# SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS: TRANSFORM AND ITERATIVE METHODS APPROACH

by

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Thesis Title: Solution of Fractional Differential Equations: Transform and Iterative Methods Approach

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# Dedication

First of all, thanks to Allah Almighty my creator, my source of inspiration, knowledge and understanding. I am dedicating this thesis to my beloved family: My parents, sisters and brother, who have loved me unconditionally. Thank you for all your love, support and kindness .. without you I would have never been here today!

This work is also dedicated to my fiance Ahmad, who has been a constant source of support and encouragement during this time of my career. I am genuinely thankful for having you in my life. Thank you for your endless wisdom, love and patience.

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# Abstract

Fractional calculus is a novel and highly active area of research in the literature as it has a wide spectrum of applications in the physical sciences and engineering.

In this thesis, we study the numerical solution of fractional differential equations subject to initial or boundary conditions. We employ two iterative methods for the solution of such equations. First, we implement a Laplace decomposition method (LDM) which is a combination of two methods: Laplace transform and a decomposition scheme. The nonlinear term is decomposed and a recursive algorithm is composed for the determination of the proposed infinite series solution. Second, we present another iterative technique that is based on the incorporation of Green's functions into well-established fixed-point iterations, including Krasnoselskii-Mann's and Picard's schemes. Numerical experiments are conducted to demonstrate the efficiency, accuracy and applicability of the proposed methods, and then, to compare them with other schemes. In addition, we present graphs to understand the behavior of the numerical solutions. A patching strategy, based on domain decomposition, is suggested to overcome a deficiency of the LDM. Fractional differential equations are widely investigated by several researchers.

**Search Terms** *Fractional derivatives, fractional integration, Caputo fractional derivative, Liouville fractional derivative, Green's function, fixed-point iteration schemes, Laplace transform.*

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# Chapter 1: Introduction

While Fractional Calculus (FC) is an ancient subject, it continues to be a novel one. It is an ancient subject because it has been gradually developed up to now, starting from some speculations. It can also be considered a novel topic as well. FC is a field of study in mathematics that raised out of the classical definitions of the calculus derivative and integral operators, likewise, fractional exponents are the outgrowth of integer-value exponents. The concepts of fractional operators have been introduced almost concurrently with the development of the classical ones. FC has been used to model procedures in physics and engineering that are best characterized by fractional differential equations. The major reason for the success of FC applications is that these new fractional-order models are usually more accurate than integer-order ones. It is worth nothing that in many cases the standard mathematical models of integer-order derivatives, including nonlinear models, do not work sufficiently [1, 2, 3]. Two criteria are developed; to be needed by a fractional operator. Two criteria, needed by a fractional operator, were developed. Riemann-Liouville and Caputo fractional derivatives are accessed in the light of the proposed criteria [4]. It is a topic that has acquired considerable popularity and significance in various fields of science and engineering over the previous few decades. Various efficient analytical and numerical methods have been introduced and presented, but there is still a need for more investigations and research [5].

## 1.1 Historical Development

Fractional calculus (FC) is an extension, with more than $300$ years of history, of ordinary calculus. Isaac Newton and Gottfried Leibniz invented the differential calculus independently and it was understood that the notion of the $n$th order derivative, that is applying the differentiation operation $n$ times in succession, was meaningful. FC was launched by Leibniz and L'Hospital as a result of a correspondence that lasted several months in $1695$. That date is considered as the exact birthday of the fractional calculus. The problem raised by Leibniz for a fractional derivative was an ongoing subject in decades to come [6]. In applied mathematics and mathematical analysis, fractional derivative is a derivative of any arbitrary order $\alpha$ real or complex. In $1695$, Leibniz described in his letter to L'Hospital, the derivative of $\alpha = \frac{1}{2}$ [7, 8, 9]. However, Lacroix found a formula for the derivative of arbitrary order of monomials in $1819$. The $n$th integer order derivative is [10]:

$$\frac{d^n}{dx^n}\, x^p \;=\; \frac{p!}{(p-n)!}\, x^{p-n}. \tag{1.1}$$

After that, Lacroix extended Eq.(1.1) from the derivative of integer order to the derivative of arbitrary order $\alpha \in \mathbb{R}$. He replaced the factorial by the Gamma function to get:

$$D^{\alpha} x^p \;=\; \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}\, x^{p-\alpha}. \tag{1.2}$$

Many others have been contributing throughout the history of fractional calculus, see [4, 11, 12, 13]. Understanding definitions with fractional calculus subject will be more clear by discussing rapidly some needed but comparatively easy mathematical definitions to understand these concepts. These functions are: The Gamma function, the Laplace transform, the Error function and the Mittag-Leffler function. They will be discussed in Chapter $2$.

