SOLVING FOURTH ORDER DIFFERENTIAL NON-LINEAR EQUATIONS, EXISTENCE AND UNIQUENESS

by

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Approval Signatures

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Dedication

I find myself humbly obligated to dedicate this work to my beloved parents in my home country. They provided true support without which I would never have dreamed of reaching this stage in my life, education, and career. I am certain they would have been deeply delighted to witness my graduation. Also, to my brothers and sister, I am immeasurably thankful for their sacrifices. They put me ahead of themselves in many critical stages in our lives.

Abstract

The aim of this thesis is to present a novel numerical approach for the solution of a class of non-linear fourth-order boundary value problems. The method is based on embedding Green's function into some fixed point iteration schemes, an idea previously used in other works to investigate nonlinear boundary value problems of lower order. To this end, the thesis is divided in 6 chapters. The first chapter is a short description of the main ideas of this thesis. The second chapter represents a review on Green's functions for differential equations. In Chapter 3 some existence and uniqueness results for the fourth order boundary value problems are presented. The proposed numerical method is then explained and applied on numerical examples in Chapter 4, where a comparison with the Spline method is also given, demonstrating thus that our method yields accurate results up to 10^{-20}, compared to the Spline method, where the results are accurate up to 10^{-13}. The results obtained by our method were achieved within a reasonable time limit compared with other methods. Chapter 5 is concerned with the convergence analysis of our method. More precisely, some conditions which guarantees the convergence of the solution under specific conditions is given. For the proof we used the Banach-Picard theorem along with the Green's function. Finally, in Chapter 6 we present a short conclusions and a summary of the whole thesis.

Search Terms Fourth Order BVPs, Green's Function, Fixed Point Iteration, Picard Iteration, Mann Iteration, Beam Theory, Dynamics, Existence, Uniqueness, Iterations Convergence.

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Chapter 1: INTRODUCTION

Fourth order differential equations play a vital role in many physical applications, such as, for instance, traveling waves (Chen, & Mckenna, 1997), (Lin, Kondic, Thiele, & Cummings, 2013), longitudinal and transverse vibrations (Shum, & Lin, 2010), image noise removal technology (Lysaker, Lundervold, & Tai, 2003), (You & Kaveh, 2000), movement of a beam deflected under its weight or under the influence of some external forces (Saker, Agarwal, O'Regan, 2010) etc.

Before discussing the general theories and the conditions that are needed to ensure the existence and the uniqueness of solutions to non-linear differential equations, part of this thesis will be dedicated to several important, basic and necessary, definitions and prerequisites. These will be explained thoroughly and then applied on numerical and applicable examples. Concepts such as Green's functions and fixed-point iteration schemes, will be provided and applied to equations that will be used in subsequent chapters, especially when solving selected test examples.

Methods will be presented that have been used to solve the proposed class of fourth order differential equations. We will present different strategies such as the Runge-Kutta, homotopy asymptotic, Spline collocation and iterative methods. It will be shown that these methods, which have been implemented in recent years, can be improved in terms of the number of iterations needed and processing time (CPU time). The disadvantages and/or deficiencies of a number of these methods to produce a reliable solution will also be discussed.

This thesis presents a novel method for solving fourth order non-linear differential equations; the proposed method will depend on embedding Green's functions into fixed point iteration schemes. To confirm the efficiency, applicability and high accuracy of the proposed method, several examples will be presented. Comparison with other techniques that exist in the literature will also be given. Finally, a summary will be given that includes solutions for several famed equations. The relevant tables for each situation will accompany the numerical solutions.

Chapter 2: GREEN'S FUNCTIONS

In this chapter we introduce the notion of Green's function for a nonhomogeneous linear differential equation and show how to derive the general solution of some problems in terms of an integral involving the Green's functions. Historically speaking, the Green's functions date back to 1928, when the British mathematician George Green (1793−1841) published the *"Essay on the Application of Mathematical Analysis to the Theory of Electricity and Magnetism"*. In this seminal work of mathematical physics, G. Green sought to determine the electric potential $u(x)$ within a vacuum bounded by conductors with specified potentials. In nowadays notation we would say that he examined the solutions of the Poisson equation $\Delta u = f$, within a volume V, that satisfy certain boundary conditions along the boundary S. In what follows we use his idea to investigate several classes of problems for linear nonhomogeneous differential equations.

2.1 Green's Function For First Order Differential Equations

Let us consider the following first order differential equation

$$L[u] \equiv u'(x) + p(x)u(x) = f(x) \text{ for } x > a, \tag{2.1}$$

subject to the initial condition

$$B[u] \equiv u(a) = 0. \tag{2.2}$$

The Green function $G(x, \xi)$ is then defined as the solution of the following initial-value problem

$$\begin{cases} L[G(x,\xi)] = \delta(x-s) \text{ for } x > a, \\ B[G(x,s)] \equiv G(x,s) = 0, \end{cases} \tag{2.3}$$

where $\delta(x - s)$ is *the Dirac delta function*, defined by the following two conditions:

$$\delta(x-s) = 0 \text{ if } x \neq s \qquad \text{and} \qquad \int\limits_{s-c}^{s+d} \delta(x-s)dx = 1 \text{ for any } c, d > 0. \tag{2.4}$$

The Dirac delta function $\delta(x - s)$ is not a function $\delta(.) : \mathbb{R} \to \mathbb{R}$, but may be treated like one for some purposes. In fact, the Dirac delta function is called a *generalized function* and can be regarded as the limit of a sequence of functions. More precisely, if

$$\delta_k(x-s) = \begin{cases} k \text{ if } |x-s| \leq \frac{1}{2k} \\ 0 \text{ if } |x-s| > \frac{1}{2k} \end{cases}, \tag{2.5}$$

then the area under each curve $y = \delta_k(x-s)$ is equal to 1 and

$$\delta(x-s)=\lim_{k\to\infty} \delta_k(x-s). \tag{2.6}$$

Next, let us introduce the following integral function

$$u(x) = \int_a^\infty G(x,s)f(s)ds. \tag{2.7}$$

If we apply the linear differential operator L to the above integral, assuming that the integral is uniformly convergent, we get

$$L\left[\int_a^\infty G(x,s)f(s)ds\right] = \int_a^\infty L\left[G(x,s)\right]f(s)ds = \int_a^\infty \delta(x-s)f(s)ds. \tag{2.8}$$

Combining (2.8) with the following shifting property of the Dirac-Delta function

$$\int_{s-c}^{s+d} \delta(x-s)f(s)\,dx = f(x) \text{ for any } c,d>0, \tag{2.9}$$

one may easily notice that the integral function $u(x)$, defined in (2.7), is in fact a solution of the differential equation (2.1). Moreover, the initial condition (2.2) is also satisfied, since we have

$$B\left[\int_a^\infty G(x,s)f(s)ds\right] = \int_a^\infty B\left[G(x,s)\right]f(s)ds = \int_a^\infty 0f(s)ds = 0. \tag{2.10}$$

In conclusion, the integral function introduced in (2.7) is a solution of the initial value problem (2.1)-(2.2).

Now, if one consider the qualitative behavior of the Green function, we may clearly notice that for $x \neq s$ the Green function is a solution for the homogeneous equation $L[u] = 0$. However, at $x = s$ we expect some singular behavior. Integrating the equation

$$G' + p(x)G = \delta(x-s), \tag{2.11}$$

on the vanishing interval (s^-, s^+), we get

$$G(s^+, s) - G(s^-, s) + \int_{s^-}^{s^+} p(x)G(x, s)dx = 1, \tag{2.12}$$

or

$$G(s^+, s) - G(s^-, s) = 1. \tag{2.13}$$

On the orther hand, since the Green function satisfies the homogeneous equation $L[u] = 0$, then it has the following form

$$G(x, s) = \begin{cases} c_1 e^{-\int p(x)dx} \text{ for } a < x < s \\ c_2 e^{-\int p(x)dx} \text{ for } s < x \end{cases}, \tag{2.14}$$

where c_1 and c_2 are some arbitrary constants. Moreover, since $G(x, s)$ also satisfies the initial condition $G(a, s) = 0$, then it must vanish for $x \in [a, s)$, so that we have

$$G(x, s) = \begin{cases} 0 \text{ for } a < x < s \\ c_2 e^{-\int p(x)dx} \text{ for } s < x \end{cases}. \tag{2.15}$$

Also, the jump condition (2.13) gives us the constrain $G(s^+, s) = 1$, which implies that $c_2 = 1$. We can thus write the Green function in the following explicit form

$$G(x, s) = H(x - s)e^{-\int_s^x p(t)dt}, \tag{2.16}$$

where $H(x - s)$ is *the Heaviside function*, defined as follows:

$$H(x - s) = \begin{cases} 0 \text{ if } x < s \\ 1 \text{ if } x \geq s \end{cases}. \tag{2.17}$$

To summarize, we have the following theorem:

