{-21}
0.8	0.587786664902119008189731140619	0.587786664902119008191650076754	1.9×10^{-21}
0.9	0.641853886172394775991035977203	0.641853886172394775992248067411	1.2×10^{-21}
1.0	0.693147180559945309417232121458	0.693147180559945309417232120582	8.8×10^{-28}

Table 3.2 (c) Comparison between the exact and numerical solutions using 12 iterations.

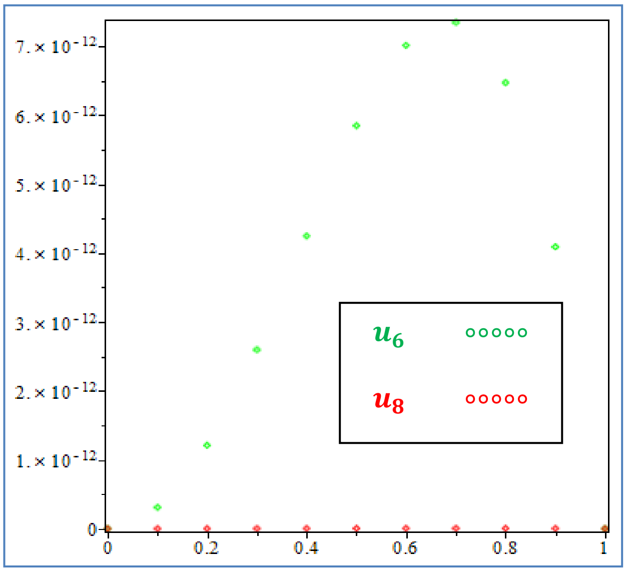

Figure 3.2 (b) Absolute errors using 6 and 8 iterations.

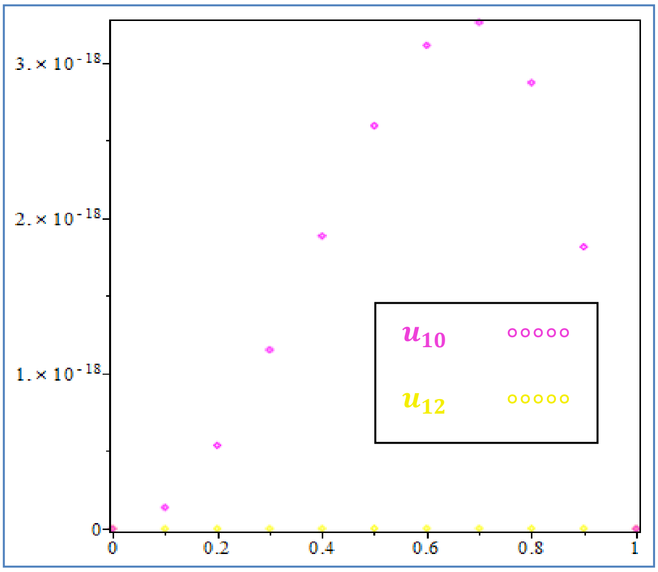

Figure 3.2 (c) Absolute errors using 10 and 12 iterations.

Problem 3.3 Consider the following third order nonlinear (BVP)

$$u'''(t) - u''(t) = 4e^{-2t}u^2(t), \tag{3.206}$$

with boundary conditions

$$u(0) = 1, u'(0) = 2, u(1) = e^2. \tag{3.207}$$

The exact solution is $u(t) = e^{2t}$.

Applying the properties described before, the operator $L[y] = u''' - u''$ on $(0,1)$ has the following Green's function

$$G(t|s) = \begin{cases} \dfrac{ee^s s - ee^s - e^s s + e}{e^s(e-2)} - \dfrac{ee^s - e^s s - e}{e^s(e-2)}t + \dfrac{e^s s - 2e^s + 2}{e^s(e-2)}e^t, & 0 < t < s \\ \dfrac{e^s + e - 2e^s}{e^s(e-2)}(1 + t - e^t), & s < t < 1 \end{cases}. \tag{3.208}$$

We apply Picard's fixed point iteration, where u_0 is the solution of $u''' - u'' = 0$ subject to

$$u(0) = 1, u'(0) = 2, u(1) = e^2.$$

Therefore, the iterative scheme becomes:

$$u_0 = \frac{-e - 1 + e^2}{e - 2} - \frac{-2e + 1 + e^2}{e - 2}t + \frac{-3 + e^2}{e - 2}e^t,$$

$$u_{n+1} = u_n + \int_0^t \left(\frac{ee^s s - ee^s - e^s s + e}{e^s(e-2)} - \frac{ee^s - e^s s - e}{e^s(e-2)} t + \frac{e^s s - 2e^s + 2}{e^s(e-2)} e^t \right) \left(u_n'''(s) - u_n''(s) - 4e^{-2s} u_n^{\,2}(s) \right) ds + \int_t^1 \left(\frac{e^s + e - 2e^s}{e^s(e-2)} (1 + t - e^t) \right) \left(u_n'''(s) - u_n''(s) - 4e^{-2s} u_n^{\,2}(s) \right) ds. \tag{3.29}$$

In this problem, the Krasnoselskii–Mann iteration method yields better approximate solutions than Picard's iterations method. Table 3.3 (c) and Figure 3.3 (b) demonstrate that the matching between the approximate and exact solution is better for $\alpha = 0.94$ rather than $\alpha = 1$. From the numerical results in Table 3.3 (a) and Figure 3.3 (a) we notice that the method yields very accurate approximate solutions. The maximum errors for Mann's iterative method using $\alpha = 0.94$ for certain iterations are reported in Table 3.3 (b).

t	Exact Solution	Numerical Solution	Error
0.0	1	0.99999999999999999999996390425574351	3.6×10^{-23}
0.1	1.22140275816016983392107199463967 42	1.22140275816016983391804525627550 77	3.0×10^{-21}
0.2	1.49182469764127031782485295283722 23	1.49182469764127031781770989503435 30	7.1×10^{-21}
0.3	1.82211880039050897487536766816286 45	1.82211880039050897486599992874801 98	9.4×10^{-21}
0.4	2.22554092849246760457953753139507 68	2.22554092849246760456688499660060 17	1.3×10^{-20}
0.5	2.71828182845904523536028747135266 25	2.71828182845904523534748561056954 40	1.3×10^{-20}
0.6	3.32011692273654748953076742960164 43	3.32011692273654748951993558312085 24	1.1×10^{-20}
0.7	4.05519996684467458722410889522862 03	4.05519996684467458721552088615741 96	8.6×10^{-21}
0.8	4.95303242439511480365428635642396 43	4.95303242439511480364845761651990 68	5.8×10^{-21}
0.9	6.04964746441294608373102395302772 53	6.04964746441294608372843053141263 19	2.6×10^{-21}
1.0	7.38905609893065022723042746057500 78	7.38905609893065022723024338995037 34	1.8×10^{-22}

Table 3.3 (a) Comparison between the exact and numerical solutions for Problem 3.3 using 15 iterations.

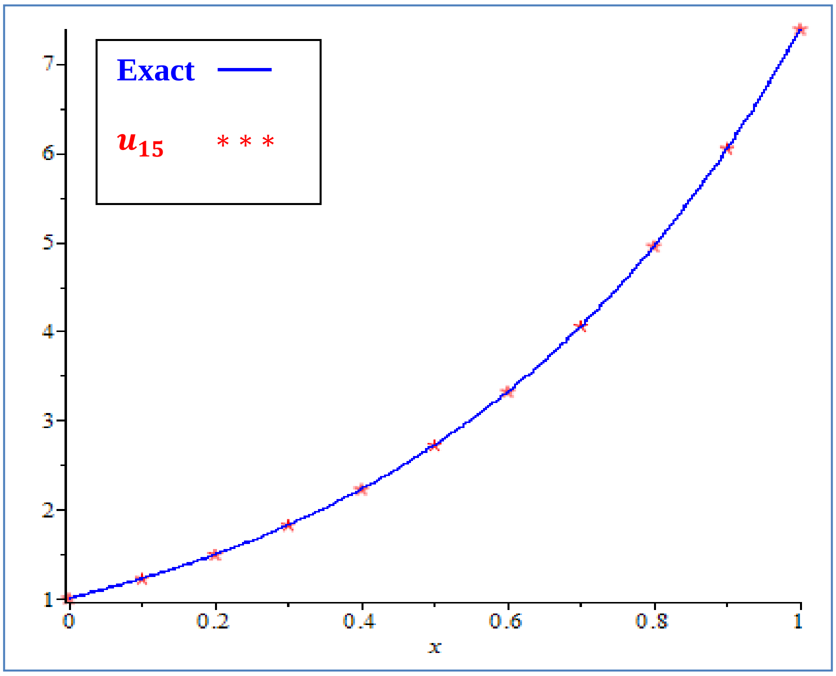

Figure 3.3 (a) Exact solution versus numerical solution using 15 iterations.

