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Research Article

The Randomized American Option as a Classical Solution to the Penalized Problem

Guillaume Leduc

American University of Sharjah, P.O. Box 26666, Sharjah, UAE

Correspondence should be addressed to Guillaume Leduc; gleduc@aus.edu

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We connect the exercisability randomized American option to the penalty method by showing that the randomized American option value u is the unique *classical* solution to the Cauchy problem corresponding to the *canonical* penalty problem for American options. We also establish a uniform bound for Au, where A is the infinitesimal generator of a geometric Brownian motion.

1. Introduction

The Black-Scholes model is the *lingua franca* [1], the vehicular language, of option pricing. Yet pricing American options in the Black-Scholes remains the topic of a significant literature because every known method requires significant numerical calculations. For surveys on American option pricing we refer to [2–4].

Among all methods, Howison et al. [5] argue that, with a quadratic convergence [6], the so-called *penalty method* for the value of the American option in the Black-Scholes model is "the most efficient numerical approximation methods presently available for American option valuation." It was used among others in [5–14]. The penalty method transforms the free boundary problem associated with the price of the American option into a partial differential equation (PDE) of the form

$$\frac{\partial}{\partial t}u(x,t) = \mathrm{A}u(x,t) - ru(x,t) + \beta(h(x), u(x,t)), \quad u(x,0) = h(x), \tag{1}$$

where

$$\mathrm{A}f(x) = rxf'(x) + \frac{1}{2}x^2\sigma^2 f''(x) \tag{2}$$

is the infinitesimal generator of a risk neutral geometric Brownian motion ξ with volatility σ and r is the risk-free rate and where the penalty term, $\beta(h(x), u(x,t))$, is a term which *typically* is zero when $h(x) \le u(x,t)$ allowing the option to behave like a European option and which drastically pushes the value of the option higher when $h(x) > u(x,t)$. Indeed, the "canonical" [5] penalty term is

$$\beta(h(x), u(x,t)) = \max(h(x) - u(x,t), 0)\, n \tag{3}$$

for some large value of $n > 0$.

In studies [5, 6, 9–12, 14] where such *canonical* penalty method was used for approximating the American option value in the Black-Scholes model, the solution to the associated PDE is seen as a *viscosity solution* or as a *weak solution* that is the solution to a *variational problem*. It is well known that, in general, viscosity and weak solutions do not possess the regularity properties of *classical solutions* which can actually be differentiated in the classical sense to solve the PDE.

In this paper, we connect the randomized American option [15] to the penalty method, showing that not only does its value u solve the *canonical* penalty problem (1), but also it is a classical solution to this Cauchy problem and, for a given maturity, Au is bounded.

Many of the above cited papers using the *canonical* penalty method as well as other papers using penalized problems such as [16, 17] were actually concerned not by estimating the value of American options but rather by determining the exact speed of convergence of option values under tree schemes approximations of the Black-Scholes model, a difficult [18] and long lasting problem still unsolved when the maturity is not allowed to float. Indeed, randomized American options can be used as a tool to help determine this exact speed of convergence. It is well known that payoff smoothness drastically affects this rate of convergence. We believe that our result may contribute to solving this problem. Yet the submitted paper answers the very natural question of whether or not the canonical penalty problem has a *classical* solution.

A randomly exercisable American option is an option which, prior to maturity, can be exercised only at some exercisable times following each other independently after an exponentially distributed waiting time of average $1/n$. Under the label "option with random intervention time," randomly exercisable American options were first introduced in Dupuis and Wang [19] for American perpetuities, then in Guo and Liu [20] for American lookback perpetuities, and then in Leduc [15] for American options. Note that the exercisability randomized American option considered in this paper differs from Carr's maturity randomized option [21] which can be exercised any time up to some random maturity. In contrast, the exercisability randomized option can be exercised only at random times up to a fixed maturity.