## 1.2 Physical Interpretation

It is usually recognized that integer-order derivatives and integrals have clear geometric and physical interpretations that greatly simplify their use to solve applied problems in widely fields of sciences. For more than 300 years, the presence of the idea of integration and differentiation of arbitrary order was not an appropriate mathematical and physical interpretation of these operations.

In 1974 in New Haven (USA), at the first international conference on the fractional calculus, the lack of these interpretations was recognized by including it in the list of open problems. The problem was unanswered, and it has been repeated in 1984 at the subsequent at the University of Strathclyde (UK) and in 1989 at the Nihon University (Tokyo, Japan). Later, in 1996 in Varna, the round-table discussion at the conference on transform methods and special functions showed that the problems were still unsolved, and in reality, the situation has not changed since that time. However, in 2002, Podlubny provided a physical interpretation of the fractional integration on two distinct time scales, namely, the inhomogeneous time scale and the homogeneous one, equably flowing scale [14].

Some authors presented the analysis of the fractional time constant and the transitory response (rise, delay, and settling times) of a resistor capacitor (RC) circuit as a physical interpretation of fractional calculus in observables terms [15]. Other physical interpretations can be found in [16] and [17].

## 1.3 Main Objectives of this Thesis

The overall aim of this study is as follows:

1. Develop the theory of fractional calculus including the fractional ordinary differential equation which has physical significance in sciences, fluid flow, and many other areas of engineering.
2. Solve fractional differential equations numerically. The definition of fractional derivative forms convolution integral, thus, the easiest way to handle this integral is to use Laplace transforms.
3. Study fractional calculus by new strategy and adjustments which are Laplace decomposition method and Green's functions including some popular fixed point iterative algorithms.
4. Study analytical and approximate solution of fractional physical problems, which are of significance in different fields, such as the fractional Ray tracing model, Bratu's problem, Lienard's equation and some other fractional boundary value problems.
5. Study the analytical and approximate solution's graphical behavior for an arbitrary fractional order derivatives.

## 1.4 Applications of Fractional Calculus

Euler and Fourier mentioned arbitrary order derivatives, but they did not provide examples or applications. In 1832, Niels Henrik Abel had the honor of making the first application [18]. This application of fractional calculus is related to the integral equation solution for the tautochrone problem. The problem deals with determining the curve shape so that the time of descent of a frictionless point mass sliding down the curve under the gravity action is independent of the starting point.

In many applications, FC provides more accurate physical system models than ordinary calculus does. Since its achievement in describing anomalous diffusion [19] non-integer order calculus both in one and multidimensional space, FC has become a significant tool in many areas of mechanics, engineering, physics, chemistry and finances. In recent decades, fractional calculus provided an excellent instrument for the description of new and recent applications in control theory, generalized voltage divider, electrical circuits with fractance and viscoelasticity [8]. Machado et al. investigated the use of FC in the fields of legged robots, digital circuit synthesis, controller tuning and many others [20]. However, various applications of FC have been developed and described. See [21, 22, 23].

## 1.5 Overview of the Structure

This Master's thesis is organized as follows. Chapter 2 involves the review of important basic definitions and significant mathematical preliminaries. In chapter 3, the Laplace transform decomposition method will be presented and applied to various differential equation problems of non-integer order, including comparisons of our numerical results with other techniques. In addition, a domain decomposition strategy for improving the accuracy of the approach will be presented. In chapter 4, we give an overview of the construction of the Green's function and some common fixed point iterations strategies, such as Mann's and Picard's; then, we describe and introduce a Green's function fixed point iterative scheme for the number solution of fractional differential equations. Some numerical experiments will be included to confirm and illustrate the convergence of the method. In chapter 5, we show the convergence of the proposed iterative algorithms. In the last chapter, we will summarize our findings with suggestions for future research work.