Number of iterations	Max. Error
5	2.4×10^{-8}
6	1.1×10^{-9}
7	3.5×10^{-11}
8	2.3×10^{-12}
9	6.0×10^{-14}
10	5.2×10^{-15}
11	1.7×10^{-16}
13	4.9×10^{-18}
15	1.3×10^{-20}

Table 3.3 (b) Maximum error arising from Mann's iterative method when $\alpha = 0.94$ for various iterations.

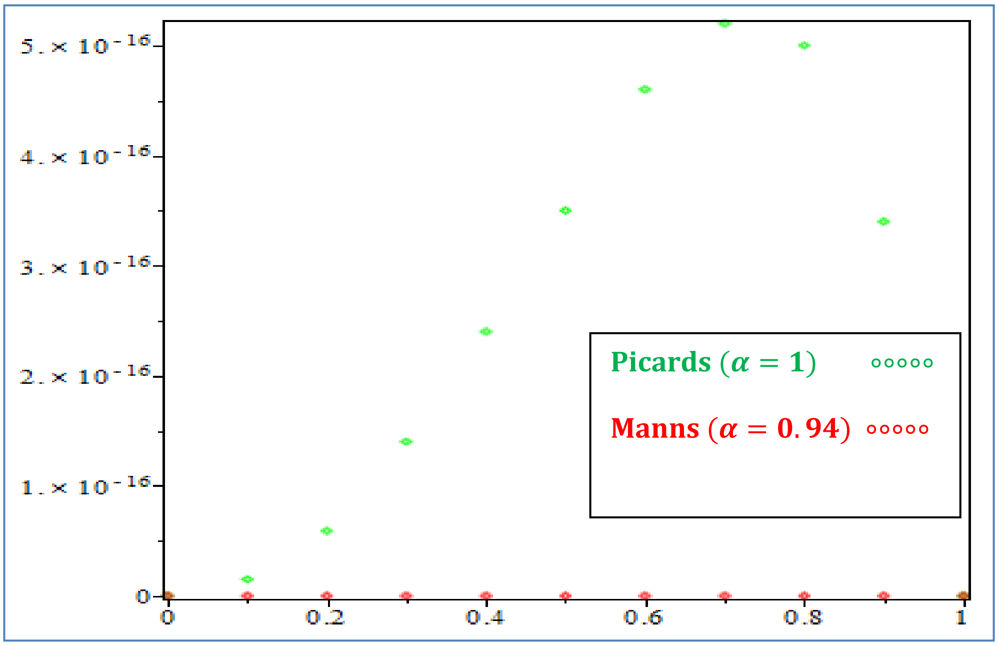

Figure 3.3 (b) Mann's iteration method with $\alpha = 0.94$ and Picard's iterative method using 15 iterations.

t	PICARD's	MANN's		
	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.92$	$\alpha = 0.94$
0.0	2.5×10^{-25}	6.9×10^{-23}	3.1×10^{-24}	3.6×10^{-23}
0.1	1.5×10^{-17}	9.0×10^{-19}	2.4×10^{-20}	3.0×10^{-21}
0.2	5.9×10^{-17}	1.5×10^{-18}	3.3×10^{-20}	7.1×10^{-21}
0.3	1.4×10^{-16}	3.8×10^{-19}	1.3×10^{-20}	9.4×10^{-21}
0.4	2.4×10^{-16}	1.6×10^{-18}	7.0×10^{-20}	1.3×10^{-20}
0.5	3.5×10^{-16}	2.6×10^{-18}	8.0×10^{-20}	1.3×10^{-20}
0.6	4.6×10^{-16}	1.4×10^{-18}	2.1×10^{-20}	1.1×10^{-20}
0.7	5.2×10^{-16}	1.8×10^{-18}	8.0×10^{-20}	8.6×10^{-21}
0.8	5.0×10^{-16}	5.6×10^{-18}	1.7×10^{-19}	5.8×10^{-21}
0.9	3.4×10^{-16}	7.2×10^{-18}	2.0×10^{-19}	2.6×10^{-21}
1.0	6.1×10^{-23}	4.1×10^{-23}	7.1×10^{-22}	1.8×10^{-22}

Table 3.3 (c) Comparison between Picard's and Mann's iteration method for certain values of $\boldsymbol{\alpha}$.

Problem 3.4 Consider the following third order nonlinear (BVP)

$$u'''(t) = e^{-2t}u^3(t), \tag{3.210}$$

with boundary conditions

$$u(0) = u'(0) = 1, u(1) = e. \tag{3.211}$$

The exact solution is given by $u(t) = e^t$.

Since the linear operator $L[u] = u'''$ and the interval $0 \leq t \leq 1$ are the same as in Problems 3.1 and 3.2, thus Green's function will be the same, which is

$$G(t|s) = \begin{cases} \left(-\frac{1}{2}s^2 + s\right)t^2 - ts + \frac{s^2}{2}, & 0 < t < s \\ \left(-\frac{1}{2}s^2 + s - \frac{1}{2}\right)t^2, & s < t < 1 \end{cases}. \tag{3.212}$$

Applying Picard's fixed point iteration, then we have the following iterative algorithm:

$$\begin{aligned} u_0 &= 1 + t + (e-2)t^2, \\ u_{n+1} &= u_n + \int_0^t \left(\left(-\frac{1}{2}s^2 + s\right)t^2 - ts + \frac{s^2}{2}\right)\left(u_n'''(s) - e^{-2t}u_n^{\,3}(s)\right)ds \\ &\quad + \int_t^1 \left(\left(-\frac{1}{2}s^2 + s - \frac{1}{2}\right)t^2\right)\left(u_n'''(s) - e^{-2t}u_n^{\,3}(s)\right)ds, \end{aligned} \tag{3.213}$$

where u_0 is the solution of $u''' = 0$ subject to $u(0) = u'(0) = 1, u(1) = e$.

The K-M iteration for this problem has no affect in improving the error. The numerical results are reported in Table 3.4 (a) and Figure 3.4 (a), which show that our method is accurate and efficient. The maximum errors for certain iterations are reported in Table 3.4 (b). Moreover, Table 3.4 (c) shows the absolute error obtained for some iterations, while Figure 3.4 (b) shows a comparison of the absolute error between the 13th and 15th iterations. Further accuracy may be achieved using more iterations.

t	Exact Solutions	Numerical Solutions	Errors
0.0	1	1.000000000000000000000000003	3.0×10^{-27}
0.1	1.10517091807564762481170782649	1.105170918075647624811706467	1.4×10^{-24}
0.2	1.22140275816016983392107199464	1.221402758160169833921066580	5.4×10^{-24}
0.3	1.34985880757600310398374431333	1.3498588075760031039837324340	1.2×10^{-23}
0.4	1.49182469764127031782485295284	1.4918246976412703178248328408	2.0×10^{-23}
0.5	1.64872127070012814684865078781	1.648721270700128146848621880	2.9×10^{-23}
0.6	1.82211880039050897487536766816	1.822118800390508974875331282	3.6×10^{-23}
0.7	2.01375270747047652162454938858	2.0137527074704765216245092865	4.0×10^{-23}
0.8	2.22554092849246760457953753140	2.2255409284924676045795004075	3.7×10^{-21}
0.9	2.45960311115694966380012656360	2.459603111156949663800102027	2.5×10^{-23}
1.0	2.71828182845904523536028747135	2.718281828459045235360287471	0

Table 3.4 (a) Comparison between the exact and the numerical solution for Problem 3.4 using 15 iterations .

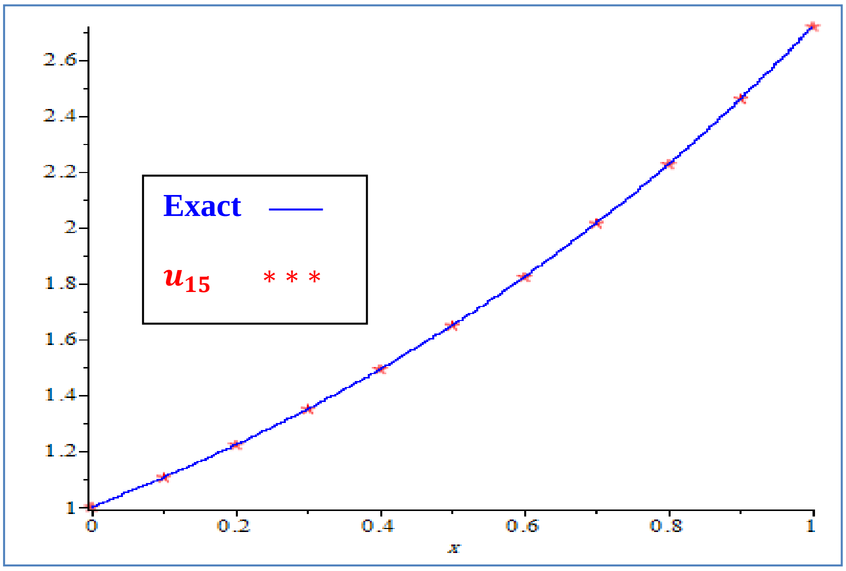

Figure 3.4 (a) Exact solution versus numerical solution using 15 iterations.

