GRAPH OF LINEAR TRANSFORMATIONS OVER A FIELD

by

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Dedication

To my husband Tariq, and my two sons, Omar and Ibrahim,

for their endless, patience, love and support.

Abstract

This research is an attempt to introduce a connection between graph theory and linear transformations of finite dimensional vector spaces over a field F (in our case we will be considering $\mathbb{R}$). Let $\mathbb{R}^m, \mathbb{R}^n$ be finite vector spaces over $\mathbb{R}$, and let L be the set of all non-trivial linear transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$. An equivalence relation $\sim$ is defined on L such that two elements $f, k \in L$ are equivalent, $f \sim k$, if and only if $\ker(f) = \ker(k)$. Let V be the set of all equivalence classes of $\sim$. We define a new graph, $G([t] : \mathbb{R}^m \to \mathbb{R}^n)$, to be the undirected graph with vertex set equal to V, such that two vertices, $[x], [y] \in G([t] : \mathbb{R}^m \to \mathbb{R}^n)$ are adjacent if and only if $\ker(x) \cap \ker(y) \neq \mathbf{0}$. The relationship between the connectivity of the graph $G([t] : \mathbb{R}^m \to \mathbb{R}^n)$ and the values of m and n has been investigated. In addition, we determine the values of m and n for a complete and totally disconnected graph, as well as the diameter and girth of the graph if connected.

Keywords: ***Graph theory; linear transformations; mathematics.***

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Chapter 1. Introduction

In this chapter, we provide a short overview of connecting graph theory with algebraic structures. Then, we present the problem investigated in this study as well as the thesis contribution. Finally, a general organization of the thesis is presented.

1.1 Overview

Graphs can be used to model different types of relations and processes in physical, biological, social and information systems. Many practical problems can be represented by graphs. To emphasize their application to real world systems, the term network is sometimes used to define a graph in which attributes are associated with the nodes and/or edges.

In mathematics, graphs are useuful in geometry and certain topics in topology such as knot theory. In addition, algebraic graph theory has close links with group theory.

Recently, there has been considerable attention in the literature to associating graphs with commutative rings (and other algebraic structures), as well as, studying the interplay between ring-theoretic and graph-theoretic properties; see the recent survey articles (10) and (40). Probably the most attention has been given to the *zero-divisor graph* $\Gamma(R)$ for a commutative ring R. The set of vertices of $\Gamma(R)$ is $Z(R)^* = Z(R) \setminus \{0\}$, and two vertices x and y are adjacent if and only if $xy = 0$. The concept of a zero-divisor graph goes back to I. Beck (25) who was interested in the notion of coloring a commutative ring R. Here R is considered as a simple graph whose vertices are the elements of R, such that two different elements x and y of R are adjacent if and only if $xy = 0$. This idea establishes a connection between graph theory and commutative ring theory which will be mutually beneficial for these two branches of mathematics.

1.2 Thesis Objectives

Driven by the developing interest in establishing links between graph theory and algebraic structures, in this research we will introduce a connection between graph theory and non-trivial linear transformations of finite dimensional vector spaces over a field F (in our case we will be considering $\mathbb{R}$). Let $\mathbb{R}^m, \mathbb{R}^n$ be finite vector spaces over $\mathbb{R}$, and let L be the set of all non-trivial linear transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$. An equivalence relation $\sim$ is defined on L such that two elements $f, k \in L$ are equivalent, $f \sim k$, if and only if $\ker(f) = \ker(k)$. Let V be the set of all equivalence classes of $\sim$. We define a new graph, $G([t] : \mathbb{R}^m \to \mathbb{R}^n)$, to be the undirected graph with vertex set equal to V, such that two vertices, $[x], [y] \in G([t] : \mathbb{R}^m \to \mathbb{R}^n)$ are adjacent if and only if $\ker(x) \cap \ker(y) \neq \mathbf{0}$.

1.3 Research Contribution

The contributions of this research work can be summarized as follows:

- Investigate the relationship between the connectivity of the graph $G([t] : \mathbb{R}^m \to \mathbb{R}^n)$ and the values of m and n.
- Determine the values of m and n for a complete and totally disconnected graph.
- Determine the diameter and girth of the graph if connected.

1.4 Thesis Organization

The rest of the thesis is organized as follows: Chapter 2 provides a historical background and relevant introductory concepts of graph theory, as well as a discussion on related work in connecting graph theory and different algebraic structures. Chapter 3 presents a detailed discussion of the problem statement, and the results obtained along with the analysis of each result. Finally, Chapter 4 concludes the thesis and outlines the future work.

Chapter 2. Background and Literature Review

This chapter is divided into three sections. In the first section, we present a brief historical background of graph theory, followed by a section on the introductory concepts on graph theory that are relevant to our research. In the final section, we discuss the literature review and previous research in connecting between graph theory and algebraic structures.

2.1 Graph Theory

2.1.1 Historical Background. In the 1700s, seven bridges were situated across the Pregolya River which passed through the city of Kenigsberg, a former German city, that is now Kaliningrad, Russia. Strangely, no resident of the city was ever able to walk a route that crossed each of these bridges exactly once. In 1736, the Swiss mathematician Leonhard Euler, learned of this frustrating problem and wrote an article called 'Kenigsberg Bridge Problem' which is considered to be the beginning of the field of graph theory.

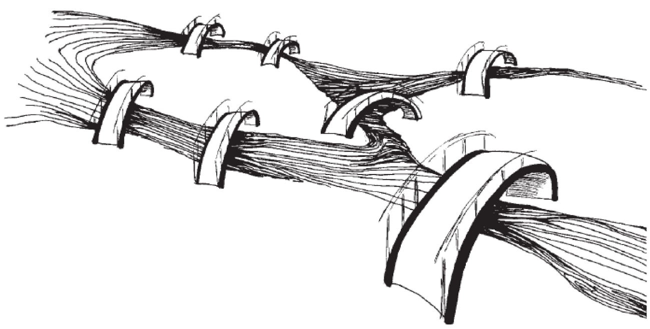

Figure 2.1: A drawing of the seven bridges of Kenigsberg.

At first, the impact of Euler's ideas in 'graph theory' was merely realized in solving puzzles and in analyzing games and other recreations. However, in the mid 1800s, the notion of graphs became of more interest as a useful tool to model social and physical phenomena. One of the most famous and stimulating problems in graph theory is the 'Four Color Map Conjecture,' introduced by De Morgan in 1852. The conjecture stated that four is the maximum number of colors required to color any map where bordering regions are colored differently.

This problem was first posed by Francis Guthrie in 1852 and its first written record is in a letter of De Morgan addressed to Hamilton the same year. Many incorrect proofs have been proposed, including those by Cayley, Kempe, and others. The study and the generalization of this problem by Tait, Heawood, Ramsey and Hadwiger led to the study of the colorings of the graphs embedded on surfaces with arbitrary genus. Tait's reformulation generated a new class of problems, the factorization problems, particularly studied by Petersen and König. The works of Ramsey on Colorations and more specially the results obtained by Turàn in 1941 was at the origin of another branch of graph theory, extremal graph theory. The four color problem remained unsolved for more than a century.