# Chapter 2: Basic Definitions Review

## 2.1 Gamma Function

### History of the Gamma Function

In the early 16th century, Leonhard Euler and others attempted to expand the domain of the factorial to the real numbers. This cannot be done with elementary functions; however, with notions of limits and integrals from calculus, there were few expressions developed. Euler [Leonhard Euler 1707–1783] came up with mathematical objects known to us as **Gamma**. One is a function and the other is a constant. The function, Gamma$(x)$, generalizes the sequence of factorial numbers. A nice history of the Gamma function is found in a $1959$ article by Philip Davis.

**Gamma function** is a generalization of the factorial function to nonintegral values, introduced by the Swiss mathematician Leonhard Euler in the 18th century. For a positive whole number $n$, the factorial (written as $n!$) is defined by $n! = 1.2.3......(n-1).n$. For example, $5! = 1.2.3.4.5 = 120$. But, this formula is meaningless if $n$ is not an integer. To extend the factorial to any real number, the Gamma function should be used.

In $1922$, Bohr and Mollerup confronted the issue of whether the expressions of the Gamma function were equivalent, and, relatedly, whether the Gamma function as defined is unique. See (Figure $2.1$).

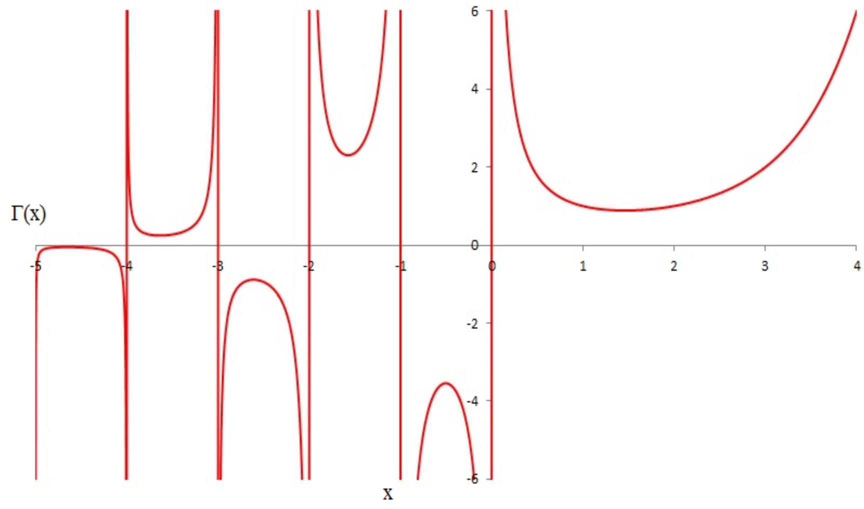


Figure 2.1: The Gamma function for real argument.

**Definition of the Gamma Function**

For any nonnegative integer $n$, we can denote factorial integrable function as $n!$, but we used to call the non-integrable $n$ by Gamma $(\Gamma)$ function.

**Definition 2.1.1. Gamma function**

$$\Gamma(P) = \int_0^\infty x^{P-1}e^{-x}\,dx, \qquad P > 0, \tag{2.1}$$

where $P$ is not necessarily an integer.

From (2.1) we have,

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx = (n-1)! \tag{2.2}$$

$$\Gamma(n+1) = \int_0^\infty x^{n}e^{-x}\,dx = n! \tag{2.3}$$

Now, we prove the above results (using induction) by expanding out $\Gamma(n)$ as follows:

*Proof.* For $n > 1$,

$$\begin{aligned}
\Gamma(n) &= (n-1)\,\Gamma(n-1)\\
\Gamma(n) &= (n-1)\,(n-2)\,\Gamma(n-2)\\
\Gamma(n) &= (n-1)\,(n-2)\,(n-3)\,\Gamma(n-3)
\end{aligned}$$

But we know that $(n-1)(n-2)(n-3)......(1) = (n-1)!$

$$\begin{aligned}
\Gamma(n) &= (n-1)\,(n-2)\,(n-3)\,.....(1)\,\Gamma(1)\\
\Gamma(n) &= (n-1)!\,\Gamma(1)
\end{aligned}$$

Its remaining to show that $\Gamma(1) = 1$

From (2.1),

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty x^{1-1}e^{-x}\,dx\\
\Gamma(1) &= \int_0^\infty e^{-x}\,dx\\
\Gamma(1) &= \frac{1}{e^{-\infty}} - (e)^{-0} = 0 - (-1) = 1
\end{aligned}$$

We complete the proof by substituting $\Gamma(1)$ into the $\Gamma(n)$ formula.

$$\begin{aligned}
\Gamma(n) &= (n-1)!\,\Gamma(1)\\
\Gamma(n) &= (n-1)!\,(1)\\
\Gamma(n) &= (n-1)!
\end{aligned}$$

□

### The Recursion Property for the Gamma Function

If we replace $P$ by $P+1$ in (2.1), we get

$$\Gamma(P+1) = \int_0^\infty x^P e^{-x}\, dx = P!, \qquad P > -1, \tag{2.4}$$

where $P$ is an integer.