Number of Iterations	Maximum Error
5	4.3×10^{-9}
7	6.7×10^{-12}
9	1.0×10^{-14}
11	1.6×10^{-17}
13	2.6×10^{-20}
15	4.0×10^{-23}

Table 3.4 (b) Maximum error of the present method for various iterations.

t	u_7	u_9	u_{11}	u_{13}	u_{15}
0.0	2.0×10^{-32}	1.9×10^{-32}	0	3.0×10^{-27}	3.0×10^{-27}
0.1	1.4×10^{-10}	3.6×10^{-16}	5.6×10^{-19}	8.7×10^{-22}	1.4×10^{-24}
0.2	5.8×10^{-10}	1.4×10^{-15}	2.2×10^{-18}	3.5×10^{-21}	5.4×10^{-24}
0.3	1.3×10^{-9}	3.1×10^{-15}	4.9×10^{-18}	7.6×10^{-21}	1.2×10^{-23}
0.4	2.1×10^{-9}	5.3×10^{-15}	8.2×10^{-18}	1.3×10^{-20}	2.0×10^{-23}
0.5	3.1×10^{-9}	7.6×10^{-15}	1.2×10^{-17}	1.8×10^{-20}	2.9×10^{-23}
0.6	3.9×10^{-9}	9.5×10^{-15}	1.5×10^{-17}	2.3×10^{-20}	3.6×10^{-23}
0.7	4.3×10^{-9}	1.0×10^{-14}	1.6×10^{-17}	2.6×10^{-20}	4.0×10^{-23}
0.8	4.0×10^{-9}	9.7×10^{-15}	1.5×10^{-17}	2.4×10^{-20}	3.7×10^{-21}
0.9	2.6×10^{-9}	6.4×10^{-15}	1.0×10^{-17}	1.6×10^{-20}	2.5×10^{-23}
1.0	0	3.5×10^{-33}	7.5×10^{-33}	0	0

Table 3.4 (c) Comparison of the numerical solutions using 7, 9, 11, 13 and 15 iterations.

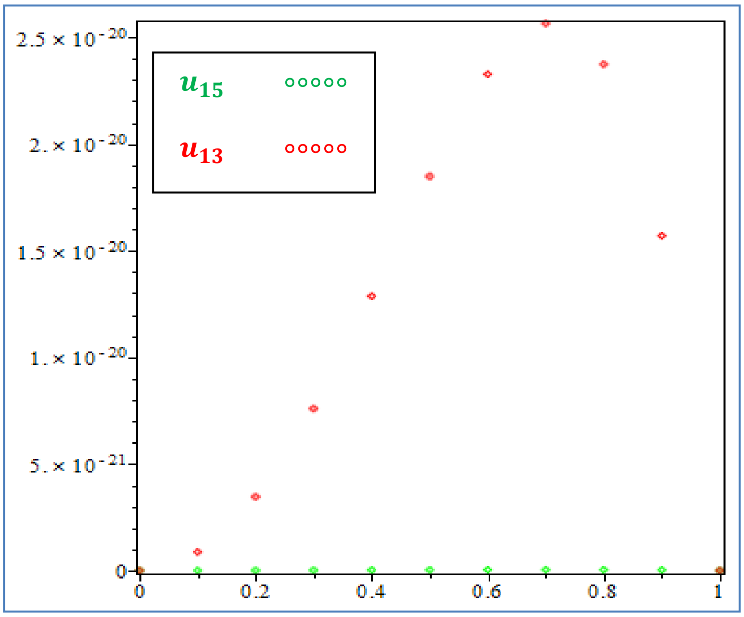

Figure 3.4 (b) Comparison of the absolute errors using 13 and 15 iterations.

Problem 3.5 Consider the following third order nonlinear (BVP)

$$u'''(t) + u^2(t) = e^t\left(3 + t(5 + t + te^t(t-1)^2)\right), \tag{3.214}$$

with boundary conditions

$$u(0) = 0, u'(0) = -1, u(1) = 0, \tag{3.215}$$

whose exact solution is given by $u(t) = t(t-1)e^t$.

Since the linear operator $L[u] = u'''$ and the interval $0 \leq t \leq 1$ are the same as in Problems 3.1, 3.2 and 3.4, thus Green's function is given by

$$G(t|s) = \begin{cases} \left(-\frac{1}{2}s^2 + s\right)t^2 - ts + \frac{s^2}{2}, & 0 < t < s \\ \left(-\frac{1}{2}s^2 + s - \frac{1}{2}\right)t^2, & s < t < 1 \end{cases}. \tag{3.216}$$

Applying Picard's fixed point iteration, we have the following iterative algorithm:

$$u_0 = t^2 + t,$$

$$
\begin{aligned}
u_{n+1} = u_n + \int_0^t & \left(\left(-\frac{1}{2}s^2 + s\right)t^2 - ts + \frac{s^2}{2}\right)\Big(u_n{}'''(s) + u_n{}^2(s) \\
& - e^s\big(3 + s(5 + s + se^s(s-1)^2)\big)\Big)\, ds \\
+ \int_t^1 & \left(\left(-\frac{1}{2}s^2 + s - \frac{1}{2}\right)t^2\right)\Big(u_n{}'''(s) + u_n{}^2(s) \\
& - e^s\big(3 + s(5 + s + se^s(s-1)^2)\big)\Big)\, ds,
\end{aligned}
\tag{3.317}
$$

where u_0 is the solution of $u''' = 0$ subject to $u(0) = 0,\ u'(0) = -1,\ u(1) = 0.$

The comparison of the absolute errors between our method and that of [42] is shown in Table 3.5 (b). The numerical results using 15 iterations are presented in Table 3.5 (a) and Figure 3.5 (a). It is clear from both of them the high accuracy and fast convergence of the method. In Table 3.5 (c), we introduce the maximum absolute error for different iterations. In addition, in Figure 3.5 (b) we compare the numerical results between those arising from the fifth iteration and the fifteenth iteration. As in Problem 3.4, the K-M iteration for this problem does not affect the accuracy or rate of convergence.

t	**Numerical Solution**	**Error**
$\mathbf{0.0}$	0	0
$\mathbf{0.1}$	0.0994653826268082862330537043841	5.0×10^{-33}
$\mathbf{0.2}$	0.195424441305627173427371519142	2.0×10^{-32}
$\mathbf{0.3}$	0.283470349590960651836586305798	4.2×10^{-32}
$\mathbf{0.4}$	0.358037927433904876277964708681	7.2×10^{-32}
$\mathbf{0.5}$	0.412180317675032036712162696953	1.1×10^{-31}
$\mathbf{0.6}$	0.437308512093722153970088240359	1.4×10^{-31}
$\mathbf{0.7}$	0.422888068568800069541155371602	1.5×10^{-31}
$\mathbf{0.8}$	0.356086548558794816732726005023	1.4×10^{-21}
$\mathbf{0.9}$	0.221364280004125469742011390724	9.3×10^{-32}
$\mathbf{1.0}$	4.0×10^{-35}	2.0×10^{-35}

Table 3.5 (a) Numerical solutions and absolute errors for Problem 3.5 using 15 iterations.

t	HPM and RKM $\lvert U_{50,3} - u \rvert$ [42]	HPM and RKM $\lvert U_{70,3} - u \rvert$ [42]	Present Method u_{15}
0.0	$-$	$-$	0
0.1	1.8×10^{-6}	7.8×10^{-7}	5.0×10^{-33}
0.2	4.9×10^{-6}	2.2×10^{-6}	2.0×10^{-32}
0.3	8.8×10^{-6}	4.1×10^{-6}	4.2×10^{-32}
0.4	1.3×10^{-5}	6.1×10^{-6}	7.2×10^{-32}
0.5	1.7×10^{-5}	8.0×10^{-6}	1.1×10^{-31}
0.6	1.9×10^{-5}	9.2×10^{-6}	1.4×10^{-31}
0.7	2.0×10^{-5}	9.6×10^{-6}	1.5×10^{-31}
0.8	1.7×10^{-5}	8.5×10^{-6}	1.4×10^{-21}
0.9	1.1×10^{-5}	5.5×10^{-6}	9.3×10^{-32}
1.0	$-$	$-$	2.0×10^{-35}

Table 3.5 (b) Comparison with other numerical methods for Problem 3.5.

Number of Iterations	Maximum Error
5	1.7×10^{-11}
6	1.7×10^{-13}
7	1.7×10^{-15}
8	1.7×10^{-17}
9	1.6×10^{-19}
10	1.6×10^{-21}
15	1.7×10^{-31}

Table 3.5 (c) Maximum Error of our method for certain iterations.