In 1969 Heinrich Heesch published a method for solving the problem using computers. A computer-aided proof produced in 1976 by Kenneth Appel and Wolfgang Haken makes fundamental use of the notion of 'discharging' developed by Heesch. The proof involved checking the properties of 1,936 configurations by computer, and was not fully accepted at the time due to its complexity. Twenty years later, a simpler proof considering only 633 configurations was given by Robertson, Seymour, Sanders and Thomas.

The autonomous development of Topology from 1860 and 1930 fertilized graph theory back through the works of Jordan, Kuratowski and Whitney. Another important factor of common development of graph theory and topology came from the use of the techniques of modern algebra. The first example of such a use is signified in the work of the physicist Gustav Kirchhoff, who published in 1845 his Kirchhoff's Circuit laws for calculating the voltage and current in electric circuits. The introduction of probabilistic methods in graph theory, especially in the study of Erdös and Renyi of the asymptotic probability of graph connectivity, gave rise to yet another branch, known as random graph theory, which has been a fruitful source of graph-theoretic results.

2.1.2 Introductory Concepts. A graph is a structure that constitutes a set of objects, which correspond to mathematical abstractions called vertices and each of the related pairs of vertices is called an edge. A *graph* G is a pair of sets (V, E), where V is the set of *vertices* and E is a set of 2-element subsets of V, called the set of *edges*.

$$G = (V, E)$$

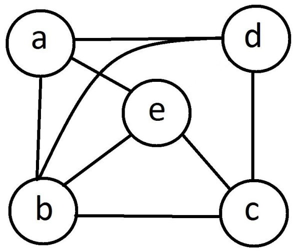

Figure 2.2: Example of a graph.

$$V = \{a, b, c, d, e\}$$

$$E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, \{a, e\}, \{b, e\}, \{c, e\}, \{b, d\}\}$$

Remark 2.1.1 *We will be considering simple and undirected graphs. A simple graph is a graph with neither loops nor multiple edges. An undirected graph is a graph where its edges have no "direction", that is the set $\{a, b\}$ is the same as the set $\{b, a\}$.*

Definition 2.1.2 *A **walk** in a graph $G = (V, E)$ is a sequence of the form*

$$\{v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\}, \ldots, v_k, \{v_k, v_{k+1}\}, v_{k+1}\}$$

$$v_1 - v_2 - \ldots - v_k - v_{k+1}$$

where $k \geq 0$, such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq k$. The **length** (number of edges) of the walk is k.

Definition 2.1.3 *A **path** in a graph $G = (V, E)$ is a walk of length $k \geq 1$, $v_1 - v_2 - \ldots - v_k - v_{k+1}$ in which the vertices $v_{1,} \ldots, v_{k+1}$ are all distinct.*

Definition 2.1.4 *A **cycle** in a graph $G = (V, E)$ is a walk of length $k \geq 1$, $v_1 - v_2 - \ldots - v_k - v_1$ in which the vertices $v_{1,} \ldots, v_k$ are distinct.*

Example 2.1.5 *Referring to Figure 2.3, we have:*
walk $= b - e - c - b - d$.
path $= b - c - e - a - d$.

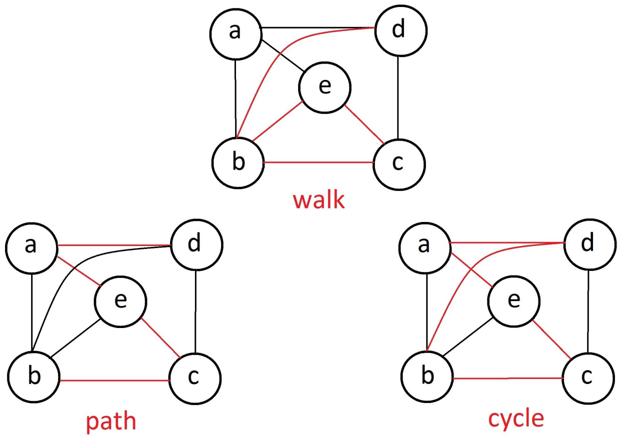

Figure 2.3: Comparison between walk, path and cycle.

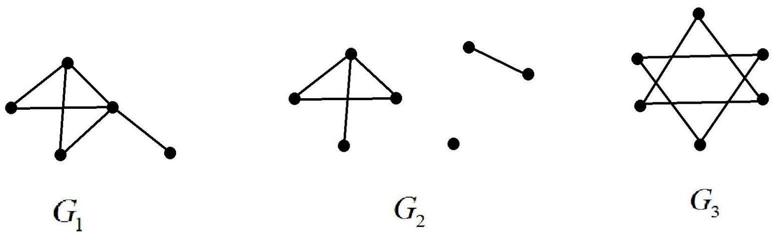

Figure 2.4: Connected and disconnected graphs.

$cycle = b - c - e - a - d - b.$

Definition 2.1.6 *Two vertices u and v of a graph G are* ***adjacent*** *if there is an edge $\{u, v\}$ joining them.*

Definition 2.1.7 *A graph $G = (V, E)$ is* ***connected*** *if there is a path in G from u to v for every $u, v \in V$. Therefore, a graph is said to be* ***disconnected****, if there exist at least two vertices $u, v \in V$ that are not joined by a path.*

Definition 2.1.8 *A graph G is* ***totally disconnected*** *if no vertices of G are adjacent.*

Definition 2.1.9 *A graph $G = (V, E)$ is* ***complete*** *if every two distinct vertices $u, v \in V$ are joined by exactly one edge. K_n is a complete graph with n vertices.*

Example 2.1.10 *Figure 2.4 gives an example of connected and disconnected graphs. The graph G_1 is connected, however the graphs G_2 and G_3 are disconnected.*

Example 2.1.11 *Figure 2.5 gives an example of complete graphs.*

Definition 2.1.12 *The* ***distance*** *in a graph $G = (V, E)$ between to vertices u and v,*

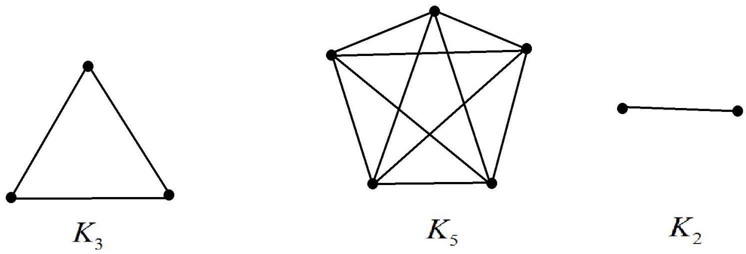

Figure 2.5: Complete graphs.

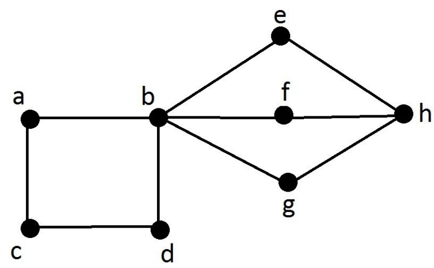

Figure 2.6: Demonstrating the diameter and girth of a graph.

denoted by $d(u,v)$ *is the length (number of edges) of a shortest* $u-v$ *path in* G *(*$d(u,u)=0$ *and* $d(u,v)=\infty$ *if there is no such path).*

Definition 2.1.13 *The **diameter** of* G*,* $\operatorname{diam}(G)=\sup\{d(u,v)\,|u$ *and* v *are vertices of* $G\}$.