Theorem 2.1. *The first order nonhomogeneous initial-value problem*

$$\begin{cases} L[u] \equiv u'(x) + p(x)u(x) = f(x) \textit{ for } x > a \\ B[u] \equiv u(a) = 0 \end{cases}, \tag{2.18}$$

has the solution given in the following form

$$u(x) = \int_a^{\infty} G(x,s) f(s) ds, \tag{2.19}$$

where the Green's function $G(x,s)$ satisfies the non-homogenous initial-value problem

$$\begin{cases} L[G(x,s)] \equiv G'(x,s) + p(x)G(x,s) = \delta(x-s) \text{ for } x > a, \\ B[G(x,s)] \equiv G(a,s) = 0. \end{cases} \tag{2.20}$$

More precisely, the Green's function $G(x,s)$ is given by the explicit formula

$$G(x,s) = H(x-s) e^{-\int_s^x p(t)dt}. \tag{2.21}$$

2.2 Green's Function For 2^{nd} Order Differential Equations

Let us now consider the following second order differential equation

$$L[u] \equiv u''(x) + p(x)u'(x) + q(x)u(x) = f(x) \text{ for } a < x < b, \tag{2.22}$$

subject to the following boundary conditions

$$\begin{cases} B_1[u] \equiv \alpha_{11}u(a) + \alpha_{12}u'(a) + \beta_{11}u(b) + \beta_{12}u'(b) = \gamma_1, \\ B_2[u] \equiv \alpha_{21}u(a) + \alpha_{22}u'(a) + \beta_{21}u(b) + \beta_{22}u'(b) = \gamma_2. \end{cases} \tag{2.23}$$

where $\alpha_{ij}, \beta_{ij},\ i,j = 1,2$ and $\gamma_k,\ k = 1,2,$ are some real constants. The general solution of the problem (2.22)-(2.23) is given by

$$u(x) = u_h(x) + u_p(x), \tag{2.24}$$

where

$$\begin{cases} L[u_h] = 0 \text{ for } a < x < b, \\ B_1[u_h] = \gamma_1,\ B_2[u_h] = \gamma_2, \end{cases} \tag{2.25}$$

and

$$\begin{cases} L[u_p] = f \text{ for } a < x < b, \\ B_1[u_p] = 0,\ B_2[u_p] = 0. \end{cases} \tag{2.26}$$

The problem (2.25) may have no solution, a unique solution or an infinite number of solutions. We consider only the case when there is a unique solution. In such a case, the homogeneous equation subject to homogeneous boundary conditions has only the trivial solution. Let $u_1(x)$ and $u_2(x)$ be two solutions of the homogeneous equation (2.25) that satisfy the homogeneous boundary conditions

$$B_1[u_1] = 0, \qquad B_2[u_2] = 0. \tag{2.27}$$

Since the completely homogeneous problem has no solution, then $B_1[u_2] \neq 0$ and $B_2[u_1] \neq 0$. Writing

$$u_h(x) = c_1 u_1(x) + c_2 u_2(x), \tag{2.28}$$

and making use of the boundary conditions (2.25), we may determine the constants c_1 and c_2, so that

$$u_h(x) = \frac{\gamma_2}{B_2[u_1]} u_1(x) + \frac{\gamma_2}{B_1[u_2]} u_2(x), \tag{2.29}$$

is the solution of problem (2.25).

On the other hand, following the idea of the previous section, one may easily note that we can represent the solution of (2.26) as an integral of a Green's function. More precisely, we have

$$u_p(x) = \int_a^b G(x,s) f(s) ds, \tag{2.30}$$

where $G(x,s)$ is the Green's function for (2.26), which means that it satisfies

$$\begin{cases} L[G(x,s)] = \delta(x-s) \text{ for } a < x < b, \\ B_1[G(x,s)] = 0, B_2[G(x,s)] = 0. \end{cases} \tag{2.31}$$

The continuity and jump conditions are, in this case, the followings

$$\begin{aligned} G(s^-, s) &= G(s^+, s), \\ G'(s^-, s) + 1 &= G'(s^+, s). \end{aligned} \tag{2.32}$$

We write the Green functions as

$$G(x,s) = H(x-s) u_s(x) + d_1 u_1 + d_2 u_2, \tag{2.33}$$

where *the casual function* $u_s(x)$ is the linear combination of the homogeneous solutions $u_1(x)$ and $u_2(x)$ which satisfies

$$u_s(s) = 0, u_s'(s) = 1. \tag{2.34}$$

Then, in this form, the continuity and jump conditions are automatically satisfied. Moreover, from the boundary conditions we have

$$\begin{aligned} B_1[G]=0 \quad &\Longleftrightarrow B_1[H(x-s)u_s(x)]+d_2B_1[u_2]=0, \\ &\Longleftrightarrow \beta_{11}u_s(b)+\beta_{12}u_s'(b)+d_2B_1[u_2]=0, \\ B_2[G]=0 \quad &\Longleftrightarrow B_2[H(x-s)u_s(x)]+d_1B_2[u_2]=0, \\ &\Longleftrightarrow \beta_{21}u_s(b)+\beta_{22}u_s'(b)+d_1B_2[u_1]=0. \end{aligned} \tag{2.35}$$

Therefore, the Green's function becomes

$$G(x,s)=H(x-s)u_s(x)-\frac{\beta_{21}u_s(b)+\beta_{22}u_s'(b)}{B_2[u_1]}u_1-\frac{\beta_{11}u_s(b)+\beta_{12}u_s'(b)}{B_1[u_2]}u_2, \tag{2.36}$$

and it is well defined, since $B_2[u_1]\neq 0$ and $B_1[u_2]\neq 0$. Thus, if there exists a unique solution u_h for (2.25) the general solution for (2.3) is

$$u(x)=\int_a^b G(x,s)f(s)ds+\frac{\gamma_2}{B_2[u_1]}u_1+\frac{\gamma_2}{B_1[u_2]}u. \tag{2.37}$$

To summarize, we have the following theorem:

Theorem 2.2. *Let us consider the a nonhomogeneous linear second order differential equation*

$$L[u]\equiv u''(x)+p(x)u'(x)+q(x)u(x)=f(x), \qquad \text{for } a<x<b, \tag{2.38}$$

subject to the following boundary conditions

$$\begin{cases} B_1[u]\equiv \alpha_{11}u(a)+\alpha_{12}u'(a)+\beta_{11}u(b)+\beta_{12}u'(b)=\gamma_1, \\ B_2[u]\equiv \alpha_{21}u(a)+\alpha_{22}u'(a)+\beta_{21}u(b)+\beta_{22}u'(b)=\gamma_2. \end{cases} \tag{2.39}$$

If the homogeneous differential equations subject to the homogeneous boundary conditions has no solution, then the problem has the unique solution

$$u(x)=\int_a^b G(x,s)f(s)ds+\frac{\gamma_2}{B_2[u_1]}u_1(x)+\frac{\gamma_2}{B_1[u_2]}u_2(x), \tag{2.40}$$

where the Green function is given by

$$G(x,s) = \begin{aligned} & H(x-s)u_s(x) - \tfrac{\beta_{21}u_s(b)+\beta_{22}u_s'(b)}{B_2[u_1]}u_1(x) \\ & -\tfrac{\beta_{11}u_s(b)+\beta_{12}u_s'(b)}{B_1[u_2]}u_2(x). \end{aligned} \tag{2.41}$$

In (2.40) and (2.41), $u_1(x)$ and $u_2(x)$ are solutions of the homogeneous equation that satisfy homogeneous boundary conditons (2.27), while u_s is the linear combination of $u_1(x)$ and $u_2(x)$ that satisfies (2.34).