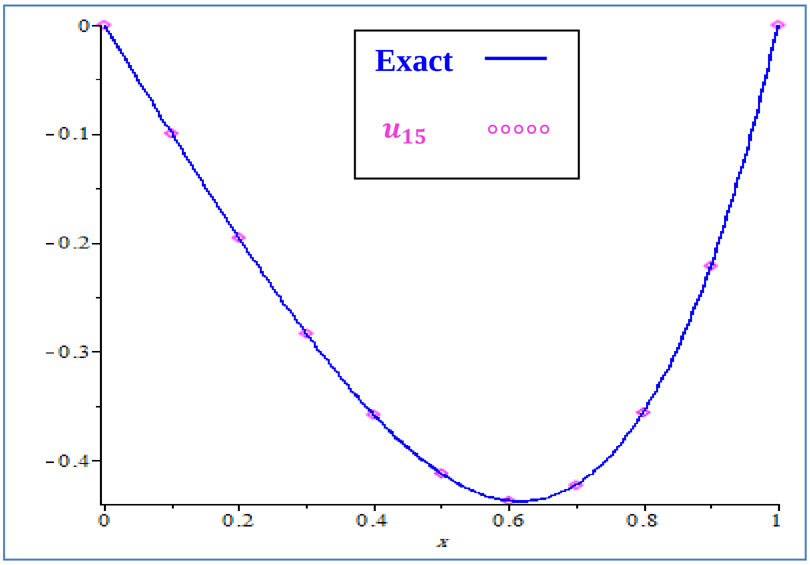

Figure 3.5 (a) Exact solution versus numerical solutions of Problem 3.5 using 15 iterations.

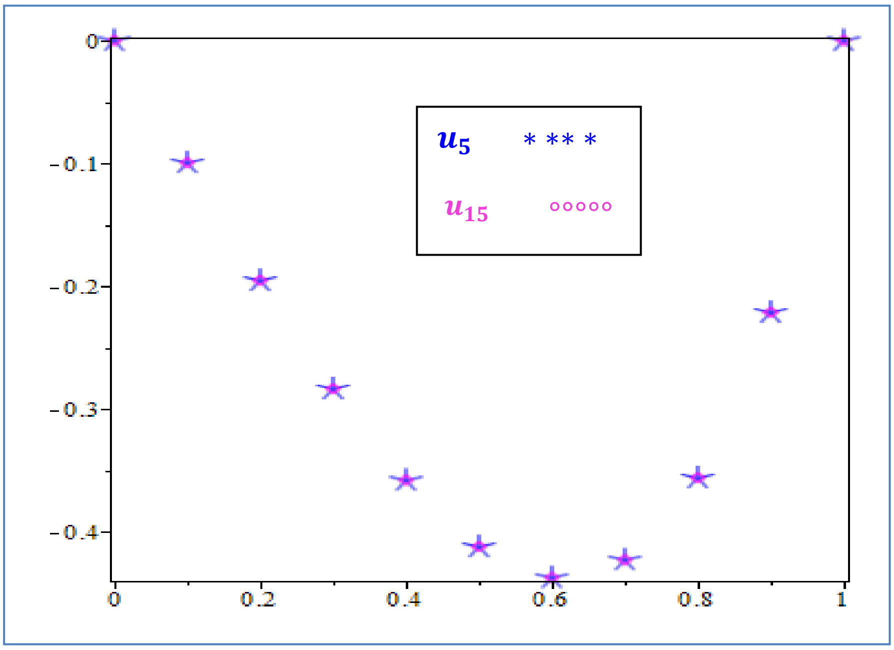

Figure 3.5 (b) Numerical solutions for Problem 3.5 using 5 and 15 iterations.

Problem 3.6 Consider the following third order nonlinear

$$u'''(t) - u'(t) - u(t)u''(t) + u^2(t) = 2t + t^2 - te^{t-1}(2e^{t-1} + t - 1) \qquad (3.318)$$

with boundary conditions

$$u(0) = 1, u(1) = -1, u'(1) = 1, \qquad (3.319)$$

whose exact solution is given by

$$u(t) = t + 1 - te^{t-1}. \tag{3.320}$$

Applying the properties described before, the operator $L[y] = u''' - u'$ on (0,1), has the following Green's function

$$G(t|s) = \begin{cases} \frac{-(e^{2s} - 2e^s + 1)e}{e^s(e^2 - 2e + 1)} + \frac{e^{2s} - 2e^s + 1}{2e^s(e^2 - 2e + 1)}e^t + \frac{(e^{2s} - 2e^s + 1)e^2}{2e^s(e^2 - 2e + 1)}e^{-t}, & 0 < t < s \\ \frac{e^2e^s - ee^{2s} - e + e^s}{e^s(e^2 - 2e + 1)} - \frac{e^2 - e^{2s} + 2e^s - 2e}{2e^s(e^2 - 2e + 1)}e^t - \frac{2e^2e^s - 2ee^{2s} - e^2 + e^{2s}}{2e^s(e^2 - 2e + 1)}e^{-t}, & s < t < 1 \end{cases}. \tag{3.321}$$

We apply Picard's fixed point iteration. Hence, the resulting fixed point iterative scheme reads:

$$\begin{aligned} u_0 &= \frac{2e(e^{-1} - 1)}{2e^{-1}e - e^{-1} - e} - \frac{e^{-1} - 1}{2e^{-1}e - e^{-1} - e}e^t + \frac{e - 1}{2e^{-1}e - e^{-1} - e}e^{-t}, \\ u_{n+1} &= u_n + \int_0^t \left(\frac{-(e^{2s} - 2e^s + 1)e}{e^s(e^2 - 2e + 1)} + \frac{e^{2s} - 2e^s + 1}{2e^s(e^2 - 2e + 1)}e^t \right. \\ &\quad \left. + \frac{(e^{2s} - 2e^s + 1)e^2}{2e^s(e^2 - 2e + 1)}e^{-t}\right)\big(u_n{'''}(s) - u_n{'}(s) - u_n(s)u_n{''}(s) \\ &\quad + u_n{}^2(s) - 2s - s^2 + se^{s-1}(2e^{s-1} + s - 1)\big)ds \\ &\quad + \int_t^1 \left(\frac{e^2e^s - ee^{2s} - e + e^s}{e^s(e^2 - 2e + 1)} - \frac{e^2 - e^{2s} + 2e^s - 2e}{2e^s(e^2 - 2e + 1)}e^t \right. \\ &\quad \left. - \frac{2e^2e^s - 2ee^{2s} - e^2 + e^{2s}}{2e^s(e^2 - 2e + 1)}e^{-t}\right)\big(u_n{'''}(s) - u_n{'}(s) \\ &\quad - u_n(s)u_n{''}(s) + u_n{}^2(s) - 2s - s^2 + se^{s-1}(2e^{s-1} + s - 1)\big)ds, \end{aligned} \tag{3.322}$$

where u_0 is the solution of $u''' - u'' = 0$ subject to $u(0) = 1, u'(0) = 2, u(1) = e^2$.

Table 3.6 (a) and Figure 3.6 (a) show the numerical results and the errors obtained by using the proposed algorithm with 20 iterations. Again in this problem, the Krasnoselskii–Mann iteration method gives better approximate solutions than the Picard's iterations method. Table 3.6 (b) and Figure 3.6 (b) show that the matching of the numerical solution with the exact solution is better for $\alpha = 0.93$ than for $\alpha = 1$.Table 3.6 (c) shows the maximum error of the Mann's iterative method when $\alpha = 0.93$ for some selected iterations. Examining that Table, it is clear that the absolute errors values are relatively very small. Higher accuracy can be achieved by taking and evaluating more iterates.

t	Exact Solution	Numerical Solution	Error
0.0	1	1.00000000000000000000000002178635023	4.2×10^{-25}
0.1	1.0593430340259400888116545760354374	1.0593430340259400890306879080948485	2.2×10^{-19}
0.2	1.1101342071765556817139795229968874	1.1101342071765556819868868250392407	2.7×10^{-19}
0.3	1.1510244088625771455885599719807413	1.1510244088625771457947078894641595	2.1×10^{-19}
0.4	1.1804753455623894269486164331069728	1.1804753455623894270416613903825568	9.3×10^{-20}
0.5	1.1967346701436832881981002325044098	1.1967346701436832881956770992810634	2.4×10^{-21}
0.6	1.1978079723786164195533402449113044	1.1978079723786164195036626018159413	5.0×10^{-20}
0.7	1.1814272455227974937531883544775282	1.1814272455227974937013901240971390	5.2×10^{-20}
0.8	1.1450153975376145130640515931047685	1.1450153975376145130336090039163179	3.0×10^{-20}
0.9	1.0856463237676363841521758464982070	1.0856463237676363841436536643266162	8.5×10^{-21}
1.0	1	0.99999999999999999999999903701671267	9.6×10^{-25}

Table 3.6 (a) Comparison between the exact and the numerical solutions for Problem 3.6 using 20 iterations