We denote by $v_t^{\mathscr{R}_n}h(x)$ the value of a randomly exercisable American option with maturity t and payoff function h, when the spot price ξ_0 of the underlying at time 0 is x. The value of this randomized American option $v_t^{\mathscr{R}_n}h(x)$ is given by

$$v_t^{\mathscr{R}_n}h(x) \stackrel{\text{def}}{=} \sup_{\tau\in\mathscr{T}^n[0,t]} E_x\left(e^{-r\tau}h(\xi_\tau)\right), \tag{4}$$

where $\mathscr{T}^n[0,t]$ is the set of exercisable stopping times in $[0,t]$ and where E_x is the expectation of ξ given that $\xi_0 = x$. As shown in [15], $v_t^{\mathscr{R}_n}h(x)$ is the only solution to the following evolution equation:

$$\begin{aligned} v_t^{\mathscr{R}_n}h(x) &= e^{-nt}\mathscr{E}_t h(x) \\ &\quad + \int_0^t \mathscr{E}_s\left(\max\left(h, v_{t-s}^{\mathscr{R}_n}h\right)\right)(x)\, ne^{-ns}ds, \end{aligned} \tag{5}$$

where, for functions $\psi : \mathbb{R} \to \mathbb{R}$, the expression $\mathscr{E}_t\psi(x)$ denotes the discounted expectation

$$\mathscr{E}_t\psi(x) \stackrel{\text{def}}{=} e^{-rt}E_x\left(\psi(\xi_t)\right). \tag{6}$$

It is also shown in [15] that $v_t^{\mathscr{R}_n}h(x)$ solves

$$v_t^{\mathscr{R}_n}h(x) = \mathrm{U}_t h(x) + \int_0^t \mathrm{U}_s\left(\mathrm{G}_{t-s}^n h\right)(x)\, ds, \tag{7}$$

where

$$\mathrm{G}_{t-s}^n h(y) \stackrel{\text{def}}{=} \max\left(h(y) - v_{t-s}^{\mathscr{R}_n}h(y), 0\right)n - v_{t-s}^{\mathscr{R}_n}h(y)\, r \tag{8}$$

and where U is the semigroup associated with ξ; that is, for functions $\psi : \mathbb{R} \to \mathbb{R}$,

$$\mathrm{U}_t\psi(x) \stackrel{\text{def}}{=} E_x\left(\psi(\xi_t)\right). \tag{9}$$

Recall that a Lipschitz function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is absolutely continuous and almost surely differentiable. In a slight abuse of notion, we replace the Lipschitz constant C of h by $\|h'\|_\infty$ so that for every $x, y \in \mathbb{R}_+$

$$\left|h(x) - h(y)\right| \le \left\|h'\right\|_\infty \left|x - y\right|. \tag{10}$$

Finally, we denote by I the identity function: $I(z) = z$ for every z.

Theorem 1. *If h is a Lipschitz function and $\|Ih'\|_\infty < \infty$, then $v_t^{\mathscr{R}_n}h(x)$ is the unique classical solution to the Cauchy problem:*

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) &= Au(x,t) - ru(x,t) \\ &\quad + \max\left(h(x) - u(x,t), 0\right)n, \\ u(x,0) &= h(x). \end{aligned} \tag{11}$$

Furthermore,

$$\left|\frac{\partial}{\partial x}v_t^{\mathscr{R}_n}h(x)\right| \le \left\|h'\right\|_\infty, \tag{12}$$

and for every T, there exists a constant Q depending only on r, σ, T, $\|h\|_\infty$, and $\|Ih'\|_\infty$ such that, for every $0 < t \le T$ and $0 \le x$,

$$\begin{aligned} \left|x\frac{\partial}{\partial x}v_t^{\mathscr{R}_n}h(x)\right| &\le Q, \\ \left|x^2\frac{\partial^2}{\partial x^2}v_t^{\mathscr{R}_n}h(x)\right| &\le \frac{Q}{\sqrt{t}}. \end{aligned} \tag{13}$$

The proof of our main result is divided into several steps. In Section 2, we show that $v_t^{\mathscr{R}_n}h(x)$ is continuous. In Section 3, we prove that $v_t^{\mathscr{R}_n}h(x)$ is Lipschitz with respect to x. In Section 4, we show that $v_t^{\mathscr{R}_n}h(x)$ is twice continuously differentiable with respect to x, and the bounds for $I(\partial/dx)v_t^{\mathscr{R}_n}$ and $I^2(\partial^2/dx^2)v_t^{\mathscr{R}_n}$ are established. In Section 5, we show that $v_t^{\mathscr{R}_n}h(x)$ is a classical solution to (11).