2.3 Green's Function For Higher Order Differential Equations

Let's now consider the general case of an n^{th} order differential equation

$$L[u] \equiv \begin{aligned} & u^{(n)}(x) + p_{n-1}(x)u^{(n-1)}(x) + ... + p_1(x)u'(x) \\ & +p_0(x)u(x) = f(x), \text{ for } a < x < b, \end{aligned} \tag{2.42}$$

subject to the following boundary conditions

$$B_j[u] \equiv \sum_{k=0}^{n-1}\alpha_{jk}u^{(k)}(a) + \sum_{k=0}^{n-1}\beta_{jk}u^{(k)}(b) = \gamma_j, \ j = 1, \dots, n, \tag{2.43}$$

where $\alpha_{ij}, \beta_{ij}, \ i,j = 1,...,n$, and $\gamma_k, \ k = 1, \dots, n$, are some real constants. If the completely homogeneous problem $L[u] = 0$ for $a < x < b$, $B_j[u] = 0, \ j = 1, \dots, n$, has only the trivial solution, then a solution of problem (2.42)-(2.43) exists and is unique, being given in the following form

$$u(x) = u_h(x) + u_p(x), \tag{2.44}$$

where

$$\begin{cases} L[u_h] = 0, \text{ for } a < x < b, \\ B_j[u_h] = \gamma_j, \ j = 1, \dots, n, \end{cases} \tag{2.45}$$

and

$$\begin{cases} L[u_p] = f, \text{ for } a < x < b, \\ B_j[u_p] = 0, \ j = 1, \dots n. \end{cases} \tag{2.46}$$

The problem (2.45) may have no solution, a unique solution or an infinite number of solutions. We consider only the case when there is an unique solution. In such a case, the homogeneous equation subject to homogeneous boundary conditions has only the trivial solution. Let $\{u_1, ..., u_n\}$ be a set of solutions.of equation (2.45), which are linearly independent, that is

$$W[u_1, ..., u_n] := \begin{vmatrix} u_1 & u_2 & ... & u_n \\ u_1' & u_2' & ... & u_n' \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(n-1)} & u_2^{(n-1)} & ... & u_n^{(n-1)} \end{vmatrix} \neq 0 \text{ for all } x \in (a, b)\,, \tag{2.47}$$

where the function $W[u_1, ..., u_2]$, defined above, is *the Wronskian of* $u_1(x), ..., u_n(x)$. Then, we can write in the form

$$u_h = c_1 u_1(x) + c_2 u_2(x) + ... + c_n u_n(x), \tag{2.48}$$

where the constants are determined by the matrix equation

$$\begin{bmatrix} B_1[u_1] & B_1[u_2] & ... & B_1[u_n] \\ B_2[u_1] & B_2[u_2] & ... & B_2[u_n] \\ \vdots & \vdots & \ddots & \vdots \\ B_n[u_1] & B_n[u_2] & ... & B_n[u_n] \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}. \tag{2.49}$$

On the other hand, as in the previous section, we represent the solution of (2.46) as an integral of a Green's function

$$u_p(x) = \int_a^b G(x, s) f(s) ds, \tag{2.50}$$

where $G(x, s)$ is the Green's function for (2.46), so it is the function satisfying

$$\begin{cases} L[G(x, s)] = \delta(x - s) \text{ for } a < x < b, \\ B_j[G(x, s)] = 0, j = 1, \ldots, n. \end{cases} \tag{2.51}$$

Let $u_s(x)$ be *the casual solution*, i.e the linear combination of $u_1(x), ..., u_n(x)$, that satisfies the following conditions

$$u_s(s) = u_s'(s) = ... = u_s^{(n-2)}(s) = 0, u_s^{(n-1)}(s) = 1. \tag{2.52}$$

The Green functions will then have the following form

$$G(x,s) = H(x-s)u_s(x) + d_1u_1 + ... + d_nu_n, \tag{2.53}$$

where the constants $d_1, ..., d_n$ are determined by the matrix equation

$$\begin{bmatrix} B_1[u_1] & B_1[u_2] & ... & B_1[u_n] \\ B_2[u_1] & B_2[u_2] & ... & B_2[u_n] \\ \vdots & \vdots & \ddots & \vdots \\ B_n[u_1] & B_n[u_2] & ... & B_n[u_n] \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} -B_1[H(x-s)u_s(x)] \\ -B_2[H(x-s)u_s(x)] \\ \vdots \\ -B_n[H(x-s)u_s(x)] \end{bmatrix}. \tag{2.54}$$

To summarize, we have the following theorem:

Theorem 2.3. *Let us consider the following second order differential equation*

$$L[u] \equiv u^{(n)} + p_{n-1}(x)u^{(n-1)} + ...p_1(x)u' + p_0(x)u = f(x), \qquad \textit{for } a < x < b, \tag{2.55}$$

subject to the following boundary conditions

$$B_j[u] \equiv \sum_{k=0}^{n-1} \alpha_{jk}u^{(k)}(a) + \sum_{k=0}^{n-1} \beta_{jk}u^{(k)}(b) = \gamma_j, \;\; j = 1, \dots, n. \tag{2.56}$$

If the homogeneous differential equation subject to the homogeneous boundary conditions has no solution, then the problem (2.55)-(2.56) has the unique solution

$$u(x) = \int_a^b G(x,s)f(s)ds + c_1u_1 + ... + c_nu_n, \tag{2.57}$$

where the Green's function is given as follows

$$G(x,s) = H(x-s)u_s(x) + d_1u_1 + ... + d_nu_n. \tag{2.58}$$

In (2.57) and (2.58), $\{u_1(x), ..., u_n(x)\}$ is a fundamental set of solutions for the homogeneous equation, $u_s(x)$ is the linear combination of $u_1(x), ..., u_n(x)$ which satisfies (2.52), while the constants $c_1, ..., c_n$ and $d_1, ..., d_n$ are obtained by solving (2.49), respectively (2.52).

Finally, we note that a Green's function depends only on the fundamental set of solutions for the associated homogeneous differential equation and not on the non-linear term on the right, namely the forcing function $f(x)$. Therefore, all the linear nonhomogeneous differential equations with the same left-hand side but with different right-hand side forcing functions $f(x)$ have the same Green's functions. In conclusion, one may alternatively say that the Green's functions for non-homogenous differential equations are Green's functions for the differential operators $L[u]$ appearing on the left-hand side of such equations.

2.4 Some examples

In this section we will use the previous theorems to solve some problems making use of Green's functions.

Example 2.1. *Use the Green's function to solve the following first order initial value problem*

$$\begin{cases} u'(x) - au(x) = e^{bx} \text{ for } x > 0, \\ u(0) = 0, \end{cases} \tag{2.59}$$

where $a, b \in \mathbb{R}^*$, $a \neq b$.

Solution:

From Theorem 2.1 we know that the general solution is given in the following integral form

$$u(x) = \int_0^\infty G(x,s) e^{bs} ds, \tag{2.60}$$

where

$$G(x,s) = H(x-s) e^{\int_s^x a dt} = \begin{cases} 0 \text{ if } x < s, \\ e^{ax-as} \text{ if } x \geq s. \end{cases} \tag{2.61}$$

Therefore, the solution of problem (2.59) is

$$u(x) = \int_0^x e^{ax-as} e^{bs} ds = e^{ax} \left. \frac{e^{(b-a)s}}{b-a} \right|_{s=0}^{s=x} = \frac{e^{bx} - e^{ax}}{b-a}. \tag{2.62}$$

Example 2.2 (Forced Harmonic Oscillator)**.** *Use the Green's function to solve the following first order initial value problem*

$$\begin{cases} u'' + u(x) = f(x) \text{ for } 0 < x < \frac{\pi}{2}, \\ u(0) = 0, \ u(\frac{\pi}{2}) = 0. \end{cases} \tag{2.63}$$

Solution:

First of all, let us note that the fundamental set of solutions for the homogeneous equations is given by

$$u_1(x) = \sin x, \qquad u_2(x) = \cos x. \tag{2.64}$$

From Theorem 2.2 we know that the general solution is given in the following integral form

$$u(x) = u_h(x) + u_p(x) = \int_0^{\frac{\pi}{2}} G(x,s)f(s)ds, \tag{2.65}$$

where

$$G(x,s) = H(x-s)u_s(x) - \frac{u_s(\frac{\pi}{2})}{u_1(\frac{\pi}{2})}u_1(x), \tag{2.66}$$

and u_s is the linear combination of the homogeneous solutions $u_1(x)$ and $u_2(x)$ that satisfies $u_s(s) = 0$, $u_s'(s) = 1$, that is

$$u_s(x) = \cos s \sin x - \sin s \cos x. \tag{2.67}$$

Therefore

$$G(x,s) = \begin{cases} -\cos s \sin x \text{ if } x < s, \\ -\sin s \cos x \text{ if } x \geq s, \end{cases} \tag{2.68}$$

and the solution of problem (2.63) is

$$u(x) = \int_0^{\frac{\pi}{2}} G(x,s)f(s)ds = -\sin x \int_x^{\frac{\pi}{2}} \cos s f(s)ds - \cos x \int_x^{\frac{\pi}{2}} \sin s f(s)ds. \tag{2.69}$$

Example 2.3. *Use the Green function to solve the following fourth order boundary value problem*

$$\begin{cases} u^{(4)}(x) = 120x \textit{ for } 0 < x < 1 \\ u(0) = 0,\ u'(0) = 0,\ u''(0) = 0,\ u'''(1) = 60 \end{cases}. \tag{2.70}$$

Solution:

First of all, let us note that the fundamental set of solutions for the homogeneous equation is given by

$$u_1(x) = 1, \qquad u_2(x) = x, \qquad u_3(x) = x^2, \qquad u_4(x) = x^3. \tag{2.71}$$

The causal solution $u_s(x)$, i.e. the linear combination of $u_1(x), ..., u_4(x)$ which satisfies

$$u_s(s) = u_s'(s) = u_s''(s) = 0, u_s'''(s) = 1, \tag{2.72}$$

is in this case

$$u_s(x) = -\frac{1}{6}s^3 + \frac{1}{2}s^2x - \frac{1}{2}sx^2 + \frac{1}{6}x^3 = \frac{1}{6}(x-s)^3. \tag{2.73}$$