Definition 2.1.14 *The **girth** of a graph* G *denoted by* $\operatorname{gr}(G)$*, is the length of a shortest cycle in* G *(*$\operatorname{gr}(G)=\infty$ *if* G *contains no cycles).*

Example 2.1.15 *In this example as shown in Figure 2.6, we have,* $d(a,f)=2$, $\operatorname{diam}(G)=4$*, and* $\operatorname{gr}(G)=4$

2.2 Literature Review

In his 1988 paper, 'Coloring of Commutative Rings', (*Journal of Algebra*), I. Beck (25), set out to establish a connection between graph theory and commutative ring theory, by introducing the notion of coloring a commutative ring R. R is considered a simple graph whose vertices are the elements of R, such that two distinct vertices x and y of R are adjacent if and only if $xy=0$ (the additive identity of R). A k-coloring of R is an assignment of k colors to the elements of R in such a way that every two adjacent elements have different colors. The main aim is to characterize and discuss the chromatic number, $\chi(R)$, that is the minimal number k where R is k-colorable, for

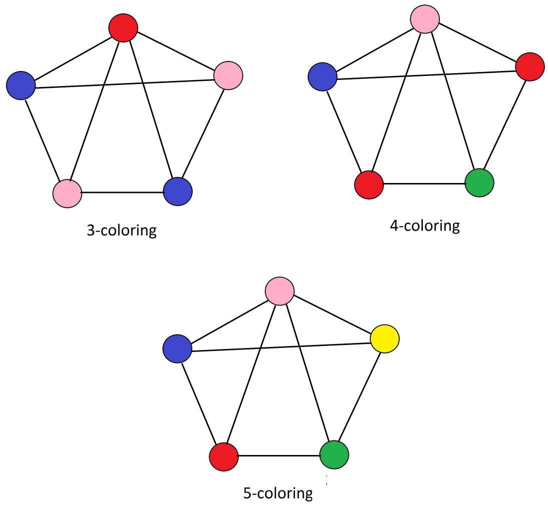

Figure 2.7: In this example $\chi(R) = 3$.

rings that are finitely colorable.

The investigation of colorings of a commutative ring was further continued by D. Anderson and M. Nasser in (3). They introduced the *zero-divisor graph*, denoted by $\Gamma_0(R)$. The vertices of $\Gamma_0(R)$ constitute all elements of R, and two distinct vertices x and y are adjacent if and only if $xy = 0$. In $\Gamma_0(R)$, the vertex 0 is adjacent to every other vertex but non-zero divisors are adjacent only to 0.

D. Anderson and P. Livingston, (*Journal of Algebra*, 1999) (16), defined a slightly different *zero-divisor graph* of R, denoted by $\Gamma(R)$, as the (undirected) graph with vertices $Z(R)^* = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of R. $\Gamma(R)$ is an induced sub-graph of $\Gamma_0(R)$, thus the results for $\Gamma(R)$ have natural analogs to $\Gamma_0(R)$, but better illustrates the zero-divisor structure of R. In $\Gamma(R)$, the vertices x and y are adjacent if and only if $xy = 0$. The objective of this paper is to study the ring-theoretic properties of R with graph-theoretic properties of $\Gamma(R)$, to help illuminate the algebraic structure of $Z(R)$. For $x, y \in Z(R)$, define $x \sim y$ if $xy = 0$ or $x = y$. The relation $\sim$ is always reflexive and symmetric, but is usually not transitive. The zero-divisor graph measures this lack of transitivity in the sense that $\sim$ is transitive if and only if $\Gamma(R)$ is complete. They have shown among other things, $\Gamma(R)$ is connected with $\operatorname{diam}(\Gamma(R)) \in \{0, 1, 2, 3\}$ and $\operatorname{gr}(\Gamma(R)) \in \{3, 4, \infty\}$. The zero-divisor graph of a ring R has been studied extensively by other numerous authors, for example see((1)-

Example showing $\Gamma(\mathbf{Z}_6)$ and $\Gamma(\mathbf{Z}_{10})$ for the commutative rings $\mathbf{Z}_6$ and $\mathbf{Z}_{10}$.

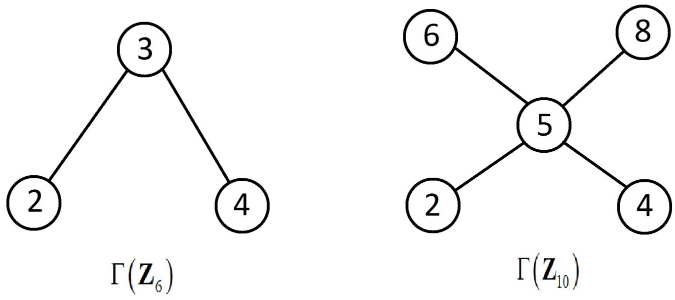

Figure 2.8: Example showing $\Gamma(\mathbf{Z}_6)$ and $\Gamma(\mathbf{Z}_{10})$ for the commutative rings $\mathbf{Z}_6$ and $\mathbf{Z}_{10}$.

(3), (11), (20)-(21), (33)-(37), (41)-(48), (52)).

Another interesting graph is the *total graph* of R, presented by D. Anderson and A. Badawi (12). The *total graph*, denoted by $T(\Gamma(R))$, is the (undirected) graph with all elements of R as vertices. For $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$. The total graph (as in (12)) has been investigated in (8), (7), (6), (5), (40), (42), (46), (31) and (50); and several variants of the total graph have been studied in (4), (13), (14), (15), (19), (24), (30), (27), (28), (29), (32), and (38). Let $a \in Z(R)$ and let $ann_R(a) = \{r \in R \mid ra = 0\}$. In 2014, A. Badawi (23) introduced the annihilator graph of R. We recall from (23) that the annihilator graph of R is the (undirected) graph $AG(R)$ with vertices $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. It follows that each edge (path) of the classical zero-divisor of R is an edge (path) of $AG(R)$. For further investigations of $AG(R)$, see (18), (45), and (51).

In 2015, A. Badawi, investigated the *total dot product graph* of R (22). In this case $R = A \times A \times \cdots \times A$ (n times), where A is a commutative ring with nonzero identity, and $1 \leq n < \infty$ is an integer. The *total dot product graph* of R is the (undirected) graph denoted by $TD(R)$, with vertices $R^* = R \setminus \{(0, 0, \ldots 0)\}$. Two distinct vertices are adjacent if and only if $x.y = 0 \in A$, where $x.y$ denote the normal dot product of x and y. The *zero-divisor dot product graph* of R is the induced subgraph $ZD(R)$ of $TD(R)$ with vertices $Z(R)^* = Z(R) \setminus \{(0, 0, ..., 0)\}$. It follows that each edge (path) of the classical zero-divisor graph $\Gamma(R)$ is an edge (path) of $ZD(R)$. In (22), both graphs $TD(R)$ and $ZD(R)$ are studied. For a commutative ring A and $n \geq 3$, $TD(R)$

$(ZD(R))$ is connected with diameter two (at most three) and with girth three. Among other things, A. Badawi, shows that for $n \geq 2$, $ZD(R)$ is identical to the *zero-divisor graph* of R if an only if either $n = 2$ and A is an integral domain or R is ring-isomorphic to $\mathrm{Z}_2 \times \mathrm{Z}_2 \times \mathrm{Z}_2$.