2. Continuity

Fix some value $T > 0$ and for every function $f : [0,T] \times [0,\infty) \to \mathbb{R}$ set

$$\left\|f(t,x)\right\|_\infty \stackrel{\text{def}}{=} \sup_{t\in[0,T]}\sup_{x\ge 0}\left|f(t,x)\right|. \tag{14}$$

Furthermore, define

$$\begin{aligned} \overline{v}_t^{\mathscr{R}_n}h(x) &\stackrel{\text{def}}{=} \overline{\lim_{(s,u)\to(t,1)}}\, v_s^{\mathscr{R}_n}h(ux), \\ \underline{v}_t^{\mathscr{R}_n}h(x) &\stackrel{\text{def}}{=} \underline{\lim}_{(s,u)\to(t,1)}\, v_s^{\mathscr{R}_n}h(ux), \end{aligned} \tag{15}$$

where $(s,u)\in[0,T]\times[0,\infty)$. One easily gets from (5) that

$$\begin{aligned}&\overline{v}_t^{\mathscr{R}_n}h(x)-\underline{v}_t^{\mathscr{R}_n}h(x)\\&\quad\le\int_0^t e^{-ns}e^{-rs}E\max\left(h(x\xi_s),\overline{v}_{t-s}^{\mathscr{R}_n}h(x\xi_s)\right)n\,ds\\&\qquad-\int_0^t e^{-ns}e^{-rs}E\max\left(h(x\xi_s),\underline{v}_{t-s}^{\mathscr{R}_n}h(x\xi_s)\right)n\,ds\\&\quad\le\int_0^T e^{-ns}\left\|\overline{v}_t^{\mathscr{R}_n}h(x)-\underline{v}_t^{\mathscr{R}_n}h(x)\right\|_\infty n\,ds\end{aligned}\tag{16}$$

yielding

$$\begin{aligned}&\left\|\overline{v}_t^{\mathscr{R}_n}h(x)-\underline{v}_t^{\mathscr{R}_n}h(x)\right\|_\infty\\&\quad\le\left(1-e^{-nT}\right)\left\|\overline{v}_t^{\mathscr{R}_n}h(x)-\underline{v}_t^{\mathscr{R}_n}h(x)\right\|_\infty\end{aligned}\tag{17}$$

which is possible only if $\|\overline{v}_t^{\mathscr{R}_n}h(x)-\underline{v}_t^{\mathscr{R}_n}h(x)\|_\infty=0$, showing that $v_t^{\mathscr{R}_n}h(x)$ is continuous.

3. Lipschitz Property

We show here that $v_t^{\mathscr{R}_n}h(x)$ is Lipschitz with respect to x.

Lemma 2. *Under the assumptions of Theorem 1, for every $x\ge 0$ and every $\varepsilon>0$,*

$$\left|v_t^{\mathscr{R}_n}h(x+\varepsilon)-v_t^{\mathscr{R}_n}h(x)\right|\le\left\|h'\right\|_\infty\varepsilon.\tag{18}$$

Proof. Note first that because h is Lipschitz, for every $0\le\alpha,\beta$ there exists a quantity γ, with $|\gamma|\le\|h'\|_\infty$ such that

$$h(\alpha)=h(\beta)+\gamma(\alpha-\beta).\tag{19}$$

Let τ_x and $\tau_{x+\varepsilon}$ be, respectively, the optimal (exercisable) stopping time for the randomized American option with maturity t when $\xi_0=x$ and when $\xi_0=x+\varepsilon$.

Step 1. Note first that when $\xi_0=x+\varepsilon$, the stopping time τ_x is suboptimal and therefore

$$\begin{aligned}v_t^{\mathscr{R}_n}h(x+\varepsilon)&=E\left(e^{-r\tau_{x+\varepsilon}}h\left((x+\varepsilon)\xi_{\tau_{x+\varepsilon}}\right)\right)\\&\ge E\left(e^{-r\tau_x}h\left((x+\varepsilon)\xi_{\tau_x}\right)\right)\\&=E\left(e^{-r\tau_x}h\left(x\xi_{\tau_x}\right)+\gamma e^{-r\tau_x}\xi_{\tau_x}\varepsilon\right)\\&=v_t^{\mathscr{R}_n}h(x)+E\left(\gamma e^{-r\tau_x}\xi_{\tau_x}\varepsilon\right)\end{aligned}\tag{20}$$

for some random variable γ satisfying $|\gamma|\le\|h'\|_\infty$. Hence, since ξ is risk neutral,

$$\begin{aligned}&v_t^{\mathscr{R}_n}h(x+\varepsilon)-v_t^{\mathscr{R}_n}h(x)\ge-\left\|h'\right\|_\infty E\left(e^{-r\tau_x}\xi_{\tau_x}\varepsilon\right)\\&\quad=-\left\|h'\right\|_\infty\varepsilon.\end{aligned}\tag{21}$$