Then, Theorem 2.3 implies that the general solution is

$$u(x) = u_h(x) + u_p(x) = 10x^3 + \int\limits_0^1 G(x,s)f(s)ds, \tag{2.74}$$

where the Green's function is given as follows

$$\begin{array}{lll} G(x,s) & = H(x-s)u_s(x) - \frac{1}{6}x^3 & \\ & = \frac{1}{6}(x-s)^3H(x-s) - \frac{1}{6}x^3 & \\ \iff & G(x,s) = \begin{cases} -\frac{1}{6}x^3 \text{ if } x < s, \\ -\frac{1}{2}x^2s + \frac{1}{2}xs^2 - \frac{1}{6}s^3 \text{ if } x \geq s. \end{cases} & . \end{array} \tag{2.75}$$

Therefore

$$\begin{array}{ll} \int_0^1 G(x,s)120sds & = \int\limits_0^x \left[-\frac{1}{2}x^2s + \frac{1}{2}xs^2 - \frac{1}{6}s^3\right]120sds + \int\limits_x^1 \left[-\frac{1}{6}x^3\right]120sds \\ & = \int\limits_0^x \left[-60x^2s^2 + 60xs^3 - 20s^4\right]ds + \int\limits_x^1 -20x^3sds \\ & = x^5 - 10x^3 \end{array} \tag{2.76}$$

In conclusion, the solution of problem (2.70) is

$$u(x) = 10x^3 + \int_0^1 G(x,s)120sds = x^5. \tag{2.77}$$

Chapter 3: EXISTENCE AND UNIQUENESS

3.1 Introduction

In this chapter we are dealing with some existence and uniqueness results for a general class of fourth order boundary value problems. Similar results for second order differential equations are well-known and may be found in the reference book of B. Bailey, L. F. Shampine, and P. E. Waltman (*Nonlinear Two Point Boundary Value Problems", 1968*). For fourth order differential equations which arise while studying the deflection of a beam, sufficient conditions for the existence and uniqueness of the solution have been obtained in (Agarwal, 1989). Our goal here is to present some extensions of Bailey's work to our more general case of fourth order differential equations, covering also Agarwal's particular results. These generalizations have been already obtained by O.A. Teterina in a recent master thesis (The Green's Function Method for Solutions of Fourth Order Nonlinear Boundary Value Problem, 2013), but for the sake of completeness we present them in what follows.

3.2 Existence and Uniqueness of Fourth Order Differential Equation

Let us consider the following fourth order differential equation

$$u^{(4)}(x) = f(x, u(x), u'(x), u''(x), u'''(x)), \tag{3.1}$$

subject to the following linearly independent boundary conditions

$$\begin{aligned}
B_1[u] &\equiv \alpha_{10}u(a) + \alpha_{11}y'(a) + \alpha_{12}y''(a) + \alpha_{13}y'''(a) &= 0,\\
B_2[u] &\equiv \alpha_{20}u(a) + \alpha_{21}u'(a) + \alpha_{22}u''(a) + \alpha_{23}y'''(a) &= 0,\\
B_3[u] &\equiv \beta_{30}u(b) + \beta_{31}u'(b) + \beta_{32}u''(b) + \beta_{33}u'''(b) &= 0,\\
B_4[u] &\equiv \beta_{40}u(b) + \beta_{41}u'(b) + \beta_{42}u''(b) + \beta_{43}u'''(b) &= 0,
\end{aligned} \tag{3.2}$$

where a test for the linear independence of the coefficients is given by:

$$Rank \begin{pmatrix} \alpha_{10} & \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 & 0 & 0 \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_{30} & \beta_{31} & \beta_{32} & \beta_{33} \\ 0 & 0 & 0 & 0 & \beta_{40} & \beta_{41} & \beta_{42} & \beta_{43} \end{pmatrix} = 4. \tag{3.3}$$

Let us define the following space of functions

$$S := \left\{C^{(4)}[a,b] : u \text{ satisfies } (3.2)\right\}, \tag{3.4}$$

which may become a subspace of a Banach space, by assigning some appropriate norm. We have:

Theorem 3.1. *(See Theorem 1.4 in Teterina, O.A. (2013))*
Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ *is a continuous function which satisfies*

$$|f(x,y_1) - f(x,y_2)| \le h(x)|y_1 - y_2|, \tag{3.5}$$

for some continuous function $h(x) \ge 0$*. Assume that a Green's function* G *for the equation*

$$y^{(4)} = g(x), \tag{3.6}$$

subject to the boundary conditions (3.2), exists. Let us define the operator $T : C[a,b] \to S \subset C[a,b]$*, as follows*

$$T[u(x)] = \int_a^b G(x,s) f(s,u(s))ds. \tag{3.7}$$

For a fixed $w(x) \in C[a,b]$*, nonnegative and nonidentically zero, assume that* $T : B_w \to B_w$*, where* $(B_w, \|\cdot\|)$ *is the Banach space defined below*

$$\begin{gathered} B_w := \{u \in C[a,b] : |u(x)| \le Cw(x) \text{ for some } C = C(u) > 0\}, \\ \|u\| = \sup_{a<x<b} \tfrac{|u(x)|}{w(x)}. \end{gathered} \tag{3.8}$$

a) If G *has constant sign on* $[a,b]$ *and*

$$\max_{x \in S_w} \left[\frac{z(x)}{w(x)}\right] < 1, \tag{3.9}$$

where z *is given by*

$$z(x) = \int_a^b |G(x,s)| h(s) w(s) ds, \tag{3.10}$$

while

$$S_w = \{x \in [a,b] : w(x) \neq 0\}, \tag{3.11}$$

then the boundary value problem (3.1)-(3.2) has a unique solution. Moreover, $z(x)$

satisfies

$$z^{(4)}(x) = (signG)h(x)w(x), \tag{3.12}$$

subject to the boundary conditions (3.2).

b) If G is possibly not of a constant sign and

$$\max_{x\in S_w}\left[\frac{1}{w(x)}\int_a^b |G(x,s)|h(s)w(s))ds\right] < 1, \tag{3.13}$$

then the boundary value problem (3.1)-(3.2) has a unique solution.

Proof. (a) Assume, without loosing the generality, that G is negative (when G is positive, the proof is similar). We have

$$\begin{array}{rcl} |T[u(x)] - T[v(x)]| & = & |\int_a^b G(x,s)\,[f(s,u(s)) - f(s,v(s))]\,ds| \\ & \leq & \int_a^b |G(x,s)||u(s) - v(s)|h(s)ds \\ & \leq & \int_a^b \|u(s) - v(s)\|^* |G(x,s)|h(s)w(s)ds \\ & = & \|u(s) - v(s)\|^* z(x), \end{array} \tag{3.14}$$

From (3.10) and the fact that G is a Green's function for (3.6), we conclude that $z(x)$ satisfies

$$z^{(4)}(x) = -h(x)w(x), \tag{3.15}$$

subject to the boundary conditions (3.2). Moreover, for $x \in S_w$, we have

$$\frac{|T[u(x)] - T[v(x)]|}{w(x)} \leq \frac{z(x)\|u(x) - v(x)\|^*}{w(x)}, \tag{3.16}$$

so

$$\|T[u] - T[v]\| \leq \|u - v\|^* \max_{x\in S_w} \frac{z(x)}{w(x)}. \tag{3.17}$$

Since, by hypothesis, $\max_{x\in S_w} \frac{z(x)}{w(x)} < 1$, then (3.17) implies that T is a contraction on B_w, so it has a unique fixed point which is the solution of the boundary value problem (3.1)-(3.2). The proof of a) is thus achieved.