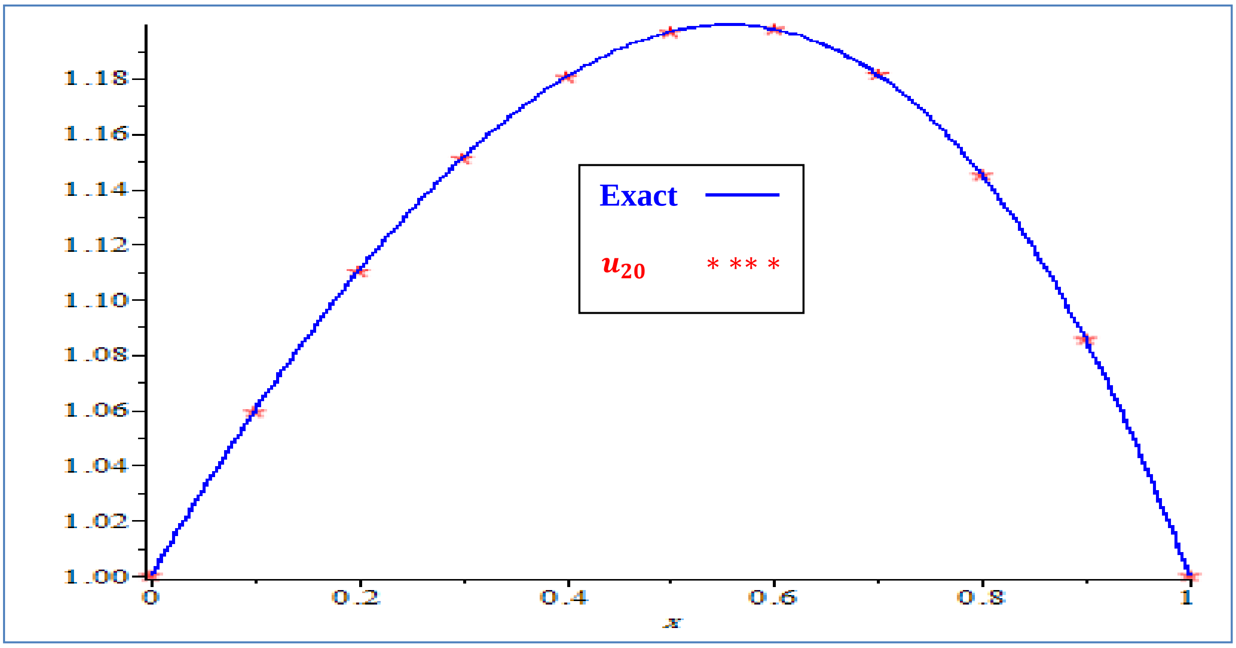

Figure 3.6 (a) Exact solution and numerical solutions for Problem 3.6 using 20 iterations.

t	PICARD's	MANN's			
	$\alpha = 1$	$\alpha = 0.97$	$\alpha = 0.95$	$\alpha = 0.93$	$\alpha = 0.9$
0.0	2.8×10^{-24}	2.3×10^{-25}	2.2×10^{-24}	4.2×10^{-25}	4.6×10^{-25}
0.1	4.0×10^{-18}	5.3×10^{-19}	3.0×10^{-19}	2.2×10^{-19}	8.5×10^{-20}
0.2	5.5×10^{-18}	5.4×10^{-19}	3.6×10^{-19}	2.7×10^{-19}	2.7×10^{-19}
0.3	5.2×10^{-18}	2.6×10^{-19}	2.5×10^{-19}	2.1×10^{-19}	4.1×10^{-19}
0.4	3.3×10^{-18}	6.8×10^{-20}	8.9×10^{-20}	9.3×10^{-20}	4.4×10^{-19}
0.5	1.4×10^{-18}	2.7×10^{-19}	3.7×10^{-21}	2.4×10^{-21}	3.7×10^{-19}
0.6	1.3×10^{-19}	3.1×10^{-19}	9.1×10^{-20}	5.0×10^{-20}	2.5×10^{-19}
0.7	3.8×10^{-19}	2.3×10^{-19}	8.3×10^{-20}	5.2×10^{-20}	1.3×10^{-19}
0.8	3.4×10^{-19}	1.2×10^{-19}	4.7×10^{-20}	3.0×10^{-20}	4.7×10^{-20}
0.9	1.1×10^{-19}	3.0×10^{-20}	1.3×10^{-20}	8.5×10^{-21}	7.7×10^{-21}
1.0	9.2×10^{-23}	1.3×10^{-23}	3.8×10^{-23}	9.6×10^{-25}	2.3×10^{-24}

Table 3.6 (b) Comparison between of Picard's and Mann's iteration methods.

Number of Iterations	Maximum Error
5	2.1×10^{-6}
10	1.5×10^{-10}
15	4.6×10^{-15}
20	2.7×10^{-19}

Table 3.6 (c) Maximum error of the Mann's iterative method when $\alpha = 0.93$ for some selected iterations.

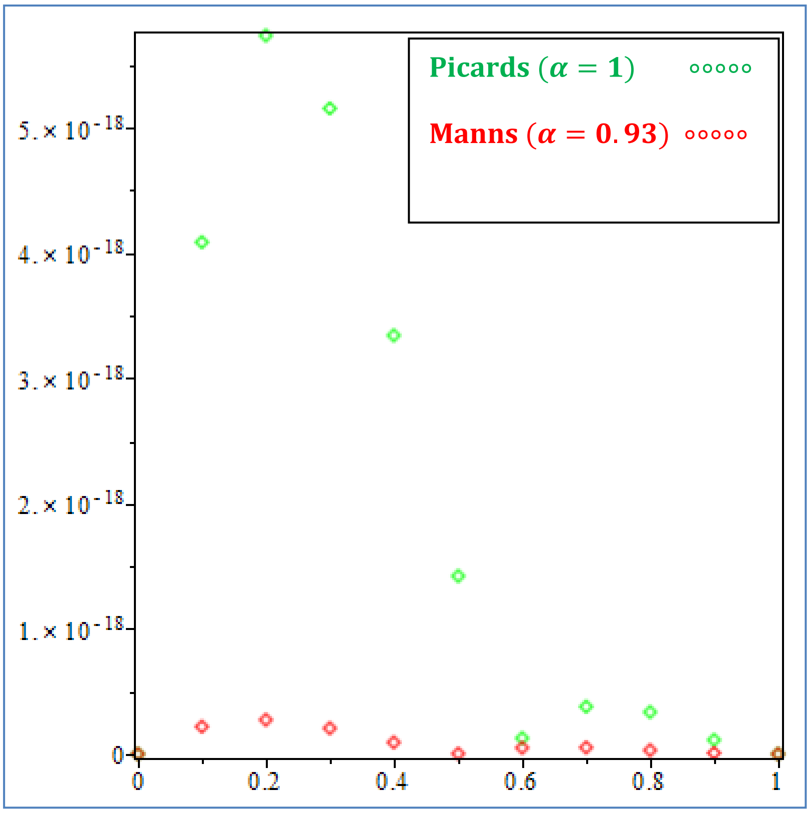

Figure 3.6 (b) Mann's iteration method with $\alpha = 0.93$ and Picard's iterative method for Problem 3.6 using 20 iterations.

Problem 3.7 Consider the following third order linear (BVP)

$$u'''(t) = tu + e^t(-3 - 5t - 2t^2 + t^3), \quad 0 \le t \le 1, \tag{3.323}$$

with boundary conditions

$$u(0) = 0, u''(0) = 0, u(1) = 0, \tag{3.324}$$

whose exact solution is given by

$$u(t) = t(1 - t)e^t. \tag{3.325}$$

We find Green's function for $L[u] = u'''$ subject to $u(0) = u(1) = u''(0) = 0$ by applying Green's properties. Green's function is found to be

$$G(t|s) = \begin{cases} \frac{1}{2}s^2 + \left(-\frac{1}{2}s^2 - \frac{1}{2}\right)t + \frac{1}{2}t^2, & 0 < t < s \\ \left(-\frac{1}{2}s^2 + s - \frac{1}{2}\right)t, & s < t < 1 \end{cases}. \tag{3.326}$$

Applying Picard's fixed point iteration we have the subsequent iterative scheme:

$$u_0 = 0,$$

$$
\begin{aligned}
u_{n+1} = u_n + \int_0^t \left(\frac{1}{2}s^2 + \left(-\frac{1}{2}s^2 - \frac{1}{2}\right)t + \frac{1}{2}t^2\right)\Big(u_n{}'''(s) - su_n \\
- e^s(-3 - 5s - 2s^2 + s^3)\Big)ds \\
+ \int_t^1 \left(\frac{1}{2}s^2 - 2s + 2 + \left(-\frac{1}{2}s^2 + 2s - 2\right)t\right)\Big(u_n{}'''(s) - su_n \\
- e^s(-3 - 5s - 2s^2 + s^3)\Big)ds, \qquad (3.327)
\end{aligned}
$$

where u_0 is the solution of $u''' = 0$ subject to $u(0) = 0,\ u''(0) = 0,\ u(1) = 0.$