Step 2. By definition,

$$\begin{aligned}v_t^{\mathscr{R}_n}h(x+\varepsilon)&=E\left(e^{-r\tau_{x+\varepsilon}}h\left((x+\varepsilon)\xi_{\tau_{x+\varepsilon}}\right)\right)\\&=E\left(e^{-r\tau_{x+\varepsilon}}h\left(x\xi_{\tau_{x+\varepsilon}}\right)+\gamma e^{-r\tau_{x+\varepsilon}}\varepsilon\xi_{\tau_{x+\varepsilon}}\right)\end{aligned}\tag{22}$$

for some random variable γ satisfying $|\gamma|\le\|h'\|_\infty$. By suboptimality of $\tau_{x+\varepsilon}$ when $\xi_0=x$, we conclude that

$$\begin{aligned}&v_t^{\mathscr{R}_n}h(x+\varepsilon)-v_t^{\mathscr{R}_n}h(x)\le E\left(\gamma e^{-r\tau_{x+\varepsilon}}\varepsilon\xi_{\tau_{x+\varepsilon}}\right)\\&\quad\le\left\|h'\right\|_\infty E\left(e^{-r\tau_{x+\varepsilon}}\varepsilon\xi_{\tau_{x+\varepsilon}}\right)=\left\|h'\right\|_\infty\varepsilon.\end{aligned}\tag{23}$$

□

4. $C^{(2)}$ Solution

Define

$$\zeta_s(z)\stackrel{\text{def}}{=}\exp\left(\sqrt{s}\sigma z+\left(r-\frac{1}{2}\sigma^2\right)s\right),\tag{24}$$

and for functions $f:\mathbb{R}\to\mathbb{R}$ and integers $\ell\ge0$ and $s>0$, let

$$\mathfrak{E}_s^{(\ell)}(f)(x)\stackrel{\text{def}}{=}e^{-rs}\int_{-\infty}^{\infty}f\left(x\zeta_s(z)\right)\phi^{(\ell)}(z)\,dz,\tag{25}$$

where ϕ is the probability density function of a standard normal random variable. Note that if f is bounded then

$$\left\|\mathfrak{E}_s^{(0)}(f)\right\|_\infty,\ \left\|\mathfrak{E}_s^{(1)}(f)\right\|_\infty\le\|f\|_\infty.\tag{26}$$

For any family of functions $f_t:\mathbb{R}\to\mathbb{R}$, $0\le t\le T$, set

$$\|f_\cdot\|_\infty\stackrel{\text{def}}{=}\sup_{0\le t\le T}\|f_t\|_\infty.\tag{27}$$

A consequence of Lemma 2 is that since h is bounded and Lipschitz,

$$\mathfrak{m}_t h(x)\stackrel{\text{def}}{=}\max\left(h(x),v_t^{\mathscr{R}_n}h(x)\right)\tag{28}$$

is also a bounded Lipschitz function. It follows from Theorem A.1 in Appendix that $E_x(\mathfrak{m}_u(\xi_u))$ is infinitely differentiable and that there exist constants α_0 such that

$$\begin{aligned}&xe^{-ru}\frac{\partial}{\partial x}E_x\left(\mathfrak{m}_u(\xi_u)\right)=\alpha_0\sqrt{u}^{-1}\mathfrak{E}_u^{(1)}\mathfrak{m}_u h(x)\\&\quad\le\left|\alpha_0\right|\sqrt{u}^{-1}\left\|\mathfrak{m}_u h(x)\right\|_\infty.\end{aligned}\tag{29}$$

Because $\|\mathfrak{m}_\cdot h\|_\infty\le\|h\|_\infty$, it follows from (5) that

$$\begin{aligned}&x\frac{\partial}{\partial x}v_t^{\mathscr{R}_n}h(x)\\&\quad=e^{-nt}x\frac{\partial}{\partial x}\mathscr{E}_t h(x)\\&\qquad+\int_0^t e^{-nu}\left(xe^{-ru}\frac{\partial}{\partial x}E_x\max\left(\mathfrak{m}_u(\xi_u)\right)\right)du\\&\quad=e^{-nt}x\frac{\partial}{\partial x}\mathscr{E}_t h(x)\\&\qquad+\int_0^t e^{-ns}\alpha_0\sqrt{u}^{-1}\mathfrak{E}_u^{(1)}\mathfrak{m}_{t-u}h(x)\,n\,du.\end{aligned}\tag{30}$$