(b) If G is possibly not of one sign, then for $x \in S_w$,

$$\frac{|T[u(x)] - T[v(x)]|}{w(x)} \leq \|u - v\|^* \frac{1}{w(x)}\int_a^b |G(x,s)|h(s)w(s)ds, \tag{3.18}$$

which implies that

$$\|T[u(x)] - T[v(x)]\| \leq \|u - v\|^* \max_{x \in S_w} \left[\frac{1}{w(x)} \int_a^b |G(x,s)| h(s) w(s) ds \right]. \tag{3.19}$$

Since, by hypothesis, the above maximum is less than 1, then T is a contraction, so it has a unique fixed point which is the solution of the BVP (3.1)-(3.2). □

Example 3.1. *Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x,y) = \frac{x^2}{1+x^2} \cdot sin(y)$ By the Mean Value Theorem, for any $y_1, y_2 \in \mathbb{R}$, there exists a number $\xi \in (y_1, y_2)$ such that*

$$|f(x,y_2) - f(x,y_1)| = \frac{\partial f}{\partial y}(x,\xi)|y_2 - y_1|. \tag{3.20}$$

Then, with

$$h(x) = \frac{x^2}{1+x^2}\ , \tag{3.21}$$

we have indeed that

$$|f(x,y_1) - f(x,y_2)| \leq h(x)|y_1 - y_2|. \tag{3.22}$$

Let us next consider a more general case:

Theorem 3.2. *(See Theorem 1.5 in Teterina, O.A. (2013))*
Assume that $f : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ satisfies the general Lipschitz condition

$$\begin{aligned} f(x,u(x),u'(x),u''(x),u'''(x)) \quad & - f(x,v(x),v'(x),v''(x),v'''(x)) \\ & \leq K|u(x) - v(x)| + L|u'(x) - v'(x)| \\ & + M|u''(x) - v''(x)| + N|u'''(x) - v'''(x)|, \end{aligned} \tag{3.23}$$

where K, L, M, N are some fixed positive constants. Assume also that the Green's function $G(x,s), a \leq x, s \leq b$, exists for the boundary value problem $u^{(4)}(x) = g(x)$, subject to (3.2). Furthermore, assume that there exist some constants, say M_1, M_2, M_3, and M_4, such that for all $x \in [a,b]$ we have

$$\begin{aligned} & \textstyle\int_a^b |G(x,s)|ds \leq M_1, \qquad \int_a^b |G_x(x,s)|ds \leq M_2, \\ & \textstyle\int_a^b |G_{xx}(x,s)|ds \leq M_3, \qquad \int_a^b |G_{xxx}(x,s)|ds \leq M_4, \\ & \text{and} \\ & LM_1 + KM_2 + MM_3 + NM_4 < 1. \end{aligned} \tag{3.24}$$

Then, there exists a unique solution to the boundary value problem (3.1)-(3.2).

Proof. Let

$$\|u\| := \max_{a\le x\le b}\left[L|u(x)| + K|u'(x)| + M|u''(x)| + N|u'''(x)|\right], \tag{3.25}$$

be a norm on $C^{(3)}[a,b]$, so that $\left(C^{(3)}[a,b], \|\cdot\|\right)$ becomes a Banach space. Let us define the operator $T\colon C^{(3)}[a,b] \to C^{(4)}[a,b]$, as follows

$$T[u(x)] = \int_a^b G(x,s) f(s,u(s),u'(s),u''(s),u'''(s))ds. \tag{3.26}$$

We first show that, indeed, T maps $C^{(3)}[a,b]$ into $C^{(4)}[a,b]$. To this end, differentiating successively (3.26), we get

$$\begin{aligned}
[T[u(x)]]' &= \textstyle\int_a^b G_x(x,s) f(s,u(s),u'(s),u''(s),u'''(s))ds,\\
[T[u(x)]]'' &= \textstyle\int_a^b G_{xx}(x,s) f(s,u(s),u'(s),u''(s),u'''(s))ds,\\
[T[u(x)]]''' &= \textstyle\int_a^b G_{xxx}(x,s) f(s,u(s),u'(s),u''(s),u'''(s))ds,
\end{aligned} \tag{3.27}$$

and

$$[T[u(x)]]^{(4)} = \int_a^b G_{xxxx}(x,s) f(s,u(s),u'(s),u''(s),u'''(s))ds. \tag{3.28}$$

Next, we want to show T is a contraction map, so we estimate

$$\begin{aligned}
T[u(x)] - T[v(x)] &\le \textstyle\int_a^b |G(x,s)|\cdot|f(s,u(s),u'(s),u''(s),u'''(s))\\
&\quad -f(s,v(s),v'(s),v''(s),v'''(s))|ds\\
&\le \textstyle\int_a^b |G_x(x,s)|\cdot|[L|u(s)-v(s)| + K|u'(s)-v'(s)|\\
&\quad +M|u''(s)-v''(s)| + N|u'''(s)-v'''(s)|\\
&\le \|u-v\| \textstyle\int_a^b |G(x,s)|ds\\
&\le \|u-v\| M_1.
\end{aligned} \tag{3.29}$$

In a similar way, one may also show that

$$\begin{aligned}
T[u(x)]' - T[v(x)]' &\le \|u-v\| \textstyle\int_a^b |G_x(x,s)|ds \le \|u-v\|M_2\\
T[u(x)]'' - T[v(x)]'' &\le \|u-v\| \textstyle\int_a^b |G_{xx}(x,s)|ds \le \|u-v\|M_3\\
T[u(x)]''' - T[v(x)]''' &\le \|u-v\| \textstyle\int_a^b |G_{xxx}(x,s)|ds \le \|u-v\|M_4.
\end{aligned} \tag{3.30}$$

Since x is an arbitrary in the above four inequalities, it follows that

$$\|T[u(x)] - T[v(x)]\| \leq \|u - v\|(LM_1 + KM_2 + MM_3 + NM_4). \tag{3.31}$$

Since, by hypothesis, $LM_1 + KM_2 + MM_3 + NM_4 < 1$, then (3.31) implies that T is a contraction. Consequently, it has a unique fixed point $u(x)$, which is the desired solution of the boundary value problem (3.1)-(3.2). $\square$

Chapter 4: ITERATIVE METHODS: DESCRIPTION AND RESULTS

In this chapter we will describe some iterative methods usually used to get numerical solutions for differential equations. Then we will describe and use our method, based on embedding Green's functions into some well-established fixed point iterations, such as, for instance, Picard's and Krasnoselskii-Mann's iterative schemes. The effectiveness of the proposed iterative method is established by implementing it on several numerical examples, including linear and nonlinear fourth order boundary value problems. We then compare our results with the analytical solution or with other numerical solutions obtained in the literature using different methods.

4.1 Fixed Point Iteration Methods

The Fixed Point Iteration methods are some mathematical methods widely used to solve numerically differential equations with initial and boundary value problems. As the name suggests, the idea is to repeat a certain number of steps until when the desired fixed point condition is met. More precisely, let us consider the Banach space X and a given operator $T : X \longrightarrow X$. We say that $x \in X$ is a fixed point for T if

$$T(x) = x. \tag{4.1}$$

Let us now consider and arbitrary point $x_0 \in X$. Then a sequence $\{x_n\} \subset X$, defined as

$$x_{n+1} = T(x_n), \; n = 0, 1, 2, ... \tag{4.2}$$

is called the *fixed point iteration* or *Picard iteration procedure* and it is the most basic iterative method. More general iterative methods can be derived easily and we mentioned below two such more general ierative method:

1) **The Mann Iterative procedure**:

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T x_n, \; n = 0, 1, 2, ..., \tag{4.3}$$

where $\{\alpha_n\} \subset [0, 1]$.

2) **The Ishikawa Iterative procedure:**

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T y_n, n = 0, 1, 2, ..., \\ y_n &= (1 - \beta_n) x_n + \beta_n T x_n, n = 0, 1, 2, ..., \end{aligned} \tag{4.4}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. We note that when $\alpha_n \equiv a \neq 1$ (*const.*), the iteration (4.3) reduces to the so-called *Krasnoselskii iteration*, while for $\alpha_n \equiv 1$ we obtain the *Picard iteration* (4.2). Also, for $\beta_n \equiv 0$ in (4.4) we obtain (4.3). Picard, Mann and Ishikawa iterations are all known to be convergent to a unique fixed point (see (Berinde, 2014) for a comparison between their convergence's conditions).

However, in this thesis we will deal with iterative methods for solving differential equations, in which the given operator T involves the Green's function of the differential equation.

4.2 Green's Function-Picard's fixed point iteration

We will describe our method by applying it directly to find the solution of the following boundary value problem:

Example 4.1. *Let us solve*

$$\begin{cases} u^{(4)}(x) - 2u''(x) + u(x) = -8e^x, \\ u(0) = u(1) = 0, u''(0) = 0, u''(1) = -4e. \end{cases} \tag{4.5}$$

One may easily verify that the problem has an exact solution $u(x) = x(1-x)e^x$. Moreover, using the techniques of Chapter 2, one may find that the Green's function for the above problem is

$$G(x,s) = \begin{cases} \left(\frac{1}{6}s - \frac{1}{6}\right)x^3 + \left(\frac{1}{6}s^3 - \frac{1}{2}s^2 + \frac{1}{3}s\right)x \text{ if } 0 < x < s, \\ \left(\frac{1}{6}s\right)x^3 + \left(-\frac{1}{2}s\right)x^2 + \left(\frac{1}{3}s + \frac{1}{3}s^3\right)x - \frac{1}{6}s^3 \text{ if } s < x < 1. \end{cases} \tag{4.6}$$

Next, we create some iterations as follows. The first term will be acquired from solving the homogeneous problem

$$\begin{cases} u^{(4)}(x) = 0, \\ u(0) = u(1) = 0, u''(0) = 0, u''(1) = -4e. \end{cases} \tag{4.7}$$

The solution of this problem is

$$w[0] := -\frac{2}{3}ex^3 + \frac{2}{3}ex. \tag{4.8}$$

The iteration will continue using the coefficients of Green's function

$$\begin{aligned} H[k+1] \quad := \quad & \int_0^s \left[\left(\tfrac{1}{6}s - \tfrac{1}{6}\right) x^3 + \left(\tfrac{1}{6}s^3 - \tfrac{1}{2}s^2 + \tfrac{1}{3}s\right) x \right] \cdot \\ & \cdot \left(w[k]^{(4)} - 2w[k]'' + w[k] + 8e^x \right) dx; \end{aligned} \tag{4.9}$$

$$\begin{aligned} G[k+1] \quad := \quad & \int_s^1 \left[\left(\tfrac{1}{6}s\right) x^3 - \left(\tfrac{1}{2}s\right) x^2 + \left(\tfrac{1}{3}s + \tfrac{1}{6}s^3\right) x - \left(\tfrac{1}{6}s^3\right) \right] \cdot \\ & \cdot \left(w[k]^{(4)} - 2w[k]'' + w[k] + 8e^x \right) dx; \end{aligned} \tag{4.10}$$

Then, the next term of the iteration is given as follows

$$w[k+1] = w[k] - \alpha(G[k+1] + H[k+1]), \tag{4.11}$$

where α may depend on the parameters and the nature of the boundary value problem.