Comparison of errors between our method and the numerical method presented in [44] is demonstrated in Table 3.7 (a), which shows that our method yields better results. Figure 3.7 (a) exhibits the exact and numerical solutions using 30 iterations. In addition, in Table 3.7 (b), we computed the maximum absolute errors for different iterates. From Table 3.7 (c), it is obvious that evaluating more iterates will enhance the numerical solution dramatically. It is important to note that the K-M iteration for this problem has no affect in improving the error.

t	Example 1 in [44]		Present Method
	$h = 0.010$	$h = 0.005$	
0.0	0	0	0
0.1	2.99×10^{-6}	7.67×10^{-7}	1.7×10^{-16}
0.2	5.33×10^{-6}	1.37×10^{-6}	3.4×10^{-16}
0.3	7.97×10^{-6}	1.81×10^{-6}	5.1×10^{-16}
0.4	7.98×10^{-6}	2.08×10^{-6}	6.7×10^{-16}
0.5	8.28×10^{-6}	2.11×10^{-6}	8.0×10^{-16}
0.6	7.89×10^{-6}	2.11×10^{-6}	8.8×10^{-16}
0.7	6.91×10^{-6}	1.87×10^{-6}	8.8×10^{-16}
0.8	5.18×10^{-6}	1.46×10^{-6}	7.6×10^{-16}
0.9	3.15×10^{-6}	1.31×10^{-7}	4.8×10^{-16}
1.0	0	0	0

Table 3.7 (a) Comparison with another method for Problem 3.7 using 8 iterations.

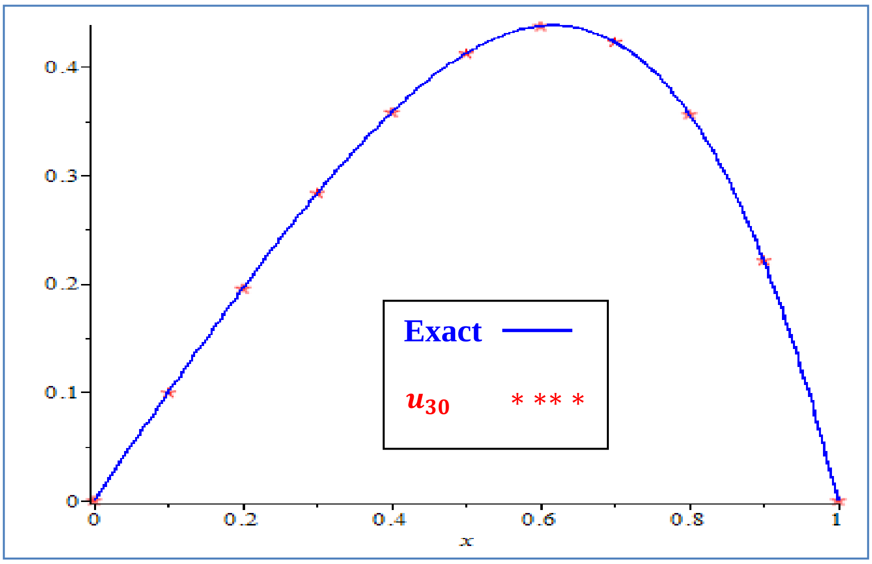

Figure 3.7 (a) Exact versus numerical solution using 30 iterations.

Number of Iterations	Maximum Error
5	3.0×10^{-10}
10	1.8×10^{-19}
15	1.2×10^{-28}
20	7.4×10^{-38}
25	4.7×10^{-47}
30	3.0×10^{-56}

Table 3.7 (b) Maximum error of the present method for certain iterations.

t	10 iterations	15 iterations	20 iterations	25 iterations	30 iterations
0.0	0	0	0	0	0
0.1	3.6×10^{-20}	2.3×10^{-29}	1.4×10^{-38}	9.2×10^{-48}	5.8×10^{-57}
0.2	7.2×10^{-20}	4.6×10^{-29}	2.9×10^{-38}	1.8×10^{-47}	1.2×10^{-56}
0.3	1.1×10^{-19}	6.8×10^{-29}	4.3×10^{-38}	2.7×10^{-47}	1.7×10^{-56}
0.4	1.4×10^{-19}	8.9×10^{-29}	5.6×10^{-38}	3.6×10^{-47}	2.3×10^{-56}
0.5	1.7×10^{-19}	1.1×10^{-28}	6.7×10^{-38}	4.3×10^{-47}	2.7×10^{-56}
0.6	1.8×10^{-19}	1.2×10^{-28}	7.4×10^{-38}	4.7×10^{-47}	3.0×10^{-56}
0.7	1.8×10^{-19}	1.2×10^{-28}	7.4×10^{-38}	$4.7 \times 10^{-4.7}$	3.0×10^{-56}
0.8	1.6×10^{-19}	1.0×10^{-28}	6.4×10^{-38}	4.1×10^{-47}	2.6×10^{-56}
0.9	1.0×10^{-19}	6.4×10^{-29}	4.0×10^{-38}	2.6×10^{-47}	1.6×10^{-56}
1.0	0	4.7×10^{-35}	7.2×10^{-47}	0	2.6×10^{-66}

Table 3.7 (c) Absolute Errors resulting from various iterations.

Problem 3.8 Consider the following third order linear (BVP)

$$u'''(t) = tu(t) + e^t(3 + t - 61t^2 - 60t^3 - 15t^4 - t^5 + t^6), \quad -1 \le t \le 1, \tag{3.328}$$

with boundary conditions

$$u(-1) = 0, \qquad u''(-1) = \frac{12}{e}, \qquad u(1) = 0. \tag{3.329}$$

The exact solution is given by

$$u(t) = t(1 - t^4)e^t. \tag{3.330}$$

We find Green's function for $L[u] = u'''$ subject to $u(0) = u(1) = u''(0) = 0$ by applying Green's function's properties. This yield

$$G(t|s) = \begin{cases} \left(\frac{1}{4}s^2 + \frac{1}{2}s - \frac{1}{4}\right) + \left(-\frac{1}{4}s^2 - \frac{1}{2}s - \frac{1}{4}\right)t + \frac{1}{2}t^2, & -1 < t < s \\ \left(-\frac{1}{4}s^2 + \frac{1}{2}s - \frac{1}{4}\right) + \left(-\frac{1}{4}s^2 + \frac{1}{2}s - \frac{1}{4}\right)t, & s < t < 1 \end{cases}. \tag{3.331}$$

Applying Picard's fixed point iteration, we have the subsequent iterative scheme:

$$u_0 = -\frac{6}{e}(1-t^2),$$

$$\begin{aligned} u_{n+1} = u_n + \int_{-1}^{t} \left(\left(\frac{1}{4}s^2 + \frac{1}{2}s - \frac{1}{4} \right) + \left(-\frac{1}{4}s^2 - \frac{1}{2}s - \frac{1}{4} \right) t + \frac{1}{2}t^2 \right) (u_n'''(s) \\ - s u_n(s) - e^s(3 + s - 61s^2 - 60s^3 - 15s^4 - s^5 + s^6)) ds \\ + \int_{t}^{1} \left(\left(-\frac{1}{4}s^2 + \frac{1}{2}s - \frac{1}{4} \right) + \left(-\frac{1}{4}s^2 + \frac{1}{2}s - \frac{1}{4} \right) t \right) (u_n'''(s) \\ - s u_n(s) - e^s(3 + s - 61s^2 - 60s^3 - 15s^4 - s^5 + s^6)) ds, \end{aligned} \qquad (3.332)$$

where u_0 is the solution of $u''' = 0$ subject to $u(-1) = 0,\ u''(-1) = \frac{12}{e},\ u(1) = 0.$

Numerical results of this linear third-order differential equation confirm that our approach is more accurate than the method in [44] as observed in table 3.8 (a). Table 3.8 (b) shows the errors of the numerical solutions for different iterations; higher accuracy can be obtained by evaluating more iterates. The K–M iteration is not implemented in this problem since it did not show any noteworthy improvement. Moreover, the maximum errors for some iterates are reported in Table 3.8 (c). Figure 3.8 (a) shows the approximate solution which is clearly highly accurate.

t	Example 2 in [44]		Present Method
	$h = 0.010$	$h = 0.005$	
-1.0	0	0	3.0×10^{-13}
-0.8	8.15×10^{-6}	1.93×10^{-6}	8.8×10^{-10}
-0.6	1.43×10^{-5}	3.30×10^{-6}	1.7×10^{-9}
-0.4	1.87×10^{-5}	4.41×10^{-6}	2.4×10^{-9}
-0.2	2.10×10^{-5}	4.76×10^{-6}	2.9×10^{-9}
0.0	2.16×10^{-5}	5.18×10^{-5}	3.1×10^{-9}
0.2	2.06×10^{-5}	4.66×10^{-5}	2.9×10^{-9}
0.4	1.78×10^{-5}	3.03×10^{-5}	2.4×10^{-9}
0.6	1.29×10^{-5}	1.96×10^{-6}	1.7×10^{-9}
0.8	6.31×10^{-6}	5.96×10^{-7}	8.9×10^{-10}
1.0	0	0	0

Table 3.8 (a) Comparison with the method in [44] for Problem 3.8 using 8 iterations.