Chapter 3. Problem Analysis

In this chapter we present a detailed discussion of the problem statement, then followed by the results obtained along with the analysis of each result.

3.1 Problem Statement

This research is an attempt to introduce a connection between graph theory and linear transformations of finite dimensional vector spaces over a field $\mathbf{F}$ (in our case we will be considering $\mathbb{R}$). Let U and W be finite dimensional vector spaces over $\mathbb{R}$, where $m = \dim(U)$ and $n = \dim(W)$. Since every finite dimensional vector space over $\mathbb{R}$, with dimension k is isomorphic to $\mathbb{R}^k$, we have U isomorphic to $\mathbb{R}^m$ and W isomorphic to $\mathbb{R}^n$.

First we will define a set L as follows; $L := \{$set of all non-trivial linear transformations, t, from $\mathbb{R}^m$ into $\mathbb{R}^n\}$. Let $s, t \in L$, we say that s is equivalent to t, and write $s \sim t$ if and only if $\ker(s) = \ker(t)$.

Clearly, $\sim$ is an equivalence relation on L. For each $t \in L$, the set $[t] := \{s \in L | s \sim t\}$ is called the **equivalence class** of t. We recall the following properties of equivalence classes:

- For all $t \in L$, $[t] \neq \emptyset$.
- If $s \in [t]$, then $[s] = [t]$, where $s, t \in L$.
- For all $s, t \in L$ either $[s] = [t]$ or $[s] \cap [t] = \emptyset$.
- $L = \cup_{t \in L} [t]$, that is, L is the union of all equivalence classes under $\sim$.

Definition 3.1.1 *Let,* V :=$\{$*set of all equivalence classes of linear transformations,* $[t]$, *from* $\mathbb{R}^m$ *into* $\mathbb{R}^n\}$. *We introduce a new undirected graph denoted by* $G([t] : \mathbb{R}^m \to \mathbb{R}^n)$, *with vertex set equals to* V, *where two distinct vertices* $[f], [k] \in V$ *are adjacent if and only if* $\ker(f) \cap \ker(k) \neq \mathbf{0}$.

A linear transformation $s : \mathbb{R}^m \to \mathbb{R}^n$, can be represented by a standard $n \times m$ matrix $\mathbf{M}_s$ over $\mathbb{R}$. Therefore, $\ker(s) = \text{null}(\mathbf{M}_s)$, is the solution set of the homogeneous system of linear equations $\mathbf{M}_s\mathbf{x} = \mathbf{0}$.

Let, $a : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation, so it can be represented by the standard

$n \times m$ matrix M_a.

Let, $b : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation, so it can be represented by the standard $n \times m$ matrix M_b.

Definition 3.1.2 *M_b is **row-equivalent** to M_a, if M_b can be obtained from M_a by a finite sequence of elementary row operations.*

We recall the following results:

- **Result 1:** If M_a and M_b are *row-equivalent* $n \times m$ matrices, then $\mathrm{null}\,(\mathbf{M}_a) = \mathrm{null}\,(\mathbf{M}_b)$.
- **Result 2:** Every $n \times m$ matrix over a field **F** is *row-equivalent* to a **unique** row-reduced echelon matrix.

Therefore, by using **Results 1** and **2**, if M_a is *row-equivalent* to M_b, then the linear transformations a and b lie in the same equivalence class, say $[f] \in V$. However, if M_a and M_b are not *row-equivalent*, then the linear transformations a and b lie in two distinct equivalence classes, say $a \in [f]$ and $b \in [k]$.

3.2 Results and Analysis

3.2.1 Generalizing the Graph $G\left([t] : \mathbb{R}^m \to \mathbb{R}\right)$. In this section, we provide a general result for the connectivity of the graph $G\left([t] : \mathbb{R}^m \to \mathbb{R}\right)$ which is demonstrated in Theorem 3.2.4. However, we first give the following examples to help us visualize the problem at hand by considering the graph $G\left([t] : \mathbb{R}^m \to \mathbb{R}\right)$, when $m = 1, 2$ and 3.

Example 3.2.1 *Let us consider the graph, $G\left([t] : \mathbb{R} \to \mathbb{R}\right)$. We have, $V = \{$set of all equivalence classes, $[t] : \mathbb{R}^2 \to \mathbb{R}\}$. Choose $[f], [k] \in V$.*

Using: $\dim(\ker) + \dim(\text{range}) = \dim(\text{domain})$.

- $\dim(\text{range}) = 1$, *since we are considering non-trivial linear transformations.*
- $\dim(\text{domain}) = 1$.

Therefore, we have $\dim(\ker) = 0$, *which implies* $\ker(f), \ker(k)$ *only contain the origin. Thus,* $\ker(f)$ *and* $\ker(k)$ *intersect at the origin. Hence,* $\ker(f) \cap \ker(k) = \mathbf{0}$. *This means the graph* $G\left([t] : \mathbb{R} \to \mathbb{R}\right)$ *is totally disconnected.*

Example 3.2.2 *Let us consider the graph, $G\left([t] : \mathbb{R}^2 \to \mathbb{R}\right)$. We have, $V = \{$set of all* *equivalence classes,* $[t] : \mathbb{R}^2 \to \mathbb{R}\}$. *Choose* $[f], [k] \in V$.

Using: $\dim(\ker) + \dim(\text{range}) = \dim(\text{domain})$.

- $\dim(\text{range}) = 1$*, since we are considering non-trivial linear transformations.*
- $\dim(\text{domain}) = 2$.

Therefore, we have $\dim(\ker) = 1$*, which implies* $\ker(f), \ker(k)$ *are straight lines passing through the origin. Thus,* $\ker(f)$ *and* $\ker(k)$ *intersect at the origin. Hence,* $\ker(f) \cap \ker(k) = \mathbf{0}$. *This means the graph* $G([t] : \mathbb{R}^2 \to \mathbb{R})$ *is totally disconnected.*

Example 3.2.3 *Let us consider the graph,* $G([t] : \mathbb{R}^3 \to \mathbb{R})$. *We have,* $V = \{$*set of all equivalence classes,* $[t] : \mathbb{R}^3 \to \mathbb{R}\}$. *Choose* $[f], [k] \in V$.

Using: $\dim(\ker) + \dim(\text{range}) = \dim(\text{domain})$.

- $\dim(\text{range}) = 1$*, since we are considering non-trivial linear transformations.*
- $\dim(\text{domain}) = 3$.