In the following table the evaluation of $|w[35] - x(1-x)e^x|$ for i from 1 to 10, is given:

x	Absolute Error
0.1	2.54630333×10^-25
0.2	4.84335676×10^-25
0.3	6.66630867×10^-25
0.4	7.83671585×10^-25
0.5	8.24001068×10^-25
0.6	7.83671585×10^-25
0.7	6.66630867×10^-25
0.8	4.84335676×10^-25
0.9	2.54630333×10^-25

Table 4.1: Absolute Error Numerical Solution Vs Exact Solution

We note that the time taken to calculate an exact number of iterations could be costly. As an alternative approach, we can make use of Taylor series approximations, when the original differential equation is replaced with an approximating polynomial, which will have a positive effect on the process efforts. More precisely, we can replace the previous iterations using Taylor series as follows

$$H[k+1] := \int_0^s \left[\left(\frac{1}{6}s - \frac{1}{6} \right) x^3 + \left(\frac{1}{6}s^3 - \frac{1}{2}s^2 + \frac{1}{3}s \right) x \right] (T(x)))\, dx; \tag{4.12}$$

$$G[k+1] := \int_s^1 \left[\left(\frac{1}{6}s\right)x^3 - \left(\frac{1}{2}s\right)x^2 + \left(\frac{1}{3}s + \frac{1}{6}s^3\right)x - \left(\frac{1}{6}s^3\right)\right](T(x))\,dx \quad (4.13)$$

where $T(x)$ is defined to be the Taylor series of

$$w[k]^{(4)} - 2w[k]'' + w[k] + 8e^s. \quad (4.14)$$

In this case, with this new iterative method, we used computed the first 60 terms and the time taken for the solution was reasonable.

In the the following example the Taylor approximation is used for iterations from the 4th to the 25th term of the iteration, since the nonlinear function f in the differential equation is cubic and calculating 25, or even a less number of iterations, using the exact function could be time consuming. On the other hand, using Taylor approximation one may see that we get better approximations of the solution.

Example 4.2. *Let us solve*

$$\begin{aligned} &u''''(x) = 3(u')^2 + 4.5u^3, \\ &u(0) = 4, u''(0) = 24, u(1) = 1, u''(1) = 1.5. \end{aligned} \quad (4.15)$$

The exact solution of problem (4.15) is

$$u(x) = \frac{4}{(1+x)^2}. \quad (4.16)$$

Next, we create some iterations as follows. The first term will be acquired from solving.the homogeneous problem

$$\begin{cases} u^{(4)}(x) = 0, \\ u(0) = 4,\ u(1) = 1,\ u''(0) = 24,\ u''(1) = 1.5. \end{cases} \quad (4.17)$$

The iteration will continue using the coefficients of Green's function mentioned before

$$\begin{aligned} H[k+1] \ := \ & \textstyle\int_0^s \left[\left(\frac{1}{6}s - \frac{1}{6}\right)x^3 + \left(\frac{1}{6}s^3 - \frac{1}{2}s^2 + \frac{1}{3}s\right)x\right] \cdot \\ & \cdot\left(w[k]^{(4)} - 3\left(w[k]'\right)^2 - 4.5\left(w[k]\right)\right)dx; \end{aligned} \quad (4.18)$$

$$
\begin{aligned}
G[k+1] \quad := \quad & \int_s^1 \left[\left(\tfrac{1}{6}s\right) x^3 - \left(\tfrac{1}{2}s\right) x^2 + \left(\tfrac{1}{3}s + \tfrac{1}{6}s^3\right) x - \left(\tfrac{1}{6}s^3\right) \right] \cdot \\
& \cdot \left(w[k]^{(4)} - 3\left(w[k]'\right)^2 - 4.5\left(w[k]\right) \right) dx;
\end{aligned} \tag{4.19}
$$

Then, the next term of the iteration is given as follows

$$
w[k+1] = w[k] - \alpha(G[k+1] + H[k+1]), \tag{4.20}
$$

where α may depend on the parameters and the nature of the boundary value problem.

Then using the following procedure, we obtain the best α which will optimise our method.

- Define $u_1 = w[n](x)$.
- Define $R_1 = u_1''''(x) - 3{u_1'}^2(x) - 4.5u_1^3(x)$
- Take $L_{norm} = \int_0^1 (R_1)^2 dx$
- Minimize L_{norm}

By applying these steps, one may find that the best α is equal to 1.23.

The Taylor approximation will be used from the 4th term to the 20th term of the iteration, to minimise the CPU timing. So, as in the previous example, we replace the previous iterations using now Taylor series

$$
H[k+1] := \int_0^x \left[\left(\frac{1}{6}s - \frac{1}{6}\right) x^3 + \left(\frac{1}{6}s^3 - \frac{1}{2}s^2 + \frac{1}{3}s\right) x \right] (T(x)))\, dx; \tag{4.21}
$$

$$
G[k+1] := \int_x^1 \left[\left(\frac{1}{6}s\right) x^3 - \left(\frac{1}{2}s\right) x^2 + \left(\frac{1}{3}s + \frac{1}{6}s^3\right) x - \left(\frac{1}{6}s^3\right) \right] (T(x))\, dx \tag{4.22}
$$

where $T(x)$ is defined to be the Taylor series of the solution of

$$
\left(w[k]^{(4)} - 3\left(w[k]'\right)^2 - 4.5\left(w[k]\right)^3 \right). \tag{4.23}
$$

The next iteration will be then calculated as

$$
w[k+1] := (w[k] - \alpha(G[k+1] + H[k+1])). \tag{4.24}
$$

We note that another factor that helped reducing the CPU time was to take Taylor approximation around $(0.5, 0.5)$, rather than around $(0, 0)$. Given the results of all the previous modifications the time taken for the solution was reasonable. The absolute error is shown by the table below:

x	Absolute Error
0.1	$1.388041910 \times 10^{-}\ 15$
0.2	$9.091535178 \times 10^{-}\ 16$
0.3	$1.228994381 \times 10^{-}\ 15$
0.4	$1.491554362 \times 10^{-}\ 15$
0.5	$4.548867987 \times 10^{-}\ 15$
0.6	$5.736664592 \times 10^{-}\ 15$
0.7	$4.855987077 \times 10^{-}\ 15$
0.8	$2.991137639 \times 10^{-}\ 15$
0.9	$1.256766229 \times 10^{-}\ 15$

Table 4.2: Absolute Error Numerical Solution Vs Exact Solution

4.3 Comparison with the Spline Method

In this subsection we will illustrate a general example of solving numerically fourth order boundary value problems. Various approaches are already known to deal with such problems. For instance, in (Graef, Qian, & Yang, 2003) the Green's function was also used, while in (Zhao, C., Chen, W., & Zhou, J. 2010) a so-called periodic method was used. Also, an efficient method for solving higher order differential equations was presented in (Malek, Beidokhti, 2006) using neural-like systems of computation, while in (M. Sakai and R. Usmani, 1983) the differential equation is solved by using Spline Method. Our aim is to compare our method with a few such known methods.

Example 4.3. *Let us consider the following boundary value problem*

$$\begin{aligned} &u^{(4)}(x) + xu(x) = -(8 + 7x + x^3)e^x, \\ &u(0) = u(1) = 0, u'(0) = 1, u'(1) = -e. \end{aligned} \tag{4.25}$$

This problem has an exact solution

$$u(x) = x(1 - x)e^x. \tag{4.26}$$

As per (M. Sakai and R. Usmani, 1983), the table below shows the maximum errors in absolute value for the above problem. Richardson's h^2-extrapolation technique was used to improve the accuracy of their solution.