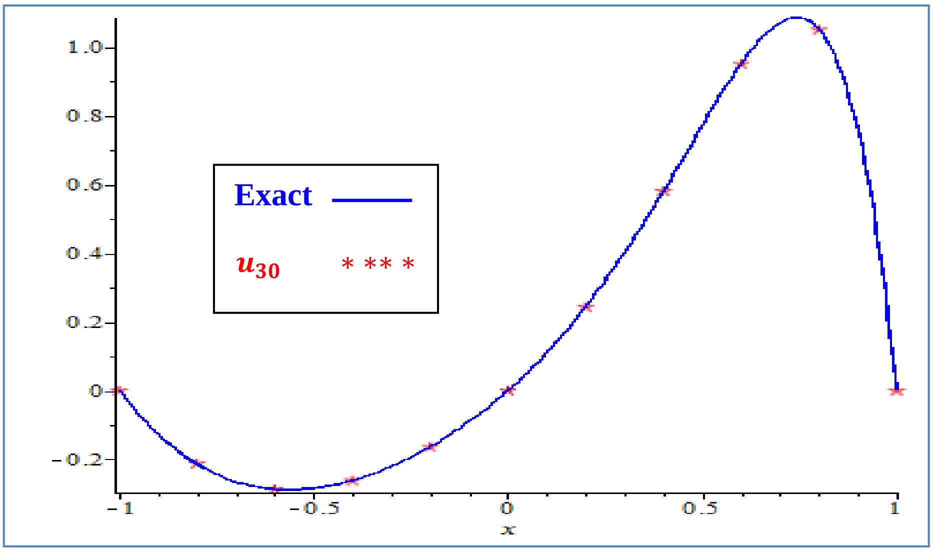

Figure 3.8 (a) Exact versus numerical solution using 30 iterations.

t	10 iterations	15 iterations	20 iterations	25 iterations	30 iterations
-1.0	8.3×10^{-20}	2.0×10^{-20}	0	2.0×10^{-30}	0
-0.8	5.6×10^{-12}	1.9×10^{-17}	5.7×10^{-23}	1.9×10^{-28}	5.9×10^{-34}
-0.6	1.1×10^{-11}	3.5×10^{-17}	1.1×10^{-22}	3.6×10^{-28}	1.1×10^{-33}
-0.4	1.5×10^{-11}	4.9×10^{-17}	1.6×10^{-22}	4.9×10^{-28}	1.6×10^{-33}
-0.2	1.8×10^{-11}	5.9×10^{-17}	1.9×10^{-22}	6.1×10^{-28}	1.9×10^{-33}
0.0	1.9×10^{-11}	6.2×10^{-17}	2.0×10^{-22}	6.4×10^{-28}	2.0×10^{-33}
0.2	1.8×10^{-11}	5.9×10^{-17}	1.9×10^{-22}	6.1×10^{-28}	1.9×10^{-33}
0.4	1.5×10^{-11}	4.9×10^{-17}	1.6×10^{-22}	5.0×10^{-28}	1.6×10^{-33}
0.6	1.1×10^{-11}	3.5×10^{-17}	1.1×10^{-22}	3.6×10^{-28}	1.1×10^{-33}
0.8	5.6×10^{-11}	1.8×10^{-17}	5.7×10^{-23}	1.9×10^{-28}	5.9×10^{-34}
1.0	0	6.0×10^{-20}	2.0×10^{-50}	3.0×10^{-30}	1.0×10^{-59}

Table 3.8 (b) Absolute Errors for Problem 3.8 for various iterations.

Number of Iterations	Maximum Error
5	6.1×10^{-6}
10	3.1×10^{-11}
15	6.2×10^{-17}
20	2.0×10^{-22}
25	6.4×10^{-28}
30	2.0×10^{-33}

Table 3.8 (c) Maximum error of the present method for some iterations.

Problem 3.9 Consider the following third order nonlinear (BVP)

$$u'''(t) = -e^{-2u(t)}(u'(t) + tu''(t) - 2t(u'(t))^2), \quad 1 \le t \le 2, \tag{3.333}$$

with boundary conditions

$$u(1) = 0, u'(1) = -1, u(2) = \ln(2). \tag{3.334}$$

The exact solution is given by

$$u(t) = \ln(t). \tag{3.335}$$

We find Green function for $L[u] = u'''$ subject to $u(1) = u(2) = u'(1) = 0$ by applying Green's function's properties. It is found to be

$$G(t|s) = \begin{cases} s^2 - 2s + 2 + \left(-\frac{1}{2}s^2 + s - 2\right)t + \frac{1}{2}t^2, & 1 < t < s \\ \frac{1}{2}s^2 - 2s + 2 + \left(-\frac{1}{2}s^2 + 2s - 2\right)t, & s < t < 2 \end{cases}. \tag{3.336}$$

Applying Picard's fixed point iteration, then we have the iterative scheme

$$u_0 = -1 - ln(2) + \left(\frac{3}{2} + ln(2)\right)t - \frac{1}{2}t^2,$$

$$
\begin{aligned}
u_{n+1} = u_n + \int_1^t \left(s^2 - 2s + 2 + \left(-\frac{1}{2}s^2 + s - 2\right)t + \frac{1}{2}t^2\right)\Big(u_n{'''}(s) \\
+ 4e^{-2u_n(s)}\left(u_n{'}(s) + s u_n{''}(s) - 2s(u_n{'}(s))^2\right)\Big)\, ds \\
+ \int_t^2 \left(\frac{1}{2}s^2 - 2s + 2 \right. \\
\left. + \left(-\frac{1}{2}s^2 + 2s - 2\right)t\right)\Big(\big(u_n{'''}(s) \\
+ e^{-2u_n(s)}\left(u_n{'}(s) + s u_n{''}(s) - 2s(u_n{'}(s))^2\right)\big)\Big)\, ds,
\end{aligned}
\tag{3.337}
$$

where u_0 is the solution of $u''' = 0$ subject to $u(1) = 0, u'(1) = -1, u(2) = \ln(2)$.

The comparison of the absolute errors between our method and that in [44] is shown in Table 3.9 (a). The approximate solutions at the mesh points $t = 1.0, 1.1, \dots, 2.0$ using 20 iterations, as well as the exact solution are depicted in Figure 3.9 (a) from which it is evident that in case we choose $\alpha = 0.77$, the approximate solution agrees very well with the exact solution. As in Problems 3.3 and 3.6, the K-M iteration for this problem has a tangible effect in improving the error. Table 3.9 (b) and Figure 3.9 (b) show that the best results are achieved using 20 iterations, when we choose $\alpha = 0.77$ in which the maximum absolute error is 1.1×10^{-17}. However, when $\alpha = 1$, $\alpha = 0.9$, $\alpha = 8$ and $\alpha = 0.7$ the maximum absolute errors are 7.8×10^{-10}, 6.8×10^{-13}, 2.6×10^{-15} and 2.6×10^{-15} using 20 iterations, respectively. Table 3.9 (c) shows that the choice $\alpha = 0.77$ gives the best accuracy.

t	Example 4 in [44]		Present Method
	$h = 0.010$	$h = 0.005$	
1.0	0	0	2.0×10^{-36}
1.1	1.54×10^{-6}	4.24×10^{-7}	3.5×10^{-18}
1.2	2.65×10^{-6}	7.15×10^{-7}	9.1×10^{-18}
1.3	3.30×10^{-6}	8.34×10^{-7}	1.1×10^{-17}
1.4	3.75×10^{-6}	9.83×10^{-7}	6.3×10^{-18}
1.5	3.81×10^{-6}	1.01×10^{-6}	2.2×10^{-19}
1.6	3.60×10^{-6}	9.23×10^{-7}	3.0×10^{-18}
1.7	3.15×10^{-6}	8.34×10^{-7}	1.2×10^{-18}
1.8	2.44×10^{-6}	6.55×10^{-7}	2.2×10^{-18}
1.9	1.31×10^{-6}	3.57×10^{-7}	3.5×10^{-18}
2.0	0	0	9.9×10^{-26}

Table 3.9 (a) Comparison with method in [44] for Problem 3.9 using 20 iterations.