Using Lemma A.2 in Appendix we get

$$\left\| I \frac{\partial}{\partial x} \mathscr{E}_t h \right\|_\infty = \left\| \mathscr{E}_t \left(Ih' \right)(x) \right\|_\infty \leq \left\| Ih' \right\|_\infty . \quad (31)$$

From this we obtain

$$\left\| I \frac{\partial}{\partial x} v_{\cdot}^{\mathscr{R}_n} h \right\|_\infty \leq \left\| Ih' \right\|_\infty + \|h\|_\infty \sqrt{T} \left| \alpha_0 \right| . \quad (32)$$

For a fixed n, set

$$\mathfrak{m}_u' h(z) \stackrel{\text{def}}{=} \frac{\partial}{\partial z} \max \left(h(z), v_u^{\mathscr{R}_n} h(z) \right), \quad (33)$$

which is Lebesgue almost everywhere since $\mathfrak{m}_u h$ is Lipschitz. Note that

$$\left\| I \mathfrak{m}_{\cdot}' h \right\|_\infty \leq \left(\left\| Ih' \right\|_\infty , \left\| I \frac{\partial}{\partial x} v_{\cdot}^{\mathscr{R}_n} h \right\|_\infty \right) < \infty . \quad (34)$$

Again, from Theorem A.1 in Appendix, there exist constants β_0 and β_1 such that

$$\begin{aligned} & x^2 e^{-ru} \frac{\partial^2}{\partial x^2} E_x \max \left(\mathfrak{m}_u \left(\xi_u \right) \right) \\ & = \beta_0 \mathfrak{E}_u^{(0)} \left(I \mathfrak{m}_u' h \right)(x) + \beta_1 \sqrt{u}^{-1} \mathfrak{E}_u^{(1)} \left(I \mathfrak{m}_u' h \right)(x) . \end{aligned} \quad (35)$$

The fact that $\|I \mathfrak{m}_{\cdot}' h\|_\infty < \infty$ implies that

$$\begin{aligned} & \left| x^2 e^{-ru} \frac{\partial^2}{\partial x^2} E_x \max \left(\mathfrak{m}_u \left(\xi_u \right) \right) \right| \\ & \quad \leq \left(\left| \beta_0 \right| + \left| \beta_1 \right| \sqrt{u}^{-1} \right) \left\| I \mathfrak{m}_{\cdot}' h \right\|_\infty . \end{aligned} \quad (36)$$

This in turn yields

$$\begin{aligned} & x^2 \frac{\partial^2}{\partial x^2} v_t^{\mathscr{R}_n} h(x) \\ & \quad = e^{-nt} x^2 \frac{\partial^2}{\partial x^2} \mathscr{E}_t h(x) \\ & \qquad + \int_0^t e^{-nu} \left(x^2 e^{-ru} \frac{\partial^2}{\partial x^2} E_x \max \left(\mathfrak{m}_u \left(\xi_u \right) \right) \right) du \\ & \quad = e^{-nt} x^2 \frac{\partial^2}{\partial x^2} \mathscr{E}_t h(x) \\ & \qquad + \int_0^t e^{-nu} \sum_{\ell=0}^{1} \beta_\ell \sqrt{u}^{-\ell} \mathfrak{E}_u^{(\ell)} \left(I \mathfrak{m}_{t-u}' \right)(x) \, du . \end{aligned} \quad (37)$$

As $E(\mathfrak{m}_u(x\xi_u))$ is infinitely differentiable with respect to $x > 0$, function $x^2 \exp(-ru)(\partial^2/\partial x^2) E_x \max(\mathfrak{m}_u(\xi_u))$ is continuous in $x > 0$ and, with (36), dominated convergence gives that $x^2(\partial^2/\partial x^2) v_t^{\mathscr{R}_n} h(x)$ is continuous. From Theorem A.1 in Appendix we obtain that, for some constant K,

$$\sup_{0 \leq t \leq T} \sup_{0 < x} \left| \sqrt{t} x^2 \frac{\partial^2}{\partial x^2} \mathscr{E}_t h(x) \right| \leq \left\| Ih' \right\|_\infty K , \quad (38)$$