Therefore, we have $\dim(\ker) = 2$*, which implies* $\ker(f), \ker(k)$ *are planes passing through the origin. Thus,* $\ker(f)$ *and* $\ker(k)$ *intersect at a line. Hence,* $\ker(f) \cap \ker(k) \neq \mathbf{0}$. *This means the graph* $G([t] : \mathbb{R}^3 \to \mathbb{R})$ *is complete.*

Theorem 3.2.4 *The undirected graph* $G([t] : \mathbb{R}^m \to \mathbb{R})$ *is totally disconnected if and only if* $m = 1$ *or* $m = 2$.

Proof: It is clear for $m = 1$, that the graph $G([t] : \mathbb{R} \to \mathbb{R})$ is totally disconnected as shown in Example 3.2.1.

'$\Leftarrow$': Let $m = 2$, and choose $[f], [k] \in V$.

We want to show $\ker(f) \cap \ker(k) = \mathbf{0}$.

Let, $\mathbf{M}_f$ be the standard 1×2 matrix representation of $[f]$.

Let, $\mathbf{M}_k$ be the standard 1×2 matrix representation of $[k]$.

By construction, $\mathbf{M}_f$ is not *row-equivalent* to $\mathbf{M}_k$. Say,

$\mathbf{M}_f = \begin{bmatrix} f_{11} & f_{12} \end{bmatrix}$

$\mathbf{M}_k = \begin{bmatrix} k_{11} & k_{12} \end{bmatrix}$

Let, $\mathbf{M}_{fk} = \begin{bmatrix} \mathbf{M}_f \\ \mathbf{M}_k \end{bmatrix}$.

Consider the system, $\mathbf{M}_{fk}\mathbf{x} = \mathbf{0}$, that is,

$$\begin{bmatrix} f_{11} & f_{12} \\ k_{11} & k_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since $\mathbf{M}_f$ is not *row-equivalent* to $\mathbf{M}_k$, the rows $\begin{bmatrix} f_{11} & f_{12} \end{bmatrix}$ and $\begin{bmatrix} k_{11} & k_{12} \end{bmatrix}$ are independent, which implies $\mathbf{M}_{fk}$ is invertible. Hence, the solution for $\mathbf{M}_{fk}\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. Therefore, $\ker(f) \cap \ker(k) = \mathbf{0}$, that is the vertices $[f]$ and $[k]$ are not adjacent. Further, since $[f]$, $[k]$ were chosen randomly we conclude that the graph $G([t] : \mathbf{R}^m \to \mathbf{R})$ is totally disconnected for $m = 2$.

'$\Rightarrow$': Using contrapositive.

Suppose, $m > 2$, show that the graph is connected.

Let, $[f], [k] \in V$.

We want to show that, $\ker(f) \cap \ker(k) \neq \mathbf{0}$.

Let, $\mathbf{M}_f$ be the standard $1 \times m$ matrix representation of $[f]$.

Let, $\mathbf{M}_k$ be the standard $1 \times m$ matrix representation of $[k]$.

By construction, $\mathbf{M}_f$ is not *row-equivalent* to $\mathbf{M}_k$. Say,

$\mathbf{M}_f = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1m} \end{bmatrix}$

$\mathbf{M}_k = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1m} \end{bmatrix}$

Let, $\mathbf{M}_{fk} = \begin{bmatrix} \mathbf{M}_f \\ \mathbf{M}_k \end{bmatrix}$.

Consider the system, $\mathbf{M}_{fk}\mathbf{x} = \mathbf{0}$, that is,

$$\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1m} \\ k_{11} & k_{12} & \cdots & k_{1m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since, $m > 2$, the number of equations $<$ the number of unknown variables. Hence, the system $\mathbf{M}_{fk}\mathbf{x} = \mathbf{0}$ has infinitely many solutions. Therefore, $\ker(f) \cap \ker(k) \neq \mathbf{0}$, that is the vertices $[f]$ and $[k]$ are adjacent. Further, since $[f]$, $[k]$ were chosen randomly we conclude that the graph $G([t] : \mathbb{R}^m \to \mathbb{R})$ is complete for $m > 2$.

3.2.2 Generalizing the Graph $G([t]:\mathbb{R}^m \to \mathbb{R}^n)$. In this section, we provide a detailed generalization for the connectivity of the graph $G([t]:\mathbb{R}^m \to \mathbb{R}^n)$ which is demonstrated in Theorems 3.2.5, 3.2.6, 3.2.7, and 3.2.10. Finally, we discuss the girth of the graph $G([t]:\mathbb{R}^m \to \mathbb{R}^n)$ in Theorem 3.2.12.

Theorem 3.2.5 *If $m = 1$ or $m = 2$, then the undirected graph $G([t]:\mathbb{R}^m \to \mathbb{R}^n)$ is totally disconnected.*

Proof: It is clear for $m = 1$, that the graph $G([t]:\mathbb{R} \to \mathbb{R}^n)$ is totally disconnected. For this case, since we are only considering non-trivial linear transformations, we can only have a single equivalence class. The kernel of each element belonging to this equivalence class is 0. Hence the graph $G([t]:\mathbb{R} \to \mathbb{R}^n)$ is totally disconnected.
'$\Rightarrow$': Let $m = 2$, and choose $[f], [k] \in V$.
We want to show $\ker(f) \cap \ker(k) = \mathbf{0}$.
Let, $\mathbf{M}_f$ be the standard $n \times 2$ matrix representation of $[f]$.
Let, $\mathbf{M}_k$ be the standard $n \times 2$ matrix representation of $[k]$.
Let, $\mathbf{M}_{fk} = \begin{bmatrix} \mathbf{M}_f \\ \mathbf{M}_k \end{bmatrix}$.

Consider the system, $\mathbf{M}_{fk}\mathbf{x} = \mathbf{0}$, that is,

$$\begin{bmatrix} \mathbf{M}_f \\ \mathbf{M}_k \end{bmatrix}_{2n\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since $\mathbf{M}_f$ is not *row-equivalent* to $\mathbf{M}_k$ by construction, then there is at most one row from $\mathbf{M}_f$ and one row from $\mathbf{M}_k$ that are independent. Therefore, $\operatorname{rank}(\mathbf{M}_{fk}) = 2$, that is $\mathbf{M}_{fk}$ has two independent rows say, R_1 and R_2. This means our system can be reduced to the following:

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since $\begin{bmatrix} R_1 & R_2 \end{bmatrix}^T$ is a 2×2 invertible matrix, we have $\operatorname{null}\left(\begin{bmatrix} R_1 & R_2 \end{bmatrix}^T\right) = (0, 0)$. This implies $\ker(f) \cap \ker(k) = \mathbf{0}$, that is the vertices $[f]$ and $[k]$ are not adjacent. Since $[f]$, $[k]$ were chosen randomly we conclude that the graph $G([t]:\mathbb{R}^m \to \mathbb{R}^n)$ is totally disconnected for $m = 2$.

We will be showing in our subsequent discussion, that Theorem 3.2.5 is actually both a necessary and sufficient condition.