First, the values without Richardson's technique are shown below: Next, the table

Maximum Errors		
Values of h	quintic	sextic
$h = 1/4$	$0.689 \times 10^{-}3$	$0.380 \times 10^{-}5$
$h = 1/8$	$0.172 \times 10^{-}3$	$0.222 \times 10^{-}6$
$h = 1/16$	$0.429 \times 10^{-}4$	$0.137 \times 10^{-}7$
$h = 1/32$	$0.107 \times 10^{-}4$	$0.854 \times 10^{-}9$
$h = 1/64$	$0.268 \times 10^{-}5$	$0.536 \times 10^{-}10$

Table 4.3: Comparison Between Quintic and Sextic Techniques

below shows the errors on Richardson's extrapolation at $t = 1/2$.

Maximum Errors		
Values of h	quintic	sextic
$h = 1/4$		
$h = 1/8$	$0.859 \times 10^{-}6$	$0.168 \times 10^{-}7$
$h = 1/16$	$0.452 \times 10^{-}7$	$0.168 \times 10^{-}9$
$h = 1/32$	$0.269 \times 10^{-}8$	$0.177 \times 10^{-}11$
$h = 1/64$	$0.158 \times 10^{-}9$	$0.853 \times 10^{-}13$

Table 4.4: Richardson's Extrapolation with $t = \frac{1}{2}$

We note that he iteration used for Quintic Splines is

$$4\bar{x}_{h/2}(1/2) - \bar{x}_h(1/2)/3 - \hat{x}(1/2), \tag{4.27}$$

whereas, the iteration used for sextic Splines is

$$16\bar{z}_{h/2}(1/2) - \bar{z}_h(1/2)/15 - \hat{x}(1/2). \tag{4.28}$$

The best results of Spline Method with possible combinations can be shown in the following table:

Absolute Errors		
Richardson's extrapolation	quintic	sextic
Not Applied	0.268×10^{-5}	0.536×10^{-10}
Applied	0.158×10^{-9}	0.853×10^{-13}

Table 4.5: Possible Combinations of Spline Methods

In a recently published paper (Hossain, 2015), the same problem was solved using **Legendre Polynomials Method**. The table below shows the numerical results using 11-polynomials.

x	Absolute Error
0.1	4.6490×10^{-14}
0.2	1.276×10^{-13}
0.3	1.782×10^{-13}
0.4	2.831×10^{-15}
0.5	2.501×10^{-13}
0.6	1.865×10^{-13}
0.7	8.543×10^{-14}
0.8	2.007×10^{-13}
0.9	1.443×10^{-13}

Table 4.6: Numerical Results of BVP using Legendre Polynomials

The table below shows the results found when using the proposed numerical solution by applying Green's Function with Iteration method. It is worth noting that only 25 iterations were used in the task below, and CPU timing was significantly less. The maximum absolute error in this method 4.140×10^{-20}, whereas the maximum error from Legendre Polynomials was 2.501×10^{-13}.

x	Absolute Error
0.1	$4.676 \times 10^{-}21$
0.2	$1.530 \times 10^{-}20$
0.3	$2.730 \times 10^{-}20$
0.4	$3.690 \times 10^{-}20$
0.5	$4.140 \times 10^{-}20$
0.6	$3.941 \times 10^{-}20$
0.7	$3.112 \times 10^{-}20$
0.8	$1.861 \times 10^{-}20$
0.9	$6.058 \times 10^{-}21$

Table 4.7: Absolute Error using Green's Function with Fixed Point Iteration

Chapter 5: CONVERGENCE ANALYSIS

In the previous chapter, to obtain approximations for the solutions to some fourth order problems, we have used some numerical methods based in iterations. Therefore, the convergence of such iteration should be also investigated, this final chapter being thus dedicated to the convergence of the methods employed and their rates. To this end, the contraction principle will be used. More precisely, the method we have used in the previous chapter is based on Green's functions and fixed point iterations. Such a method has been previously used by M. Abushammala, A. Khouri and Sayfy in (Abushammala, 2015) for a third order differential equations and adapted by us to fourth order equations in this thesis.

First, let us state the following classical result:

Theorem 5.1. *(Banach-Picard Theorem)*
Let (X,d) be a non-empty complete metric space with a contracting mapping $T : X \to X$. Then T admits a unique point x^* in X such that $T(x^*) = x^*$. Moreover, we can determine precisely x^* by choosing an arbitrary value x_0 in X, defining $x_n = T(_{n-1})$ and then letting $x_n \to x^*$.

The proof will be applied on nonlinear differential equations with boundary conditions.

$$u^{(4)}(t) - f(t, u(t), u'(t), u''(t), u'''(t)) = 0 \tag{5.1}$$

with boundary conditions

$$u(0) = \alpha, u(1) = \beta, u''(0) = \gamma, u''(1) = \zeta. \tag{5.2}$$

We define an operator $T[u_p]$ follows

$$\begin{aligned} T[u_p] = \quad & u_p + \textstyle\int_a^b G(t,s) \left[H(s)u_p''''(s) + I(s)u_p'''(s) + J(s)u_p''(s) + K(s)u_p'(s)\right. \\ & \left. + L(s)u_p(s) - f(t, u(t), u'(t), u''(t), u'''(t))\right] ds. \end{aligned} \tag{5.3}$$

where

$$u_p = \int_a^b G(t,s) f(s, u_p(s), u_p'(s), u_p''(s), u_p'''(s))] ds. \tag{5.4}$$

Using Green's function found in the previous chapter

$$G(t,s) = \begin{cases} (\frac{1}{6}s - \frac{1}{6})t^3 + (\frac{1}{6}s^3 - \frac{1}{2}s^2 + \frac{1}{3}s)t \text{ if } 0 < t < s, \\ (\frac{1}{6}s)t^3 - (\frac{1}{2}s)t^2 + (\frac{1}{3}s + \frac{1}{3}s^3)t - \frac{1}{6}s^3 \text{ if } s < t < 1. \end{cases} \tag{5.5}$$

Therefore, the iteration scheme can be defined as follow:

$$u_{n+1}(t) = u_n(t) + \int_0^1 G^*(t,s)[u_n''''(s) - f(s, u_n(s), u_n'(s), u_n''(s), u_n'''(s))]ds, \tag{5.6}$$

where $G^*(t,s)$ is the adjoint Green's function of the previously defined $G(t,s)$:

$$G^*(t,s) = \begin{cases} (\frac{1}{6}s)t^3 - (\frac{1}{2}s)t^2 + \left(\frac{1}{3}s + \frac{1}{3}s^3\right)t - \frac{1}{6}s^3 \text{ if } 0 < t < s, \\ (\frac{1}{6}s - \frac{1}{6})t^3 + (\frac{1}{6}s^3 - \frac{1}{2}s^2 + \frac{1}{3}s)t \text{ if } s < t < 1. \end{cases} \tag{5.7}$$

On some occasions the adjoint Green's function had to be replaced with the Green's function, which should also satisfy the conditions imposed on the Green's function, i.e. $LG(t,s) = \delta(t-s)$, so the adjoint Green's function satisfies $L^*G^*(t,s) = \delta(t-s)$.

Theorem 5.2. *Assume that $f(t,u,u',u'',u''')$ is a continuous function on $[a,b]$, whose derivative is bounded with respect to u. Assume that $K := \frac{1}{24}L_c < 1$, where $L_c = max\left|\frac{\delta f}{\delta u}\right|$. Then the iterative sequence $\{u_n(t)\}_{n=1}^{\infty}$, given by*

$$\begin{aligned} u_{n+1}(t) \quad = \quad & u_n(t) - \textstyle\int_0^t \left[(\frac{1}{6}s)t^3 - (\frac{1}{2}s)t^2 + (\frac{1}{3}s + \frac{1}{3}s^3)t - \frac{1}{6}s^3\right] \\ & \cdot[u_n''''(s) - f(s, u_n(s), u_n'(s), u_n''(s), u_n'''(s))]ds \\ & - \textstyle\int_t^1 \left[(\frac{1}{6}s - \frac{1}{6})t^3 + (\frac{1}{6}s^3 - \frac{1}{2}s^2 + \frac{1}{3}s)t\right] \\ & \cdot[u_n''''(s) - f(s, u_n(s), u_n'(s), u_n''(s), u_n'''(s))]ds, \end{aligned} \tag{5.8}$$

converges uniformly to the exact solution $u(t)$ of the problem.