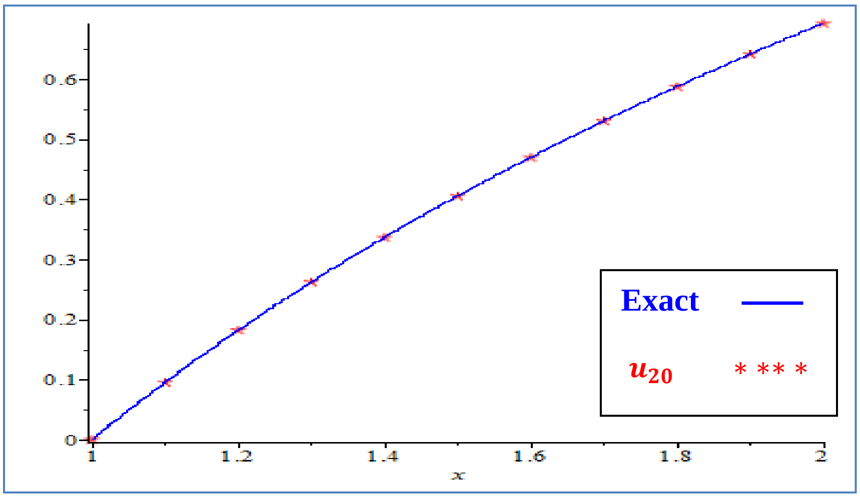

Figure 3.9 (a) Exact versus numerical solution using 20 iterations.

t	PICARD's	MANN's			
	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.77$	$\alpha = 0.7$
1.0	2.0×10^{-36}	1.0×10^{-36}	0	2.0×10^{-36}	0
1.1	2.1×10^{-10}	2.5×10^{-13}	1.0×10^{-15}	3.5×10^{-18}	5.2×10^{-16}
1.2	4.1×10^{-10}	4.4×10^{-13}	1.8×10^{-15}	9.1×10^{-18}	2.6×10^{-16}
1.3	5.7×10^{-10}	5.8×10^{-13}	2.3×10^{-15}	1.1×10^{-17}	1.4×10^{-15}
1.4	7.0×10^{-10}	6.5×10^{-13}	2.3×10^{-15}	6.3×10^{-18}	2.6×10^{-15}
1.5	7.7×10^{-10}	6.8×10^{-13}	2.6×10^{-15}	2.2×10^{-19}	2.2×10^{-15}
1.6	7.8×10^{-10}	6.5×10^{-13}	2.5×10^{-15}	3.0×10^{-18}	6.6×10^{-16}
1.7	7.1×10^{-10}	5.7×10^{-13}	2.1×10^{-15}	1.2×10^{-18}	1.0×10^{-15}
1.8	5.7×10^{-10}	4.3×10^{-13}	1.6×10^{-15}	2.2×10^{-18}	2.0×10^{-15}
1.9	3.3×10^{-19}	2.4×10^{-13}	9.0×10^{-16}	3.5×10^{-18}	1.7×10^{-15}
2.0	5.0×10^{-30}	9.0×10^{-30}	2.3×10^{-37}	9.9×10^{-26}	4.5×10^{-36}

Table 3.9 (b) Comparing Picard's and Mann's schemes for Problem 3.9 using 20 iterations.

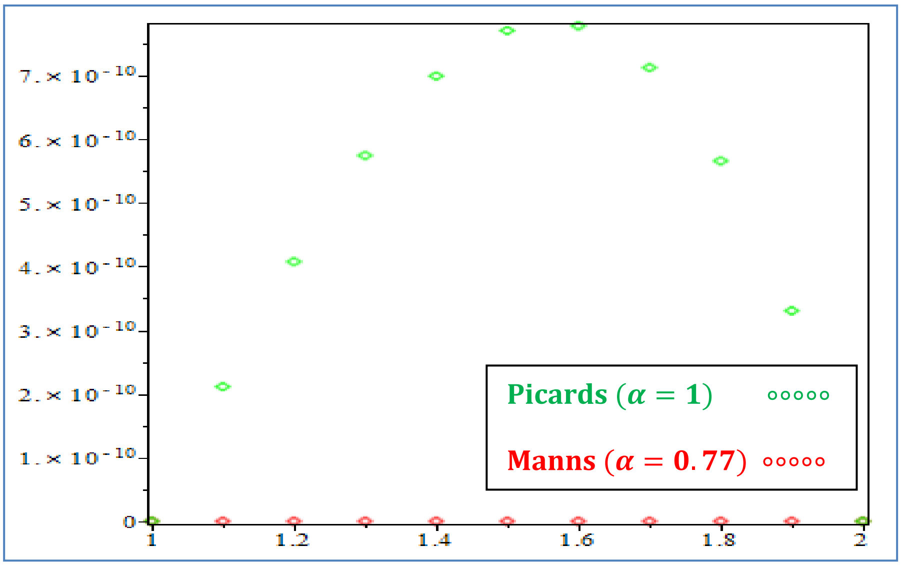

Figure 3.9 (b) Mann's with $\alpha = 0.77$ and Picard's for Problem 3.9 using 20 iterations.

t	5 iterations	10 iterations	15 iterations	20 iterations	25 iterations
1.0	10.0×10^{-37}	10.0×10^{-32}	1.0×10^{-36}	2.0×10^{-36}	2.0×10^{-45}
1.1	8.6×10^{-8}	6.5×10^{-12}	3.7×10^{-15}	3.5×10^{-18}	2.3×10^{-20}
1.2	2.0×10^{-7}	1.0×10^{-11}	1.3×10^{-15}	9.1×10^{-18}	4.1×10^{-20}
1.3	2.9×10^{-7}	6.9×10^{-12}	5.7×10^{-15}	1.1×10^{-17}	5.1×10^{-20}
1.4	3.2×10^{-7}	3.1×10^{-11}	4.0×10^{-15}	6.3×10^{-18}	5.4×10^{-20}
1.5	3.2×10^{-7}	4.1×10^{-11}	8.2×10^{-15}	2.2×10^{-19}	5.6×10^{-20}
1.6	2.9×10^{-7}	3.2×10^{-11}	2.3×10^{-14}	3.0×10^{-18}	5.6×10^{-20}
1.7	2.6×10^{-7}	8.8×10^{-12}	3.3×10^{-14}	1.2×10^{-18}	5.2×10^{-20}
1.8	2.2×10^{-7}	1.4×10^{-11}	3.2×10^{-14}	2.2×10^{-18}	4.2×10^{-20}
1.9	1.5×10^{-7}	2.2×10^{-11}	2.0×10^{-14}	3.5×10^{-18}	2.4×10^{-20}
2.0	5.0×10^{-28}	5.8×10^{-25}	1.4×10^{-25}	9.9×10^{-26}	5.9×10^{-34}

Table 3.9 (c) Errors of numerical solutions for some iterations using Mann's with $\alpha = 0.77$.

CHAPTER 4: CONCLUSIONS

4.1 Summery

This thesis was divided into three chapters in which we surveyed two existing techniques and introduced a novel one to obtain numerical solutions for various types of differential equations, particularly for a wide class of boundary value problems. In the first chapter we started with the Adomian Decomposition Method (ADM) for solving ordinary differential equations, partial differential equations, algebraic equations, delay differential equations, integral equations, and integro-differential equations. This method tackles many differential equations, whether they are homogeneous, inhomogeneous, linear, or nonlinear, in a straightforward manner without any restrictive assumptions, such as linearization or perturbation. In addition, the (ADM) often converges to the exact solution if it exists. On the other hand, we noticed that the method gives highly accurate approximations only close to the initial condition but not as we move away from it.

In the second chapter, we introduced the Variational Iteration Method (VIM) for solving ordinary differential equations, partial differential equations, calculus of variations, integral equations, and integro-differential equations. This method gives rapidly convergent successive approximations of the exact solution if such a solution exists. It also provides an approximation with high level of accuracy by applying few iterations only. Actually, the variational iteration method proved to be an effective tool to handle nonlinear equations without the use of Adomian polynomials. However, we noticed that for this method, as we increase the values of x, the error slowly deteriorates over the entire domain.

Finally, in the third chapter, we presented a new approach for obtaining numerical solutions for third order linear and nonlinear boundary value problems by utilizing Green's functions and manipulating fixed point iterations, such as Picard's and Krasnoselskii-Mann's schemes. The aim of our alternative strategy is to overcome the major deficiency of both the ADM and VIM, particularly the local convergence of the method and the deterioration of the error as the domain increases. A number of examples were solved to illustrate the method and demonstrate its reliability and accuracy. Moreover, we compared our results with both the analytical and the numerical solutions obtained by other methods that exist in the literature.

4.2 Future Work

In future research, we will work on deriving the necessary conditions for the convergence of the iterative method as well as its rate of convergence. Also, we will try to generalize and apply the scheme to BVPs other than third order, such as fourth order. In the approach we used Picard's and Mann's fixed point schemes, therefore in the future work, we will explore other schemes such as the Ishikawa iteration and try to embed it into our method. Moreover, my ultimate goal is to publish all this work in an international and reputed journal.

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Vita

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