Theorem 3.2.6 *Consider the undirected graph* $G([t] : \mathbb{R}^m \to \mathbb{R}^n)$*, then the graph is complete if and only if* $m \geq 2n + 1$.

Proof: We will choose $[f], [k] \in V$, such that $[f] \neq 0$, and $[k] \neq 0$.
Let, M_f be the standard $n \times m$ matrix representation of $[f]$.
Let, M_k be the standard $n \times m$ matrix representation of $[k]$.
Let, $\mathrm{M}_{fk} = \begin{bmatrix} \mathbf{M}_f \\ \mathbf{M}_k \end{bmatrix}$.
Assume, $(x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$ is a solution to $\mathbf{M}_{fk}\mathbf{x} = \mathbf{0}$, that is,

$$\begin{bmatrix} \mathbf{M}_f \\ \mathbf{M}_k \end{bmatrix}_{2n \times m} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}_{m \times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{2n \times 1}$$

Let, $r = \operatorname{rank}(\mathbf{M}_{fk})$.

'$\Leftarrow$': Let, $m \geq 2n + 1$, we want to show $\ker(f) \cap \ker(k) \neq \mathbf{0}$.
For this situation, $r \leq 2n$, and $m \geq 2n + 1$, that is, we have number of equations $<$ number of unknown variables. Hence, the system $\mathbf{M}_{fk}\mathbf{x} = \mathbf{0}$ has infinitely many solutions, or $\operatorname{null}(\mathbf{M}_{fk}) \neq \mathbf{0}$. Therefore, $\ker(f) \cap \ker(k) \neq \mathbf{0}$, that is the vertices $[f]$ and $[k]$ are adjacent. Further, since this is the only case and $[f]$, $[k]$ were chosen randomly we conclude that the graph $G([t] : \mathbb{R}^m \to \mathbb{R}^n)$ is complete for $m \geq 2n + 1$.

'$\Rightarrow$': Using contrapositive.
Suppose $m < 2n + 1$, we want to show that the graph is not complete.
Let $[f], [k] \in V$, such that $[f] \neq 0$, and $[k] \neq 0$.

Case I: Suppose $r = m$.
We conclude that $\mathbf{M}_{fk}$ has m independent rows, say $R_1, R_2, \cdots, R_m$.

Consider the system,

$$\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since $\begin{bmatrix} R_1 & R_2 & \cdots & R_m \end{bmatrix}^T$ is an invertible $m \times m$ matrix, we have $\text{null}\left(\begin{bmatrix} R_1 & \cdots & R_m \end{bmatrix}\right)^T = (0, \cdots, 0)$. This implies $\ker(f) \cap \ker(k) = \mathbf{0}$, hence the vertices $[f]$ and $[k]$ are disconnected.

Case II: Suppose $r < m$. Thus we have the following system:

$$\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since number of equations < number of unknown variables, we conclude that $\text{null}\left(\begin{bmatrix} R_1 & \cdots & R_r \end{bmatrix}^T\right) \neq (0, \cdots, 0)$. This implies $\ker(f) \cap \ker(k) \neq \mathbf{0}$, hence the vertices $[f]$ and $[k]$ are connected.

Since, the vertices $[f]$ and $[k]$ can either be connected or disconnected, we can say that the graph is incomplete for $m < 2n + 1$.

Theorem 3.2.7 *Consider the undirected graph $G([t] : \mathbb{R}^m \to \mathbb{R}^n)$. Assume $m \leq n$ and $m > 2$. Then: (i) Graph is connected. (ii) Graph diameter, $d = 2$.*

Proof: Let $[T], [L] \in V$, such that $[T]$ and $[L]$ are not adjacent ($\ker(T) \cap \ker(L) = \mathbf{0}_m$), and $[T] \neq 0$, $[L] \neq 0$. In addition, $[T]$ and $[L]$ are non-trivial vertices, that is if $f \in [T]$ and $k \in [L]$, then $\text{rank}(M_f) \neq m$ and $\text{rank}(M_k) \neq m$ where, M_f and M_k are the standard matrix representations of f and k, with size $n \times m$.

Remark 3.2.8 *If $\text{rank}(M_f) = 1$ and $\text{rank}(M_k) = 1$, then $\text{null}(M_{fk}) \neq 0$, since by construction M_f and M_k are not row-equivalent, and $m > 2$. This implies that $\ker(f) \cap \ker(k) \neq \mathbf{0}$, hence $[T]$ and $[L]$ are connected. Therefore, this case is intuitively* *not considered in the next discussion.*

- If we have:

  $\text{rank}\,(M_f) = m - i$, where $i \in \mathbf{N}$, $i \neq 1$, and

  $\text{rank}\,(M_k) = m - j$, where $j \in \mathbf{N}$, $j \neq 1$.

  Then, we can choose any non-zero row from M_f or M_k, say Y, to form the $n \times m$ matrix M_d, where:

  $$M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

  is the standard matrix representation of some $d \in [W]$, such that $[T] - [W] - [L]$.

- If we have:

  $\text{rank}\,(M_f) = m - 1$, and

  $\text{rank}\,(M_k) = m - 1$.

  Then, M_f has $m - 1$ independent rows, $R_1, R_2, \ldots, R_{m-1}$. Since, $[T]$ and $[L]$ are not adjacent, M_k has one row say R such that, $\{R_1, R_2, \ldots, R_{m-1}, R\}$ is an independent set which forms a basis for $\mathbf{R}^m$.

  Let, $K \neq R$ be a non-zero row in M_k, hence $K \in \text{rowspace}\,(M_k)$. Since $K \in \mathbf{R}^m$, we have:

  $$K = c_1R_1 + c_2R_2 + \cdots + c_{m-1}R_{m-1} + c_mR$$

  Let, $Y = K - c_mR$.

  This implies, $Y \in \text{rowspace}\,(M_k)$, (since both K and c_mR are $\in \text{rowspace}\,(M_k)$), and $Y \in \text{rowspace}\,(M_f)$.

  Let, $M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times m}$, be the standard matrix representation of some $d \in [W]$.

  Since, $Y \in \text{rowspace}\,(M_f)$, Y becomes a zero row through row operations using the rows in M_f,

$\Rightarrow \text{null}\,(M_{fd}) \neq \mathbf{0}$ since $\text{rank}\,(M_{fd}) = m - 1$,

$\Rightarrow \ker(T) \cap \ker(W) \neq \mathbf{0} \Rightarrow [T] - [W]$.

Similarly, since, $Y \in \text{rowspace}\,(M_k)$, Y becomes a zero row through row operations using the rows in M_k,

$\Rightarrow \text{null}\,(M_{kd}) \neq \mathbf{0}$ since $\text{rank}\,(M_{kd}) = m - 1$,

$\Rightarrow \ker(L) \cap \ker(W) \neq \mathbf{0} \Rightarrow [W] - [L]$.

Therefore, we have $[T] - [W] - [L]$.