Proof. Let us start by computing $I = \int_a^b Gu''''(t)ds$ by parts. Let $u = G$, so $u' = G_s$, and $dv = u''''(s)$, so $v = u'''(s)$.We then have

$$I = G(b)u'''(b) - G(a)u'''(a) - \int_a^b G_s u''' ds. \tag{5.9}$$

By repeating this argument, taking $u = G$ and $dv = u'''(s)$, we get

$$I = G(b)u'''(b) - G(a)u'''(a) - G_s(b)u''(b) + G_s(a)u''(a) + \int_a^b G_{ss}u''(t)ds. \tag{5.10}$$

Then, we repeat again the same argument until when we get

$$I = G(b)u'''(b) - G(a)u'''(a) - G_s(b)u''(b) + G_s(a)u''(a) + G_{ss}(b)u'(b)$$
$$-G_{ss}(a)u'(a) - G_{sss}(b)u(b) + G_{sss}(a)u(a) + \int_a^b G_{ssss}u(t)ds. \tag{5.11}$$

By Green's function properties, we know that $G_{ssss}(t,s) = -\delta(t-s)$, so

$$\int_a^b G_{ssss}(t,s)u(s)ds = -u(t), \tag{5.12}$$

Now, we know that the Green's Function satisfies $G(t,0) = G(t,1) = 0$. Moreover, $G_{ss}(t,0) = G_{ss}(t,1) = 0$ so that

$$\begin{aligned} u_{n+1}(t) &= u_n(t) + G(b)u'''(b) - G(a)u'''(a) - G_s(b)u''(b) + G_s(a)u''(a) + G_{ss}(b)u'(b) \\ &\quad -G_{ss}(a)u'(a) - G_{sss}(b)u(b) + G_{sss}(a)u(a) \\ &\quad +\int_a^b G_{ssss}u(t)ds - \int_a^b G^*(t,s)f(s,u,u',u'',u'''))ds. \end{aligned} \tag{5.13}$$

After some simplification we get

$$\begin{aligned} u_{n+1}(t) &= -\zeta G_s^*(t,b) + \gamma G_s^*(a) - \beta G_{sss}^*(b) + \alpha G_{sss}^*(a) \\ &\quad -\int_a^b G^*(t,s)f(s,u_n(s),u_n'(s),u_n''(s),u_n'''(s))ds. \end{aligned} \tag{5.14}$$

Based on (5.7), we can compute $G_s^*(t,b)$, $G_s^*(t,a)$, $G_{sss}^*(t,a)$, and $G_{sss}^*(t,b)$. For instance

$$G_s^*(t,s) = \begin{cases} \frac{1}{6}t^3 - \frac{1}{2}t^2 + \left(\frac{1}{3} + s^2\right)t - \frac{1}{2}s^2 \text{ if } 0 < t < s, \\ \frac{1}{6}t^3 + (\frac{1}{2}s^2 - s + \frac{1}{3})t \text{ if } s < t < 1. \end{cases} \tag{5.15}$$

Therefore, $G_s^*(t,1) = 0$ and $G_s^*(t,0) = \frac{1}{6}t^3 - \frac{1}{2}t^2 + \frac{1}{3}t$. Similarly, one may obtain

$$G_{ss}^*(t,s) = \begin{cases} 2st - s \text{ if } 0 < t < s, \\ (s-1)t \text{ if } s < t < 1, \end{cases} \tag{5.16}$$

and

$$G_{sss}^*(t,s) = \begin{cases} 2t - 1 \text{ if } 0 < t < s, \\ t \text{ if } s < t < 1, \end{cases} \tag{5.17}$$

so $G^*_s(t,1) = 0$ and $G^*_s(t,0) = \frac{1}{6}t^3 - \frac{1}{2}t^2 + \frac{1}{3}t$. In conclusion

$$y_{n+1}(t) = \gamma\left[\frac{1}{6}t^3 - \frac{1}{2}t^2 + \frac{1}{3}t\right] - \beta t + \alpha(2t-1) - \int_a^b G^*(t,s)f(s,u_n,u_n',u_n'',u_n'''))ds. \tag{5.18}$$

Next, we define a new operator $T_G : C[0,1] \to C[0,1]$, as follows:

$$T_G(y_n) = \left[\frac{\gamma}{6}\right]t^3 - \frac{\gamma}{2}t^2 + \left(2\alpha - \beta + \frac{\gamma}{3}\right)t - \alpha - \int_a^b G^*(t,s)f(s,u_n,u_n',u_n'',u_n''')ds, \tag{5.19}$$

and note that

$$u_{n+1}(t) = T_G(u_n). \tag{5.20}$$

To apply Banach-Picard fixed point theorem, we have to show that T_G is contracting mapping. To this end, we consider the following difference

$$|T_G(y) - T_G(z)| = \left|\int_a^b G^*(t,s)f(s,y,y',y'',y''')ds - \int_a^b G^*(t,s)f(s,z,z',z'',z''')ds\right|, \tag{5.21}$$

and try to find the appropriate estimate for it. Let us consider

$$g(t) = \int_a^b G^*(t,s)ds. \tag{5.22}$$

The integration of the Green's Function's branches should be then calculated separately, as follows

$$\int_0^t \left[(\frac{1}{6}s)t^3 - (\frac{1}{2}s)t^2 + (\frac{1}{3}s + \frac{1}{3}s^3)t - \frac{1}{6}s^3\right]ds = (\frac{1}{6})t^5 - (\frac{7}{24})t^4 + (\frac{1}{6})t^3, \tag{5.23}$$

respectively

$$\int_t^1 \left[(\frac{1}{6}s - \frac{1}{6})t^3 + (\frac{1}{6}s^3 - \frac{1}{2}s^2 + \frac{1}{3}s)t\right]ds = (-\frac{1}{8})t^5 + (\frac{1}{24})t + (\frac{1}{3})t^4 - (\frac{1}{4})t^3. \tag{5.24}$$

Simplifying the above integrals, one can write that

$$g(t) = \frac{1}{24}t^5 + \frac{1}{24}t^4 - \frac{1}{12}t^3 + \frac{1}{24}t. \tag{5.25}$$

Therefore, by looking for the extreme values of $g(t)$, we find that

$$|g(t)| \leq \frac{1}{24}, \tag{5.26}$$

so

$$|T_G(y) - T_G(z)| \leq \frac{1}{24}\left|\int_0^1 G^*(t,s)f(s,y,y',y'',y''') - f(s,z,z',z'',z''')\right| ds. \tag{5.27}$$

Applying next the mean value theorem for f, we get

$$|T_G(y) - T_G(z)| \leq \frac{1}{24}\max|f(s,y,y',y'',y''') - f(s,z,z',z'',z''')|. \tag{5.28}$$

By defining

$$\| y - z \| = max|y(t) - z(t)| \text{ and } L_c = max|\frac{\delta}{\delta y}f(t,y,y',y'',y''')|, \tag{5.29}$$

and making use of the fact that $K = \frac{1}{24}L_c < 1$, we have

$$\| T_G(y) - T_G(z) \| \leq K \| y - z \| . \tag{5.30}$$

The rate of convergence can be calculated by taking two consecutive terms from the iteration

$$\|u_{n+1} - u_n\| = \|T_G(u_n) - T_G(u_{n-1})\| \leq K\|u_n - u_{n-1}\| \leq K^n\|u_1 - u_0\|. \tag{5.31}$$

Using the above inequality with two terms with indices m, n such that $m > n > 0$, then

$$\begin{aligned} \|u_m - u_n\| &\leq \|u_m - u_{m-1}\| + ... + \|u_{n+1} - u_n\| \\ &\leq \left(K^{m-1} + ... + K^n\right)\|u_1 - u_0\| \\ &\leq K^n\left(1 + ... + K^{m-n}\right)\|u_1 - u_0\| \\ &\leq \tfrac{K^n}{1-K}\|u_1 - u_0\|, \end{aligned} \tag{5.32}$$

where the term $\frac{K^n}{1-K}$ is the infinite sum of the previous geometric series. In conclusion, as m grows, the error is

$$\|u - u_n\| = \frac{K^n}{1-K}\|u_1 - u_0\|. \tag{5.33}$$

Chapter 6: CONCLUSION

In this thesis we have first introduced some basic facts and properties for the Green's Functions of some general classes of linear differential equations. We have shown how to obtain the Green's Functions for equations of different orders and supported these results by several examples.

Then we have presented some Existence and Uniqueness results, based on some classical ideas found in P. B. Baley, L.F. Shampine, P. E. Waltman, 1969. Again, the theoretical results have been supported by some examples.

Next, we discussed the concept of Fixed Point Iteration and presented some classical iteration schemes, such as, for instance, Picard Iteration and Mann Iteration, which were then used in our proposed method in different contexts. Also, we presented a comparison between our proposed method and other established methods, such as, for instance, the Spline method, and the Richardson's h^2-extrapolation. The results shown demonstrate the advantage of the proposed method over these traditional methods. Another advantage of our method is the ability of controlling the parameter to improve the accuracy and the processing time. We have illustrated a step by step description of the proposed method, applied on numerical examples with exact solutions, to show the effectiveness of this method. The use of other techniques that we can eventually employ in some future works to improve our results and optimise the processing timing were also explained.

Finally, the convergence of the solution of our numerical method has been investigated, as well. Based on the Banach-Picard fixed-point theorem we found some clear conditions for the convergence of the iteration sequences considered.

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Vita

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