and hence

$$\begin{aligned} & \sup_{0 \leq t \leq T} \sup_{0 < x} \left| \sqrt{t} x^2 \frac{\partial^2}{\partial x^2} v_t^{\mathscr{R}_n} h(x) \right| \\ & \quad \leq \left\| Ih' \right\|_\infty K + \left| \beta_0 \right| \sqrt{T} \left\| I \mathfrak{m}_{\cdot}' h \right\|_\infty \\ & \qquad + \left| \beta_1 \right| T \left\| I \mathfrak{m}_{\cdot}' h \right\|_\infty . \end{aligned} \quad (39)$$

5. Classical Solution

Recall $\mathrm{G}_t^n h$ from (8). Equation (7) can be rewritten as

$$v_t^{\mathscr{R}_n} h(x) = \mathrm{U}_t(h)(x) + \int_0^t \mathrm{U}_{t-u} \left(\mathrm{G}_u^n h \right)(x) \, du , \quad (40)$$

from which it follows that, for $0 < t, t + \varepsilon \leq T$,

$$\begin{aligned} \left(\frac{\mathrm{U}_\varepsilon - I}{\varepsilon} \right) v_t^{\mathscr{R}_n} h(x) & = \frac{v_{t+\varepsilon}^{\mathscr{R}_n} h - v_t^{\mathscr{R}_n} h(x)}{\varepsilon} \\ & \quad - \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathrm{U}_{t+\varepsilon-u} \left(\mathrm{G}_u^n h \right) du . \end{aligned} \quad (41)$$

Letting ε go to zero, we get

$$\mathrm{A} v_t^{\mathscr{R}_n} h(x) = \frac{\partial}{\partial t} v_t^{\mathscr{R}_n} h(x) - \mathrm{G}_t^n h(x) , \quad (42)$$

where A is the infinitesimal generator of the GBM. In other words,

$$\begin{aligned} \frac{\partial}{\partial t} v_t^{\mathscr{R}_n} h(x) & = \mathrm{A} v_t^{\mathscr{R}_n} h(x) \\ & \quad + \max \left(h(x) - v_t^{\mathscr{R}_n} h(x), 0 \right) n \\ & \quad - v_t^{\mathscr{R}_n} h(x) r . \end{aligned} \quad (43)$$

6. Conclusion

In this paper, we showed that the exercisability randomized American option is a classical solution (and therefore the only classical solution) to the canonical penalized problem. This relationship can be extended to a broader class of models than the Black-Scholes model. Indeed, the key property is to obtain uniform bounds for $x^k(\partial^k/\partial x^k)\mathscr{E}_s h(x)$ in terms of powers of $1/\sqrt{s}$ as in Theorem A.1 in Appendix.

Appendix

Theorem A.1. *If h is a bounded Lipschitz function then for every integer $k \geq 0$ and every $s > 0$ there exist real numbers $\alpha_1, \ldots, \alpha_k$, such that*

$$x^k \frac{\partial^k}{\partial x^k} \mathscr{E}_s h(x) = \sum_{\ell=1}^{k} \alpha_\ell \sqrt{s}^{-\ell} \mathfrak{E}_s^{(\ell)} h(x) . \quad (\text{A.1})$$

If additionally $k \geq 1$, there exist real numbers $\beta_0, \ldots, \beta_{k-1}$, such that

$$x^k \frac{\partial^k}{\partial x^k} \mathscr{E}_s h(x) = \sum_{\ell=0}^{k-1} \beta_\ell \sqrt{s}^{-\ell} \mathfrak{E}_s^{(\ell)} \left(Ih' \right)(x) . \quad (\text{A.2})$$

Proof. See [22, Th. 4.1] where the assumption that h is polynomially bounded, continuous, and piecewise continuously differentiable with polynomially bounded derivative can be replaced by our assumptions on h without any change in the argument, as a Lipschitz function is Lebesgue almost everywhere differentiable and absolutely continuous allowing the required integration by parts. □

Lemma A.2. *If h is a bounded Lipschitz function then for every* $s > 0$

$$x \frac{\partial}{\partial x} \mathscr{E}_s h(x) = \mathscr{E}_s \left(I h' \right)(x). \tag{A.3}$$

Proof. As h is Lipschitz it is Lebesgue almost everywhere differentiable and the proof of [22, Lemma 6.3] can be followed without any change. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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