Example 3.2.9 *Suppose $m = 3$ and $n = 4$. So we are considering the graph $G([t] : \mathbb{R}^3 \to \mathbb{R}^4)$, where $m \leq n$, and $m > 2$, as given in Theorem 3.2.7. Let $[T], [L] \in V$, such that $[T]$ and $[L]$ are not adjacent ($\ker(T) \cap \ker(L) = \mathbf{0}$), and $[T] \neq 0, [L] \neq 0$. Let $f \in [T]$, and $k \in [L]$. Since $[T]$ and $[L]$ are non-trivial vertices, then $\text{rank}\,(M_f) \neq m$ and $\text{rank}\,(M_k) \neq m$, where M_f and M_k are the standard matrix representations of f and k.*

Suppose,

$$M_f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{4\times 3}, M_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{4\times 3}$$

Let, $M_{fk} = \begin{bmatrix} M_f \\ M_k \end{bmatrix}_{8\times 3}$.

It can be easily seen that $\text{rank}\,(M_{fk}) = 3$, which implies that $\text{null}\,(M_{fk}) = \mathbf{0}$. Therefore, $\ker(f) \cap \ker(k) = \mathbf{0}$, that is the vertices $[T]$ and $[L]$ are not adjacent. We have:

$\text{rank}\,(M_f) = 2 = 3 - 1 = m - 1$, *and*

$\text{rank}\,(M_k) = 2 = 3 - 1 = m - 1$.

Then, M_f has 2 independent rows R_1 and R_2, such that $R_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and $R_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$. The vertices $[T]$ and $[L]$ are not adjacent, thus M_k has one row R, such that $\{R_1, R_2, R\}$ are independent and form a basis for $\mathbb{R}^m$, where $m = 3$. In this example, $R = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.

Let, $K \neq R$ be a non-zero row in M_k, $K = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$. $K \in \operatorname{rowspace}(M_k)$ and since $K \in \mathbf{R}^3$ it can be written as a linear combination of $\{R_1, R_2, R\}$ as follows:

$$K = 1.R_1 + 1.R_2 - R = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

Let, $Y = K - (-1).R = K + R = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.
This implies $Y \in \operatorname{rowspace}(M_k)$ and $Y \in \operatorname{rowspace}(M_f)$.

Let, $M_d = \begin{bmatrix} Y \\ 0 \\ 0 \\ 0 \end{bmatrix}_{4\times3} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{4\times3}$, be the standard matrix representation of some $d \in [W]$.

Since, $Y \in \operatorname{rowspace}(M_f)$, Y becomes a zero row through row operations using the rows in M_f,
$\Rightarrow \operatorname{null}(M_{fd}) \neq \mathbf{0}$ *since* $\operatorname{rank}(M_{fd}) = 2$,
$\Rightarrow \ker(T) \cap \ker(W) \neq \mathbf{0} \Rightarrow [T] - [W]$.

Similarly, since, $Y \in \operatorname{rowspace}(M_k)$, Y becomes a zero row through row operations using the rows in M_k,
$\Rightarrow \operatorname{null}(M_{kd}) \neq \mathbf{0}$ *since* $\operatorname{rank}(M_{kd}) = 2$,
$\Rightarrow \ker(L) \cap \ker(W) \neq \mathbf{0} \Rightarrow [W] - [L]$.

Therefore, we have $[T] - [W] - [L]$.

Theorem 3.2.10 *Consider the undirected graph $G([t] : \mathbb{R}^m \to \mathbb{R}^n)$. Assume $n < m \leq 2n$ and $m > 2$. Then: (i) Graph is connected. (ii) Graph diameter, $d = 2$.*

Proof: Let $[T], [L] \in V$, such that $[T]$ and $[L]$ are not adjacent ($\ker(T) \cap \ker(L) = \mathbf{0}$), and $[T] \neq 0$, $[L] \neq 0$. In addition, since $n < m \leq 2n$, $[T]$ and $[L]$ will be non-trivial vertices. Let, $f \in [T]$ and $k \in [L]$, then $\operatorname{rank}(M_f) < m$ and $\operatorname{rank}(M_k) < m$ where, M_f and M_k are the standard matrix representations of f and k, with size $n \times m$.

- If we have, $n + 1 < m \leq 2n$:
  $\operatorname{rank}(M_f) = n - i$, where $i = 0, 1, 2, \ldots$, and
  $\operatorname{rank}(M_k) = n - j$, where $j = 0, 1, 2, \ldots$

Then we can choose any non-zero row from M_f or M_k, say Y, to form the $n \times m$ matrix M_d, where:

$$M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is the standard matrix representation of some $d \in [W]$, such that $[T] - [W] - [L]$.

- If $m = n + 1$, then we have 3 cases:

Case I:

$\text{rank}\,(M_f) = n = m - 1$, and

$\text{rank}\,(M_k) = n - j$, where $j = 1, 2, \ldots$

Then we can choose any non-zero row, say Y from M_f, (that is the matrix with the higher rank), to form the $n \times m$ matrix M_d, where:

$$M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is the standard matrix representation of some $d \in [W]$, such that $[T] - [W] - [L]$.

Case II:

$\text{rank}\,(M_f) = n - i$, where $i = 1, 2, \ldots$ and

$\text{rank}\,(M_k) = n - j$, where $j = 1, 2, \ldots$

In this case any non-zero row Y can be chosen either from M_f or M_k, to form M_d, where:

$$M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Case III:

$\text{rank}\,(M_f) = n$, and

$\text{rank}\,(M_k) = n.$

Then M_f has n independent rows $R_1, R_2, \ldots, R_n$. Since, $[T]$ and $[L]$ are not adjacent ($\ker(T) \cap \ker(L) = \mathbf{0}$), M_k has one row say R such that, $\{R_1, R_2, \ldots, R_{m-1}, R\}$ is an independent set which forms a basis for $\mathbb{R}^m = \mathbb{R}^{n+1}$.

Let, $K \neq R$ be a non-zero row in M_k, hence $K \in \text{rowspace}\,(M_k)$. Since $K \in \mathbb{R}^{n+1}$, we have:

$$K = c_1R_1 + c_2R_2 + \cdots + c_nR_n + c_{n+1}R$$

Let, $Y = K - c_{n+1}R$.
This implies, $Y \in \text{rowspace}\,(M_k)$, (since both K and $c_{n+1}R$ are $\in \text{rowspace}\,(M_k)$), and $Y \in \text{rowspace}\,(M_f)$.

Let, $M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times m}$, be the standard matrix representation of some $d \in [W]$.

Since, $Y \in \text{rowspace}\,(M_f)$, Y becomes a zero row through row operations using the rows in M_f,
$\Rightarrow \text{null}\,(M_{fd}) \neq \mathbf{0}$ since $\text{rank}\,(M_{fd}) = n$,
$\Rightarrow \ker(T) \cap \ker(W) \neq \mathbf{0} \Rightarrow [T] - [W]$.

Similarly, since, $Y \in \text{rowspace}\,(M_k)$, Y becomes a zero row through row operations using the rows in M_k,
$\Rightarrow \text{null}\,(M_{kd}) \neq \mathbf{0}$ since $\text{rank}\,(M_{kd}) = n$,
$\Rightarrow \ker(L) \cap \ker(W) \neq \mathbf{0} \Rightarrow [W] - [L]$.

Therefore, we have $[T] - [W] - [L]$.

Example 3.2.11 *Suppose* $m = 4$ *and* $n = 3$. *So we are considering the graph* $G([t] : \mathbb{R}^4 \to \mathbb{R}^3)$, *where* $n < m \leq 2n$, $m \neq 1$ *or* $m \neq 2$ *and* $m = n + 1$, *as given in Theorem 3.2.10. Let* $[T], [L] \in V$, *such that* $[T]$ *and* $[L]$ *are not adjacent (*$\ker(T) \cap \ker(L) = \mathbf{0}$*), and* $[T] \neq 0, [L] \neq 0$.*In addition, since* $n < m \leq 2n$, $[T]$

and $[L]$ *are non-trivial vertices. Let* $f \in [T]$, *and* $k \in [L]$, *then* $\text{rank}\,(M_f) < m$ *and* $\text{rank}\,(M_k) < m$, *where* M_f *and* M_k *are the standard matrix representations of* f *and* k, *with size* $n \times m = 3 \times 4$. *Suppose,*

$$M_f = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{3\times4}, M_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{3\times4}$$

Let, $M_{fk} = \begin{bmatrix} M_f \\ M_k \end{bmatrix}_{6\times4}$.

It can be easily seen that $\text{rank}\,(M_{fk}) = 4$, *which implies that* $\text{null}\,(M_{fk}) = \mathbf{0}$. *Therefore,* $\ker(f) \cap \ker(k) = \mathbf{0}$, *that is the vertices* $[T]$ *and* $[L]$ *are not adjacent. We have:*

$\text{rank}\,(M_f) = 3 = n$, *and*

$\text{rank}\,(M_k) = 3 = n$.

Then M_f *has* 3 *independent rows* R_1, R_2, *and* R_3, *such that* $R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$, $R_2 = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$, *and* $R_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$. *The vertices* $[T]$ *and* $[L]$ *are not adjacent, thus* M_k *has one row,* $R = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$, *such that* $\{R_1, R_2, R_3, R\}$ *is an independent set which forms a basis for* $\mathbf{R}^4$.

Let $K \neq R$ *be a non-zero row in* M_k, $K = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$.

Since $K \in \text{rowspace}\,(M_k)$ *and* $K \in \mathbf{R}^4$, *it can be written as a linear combination of* $\{R_1, R_2, R_3, R\}$ *as follows:*

$$K = 0.R_1 + 1.R_2 + 0.R_3 + (-1)\,.R = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

Let, $Y = K - (-1)\,.R = K + R = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$.

This implies $Y \in \text{rowspace}\,(M_k)$ *and* $Y \in \text{rowspace}\,(M_f)$.

Let, $M_d = \begin{bmatrix} Y \\ 0 \\ 0 \end{bmatrix}_{3\times4} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times4}$, *be the standard matrix representation of some* $d \in [W]$.

Since, $Y \in \text{rowspace}\,(M_f)$*,* Y *becomes a zero row through row operations using the rows in* M_f*,*

$\Rightarrow \text{null}\,(M_{fd}) \neq \mathbf{0}$ *since* $\text{rank}\,(M_{fd}) = 3$*,*

$\Rightarrow \ker\,(T) \cap \ker\,(W) \neq \mathbf{0} \Rightarrow [T] - [W]$*.*

Similarly, since, $Y \in \text{rowspace}\,(M_k)$*,* Y *becomes a zero row through row operations using the rows in* M_k*,*

$\Rightarrow \text{null}\,(M_{kd}) \neq \mathbf{0}$ *since* $\text{rank}\,(M_{kd}) = 3$*,*

$\Rightarrow \ker\,(L) \cap \ker\,(W) \neq \mathbf{0} \Rightarrow [W] - [L]$*.*

Therefore, we have $[T] - [W] - [L]$*.*

Theorem 3.2.12 *A connected graph* $G\,([t] : \mathbb{R}^m \to \mathbb{R}^n)$ *has a girth of* 3*.*

Proof: $[T]\,, [L] \in V$, such that $[T]$ and $[L]$ are adjacent, $\ker\,(T) \cap \ker\,(L) \neq 0$ and $[T] \neq 0, [L] \neq 0$. Let, $f \in [T]$ and $k \in [L]$, then M_f and M_k are the standard matrix representations of f and k with size $n \times m$. Suppose, that each matrix M_f and M_k, is composed of only one row, R_f and R_k that are independent of each other since f and k are in different equivalence classes $[T]$ and $[L]$. M_f and M_k can be written as follows:

$$M_f = \begin{bmatrix} R_f \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times m}, M_k = \begin{bmatrix} R_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times m}$$

Let, $Y = R_f + R_k$. Since, Y is a linear combination of two linearly independent rows, then the sets $\{Y, R_f\}$ and $\{Y, R_k\}$ are linearly independent.

Let, $M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times m}$, be the standard matrix representation of some non-trivial linear transformation d.

Since Y is independent of both R_f and R_k, M_d is not row-equivalent to either M_f or M_k, hence d is in a different equivalence class from both f and k, say $d \in [W]$.

Since, $\ker(T) \cap \ker(L) \neq 0$, we have $\text{null}(M_{fk}) \neq 0$, which implies $\text{null}(M_{fd}) \neq 0$ and $\text{null}(M_{kd}) \neq 0$. Therefore, we have, $[T] - [L] - [W] - [T]$. This forms the shortest possible cycle. Therefore, the length of the shortest cycle or girth of the connected graph $G([t] : \mathbb{R}^m \to \mathbb{R}^n)$ is 3.

Chapter 4. Conclusion and Future Work

In conclusion, the relationship between the connectivity of the graph $G([t]:\mathbb{R}^m \to \mathbb{R}^n)$ and the values of m and n had been determined. In addition, we determined the values of m and n for a complete and totally disconnected graph, as well as the diameter and girth of the graph if connected. The main results of the research can be stated as follows:

- If $m = 1$ or $m = 2$, then the undirected graph $G([t]:\mathbb{R}^m \to \mathbb{R}^n)$ is totally disconnected.
- The undirected graph $G([t]:\mathbb{R}^m \to \mathbb{R}^n)$ is complete, if and only if $m \geq 2n + 1$.
- Consider the undirected graph $G([t]:\mathbb{R}^m \to \mathbb{R}^n)$. Assume $m \leq n$ and $m > 2$. Then: (i) Graph is connected. (ii) Graph diameter, $d = 2$.
- Consider the undirected graph $G([t]:\mathbb{R}^m \to \mathbb{R}^n)$. Assume $n < m \leq 2n$ and $m > 2$. Then: (i) Graph is connected. (ii) Graph diameter, $d = 2$.
- A connected graph $G([t]:\mathbb{R}^m \to \mathbb{R}^n)$ has a girth of 3.

For future work, it would be interesting to investigate the graph of non-trivial linear transformations over a finite field. This would allow us to evaluate other features of the graph such as the chromatic number and dominating number, $n_D = \min(|D|)$, such that D is the dominating set. The dominating set, $D \subseteq V$ is such that for every $v \in V \backslash D$, $\exists w \in D$ such that $v - w$.

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