Let's integrate (2.4) by using integration by parts, calling

$$u = x^P, \qquad dv = e^{-x} dx;$$

$$du = P x^P dx, \qquad v = -e^{-x},$$

then we get,

$$\begin{aligned} \Gamma(P+1) &= uv|_0^\infty - \int_0^\infty v\, du \\ &= -x^P e^{-x}|_0^\infty - \int_0^\infty (-e^{-x})\, P\, x^{P-1}\, dx \end{aligned}$$

$\frac{x^P}{e^x}$ is indeterminate form as $x \to \infty$, using L'Hospital's rule $P$ times, we get, $\frac{P!}{e^x} \to 0$ as $x \to \infty$, thus,

$$= 0 + P \int_0^\infty e^{-x} x^{P-1}\, dx = P\, \Gamma(P).$$

Therefore, we have

$$\Gamma(P+1) = P\, \Gamma(P). \tag{2.5}$$

This is called **recursion relation** for the Gamma function.

### The Gamma function of negative numbers

For $P \leq 0$, $\Gamma(P)$ is not defined. We have to define it by using the recursion relation (2.5)

$$\Gamma(P) = \frac{1}{P}\, \Gamma(P+1). \tag{2.6}$$

**Example 2.1.**

$$\Gamma(-0.4) = \frac{1}{(-0.4)}\, \Gamma(0.6),$$

$$\Gamma(-1.4) = \frac{1}{(-0.4)(-1.4)}\, \Gamma(0.6),$$

and so on.

Since $\Gamma(1) = 1$, we can see that,

$$\Gamma(P) = \frac{1}{P}\,\Gamma(P+1) \to \infty \text{ as } P \to 0.$$

From this result and the successive use of (2.6) we conclude that $\Gamma(P)$ becomes infinite not only at zero but also for all negative integers. The poles of the Gamma functions are the negative integers.

**Convergence of the Gamma Function**

We will write up the proof of the convergence of the Gamma function as follows, where we show why the exponential function grows faster than any polynomial.

*Proof.* Converges for $P > 0$. The integral (2.1) is improper of Type I and Type II, so, we need to divide the integral as a sum of two terms

$$\Gamma(P) = \int_0^1 x^{P-1}e^{-x}\,dx + \int_1^\infty x^{P-1}e^{-x}\,dx.$$

For the first term, since the function $e^{-x}$ is decreasing, it attains its maximum on the interval $[0, 1]$ at $t = 0$, so

$$\begin{aligned}\Gamma(P) = \int_0^1 x^{P-1}e^{-x}\,dx &< \int_0^1 x^{P-1}\,dx \\ &= \frac{x^P}{p}|_0^1 \\ &= \frac{1}{P}.\end{aligned}$$

Since $P > 0$, by Direct Comparison Test (DCT) we conclude that $\displaystyle\int_0^1 x^{P-1}e^{-x}\,dx$ converges. For the second term, since the exponential grows faster than any polynomial, for every $P$ we can find $N \in \mathbb{N}$, big enough so that $e^{\frac{x}{2}} \geq x^{P-1}$, for $x \in [N, +\infty)$. Thus

$$\begin{aligned}
\int_1^\infty x^{P-1}e^{-x}\,dx &= \int_1^N x^{P-1}e^{-x}\,dx + \int_N^\infty x^{P-1}e^{-x}\,dx \\
&\leq \int_1^N x^{P-1}e^{-x}\,dx + \int_N^\infty e^{\frac{x}{2}}e^{-x}\,dx \\
&= \int_1^N x^{P-1}e^{-x}\,dx + \int_N^\infty e^{\frac{-x}{2}}\,dx
\end{aligned}$$

The first term $\int_1^N x^{P-1}e^{-x}\,dx$ is finite real number because the function $x^{P-1}e^{x}$ is continuous on $[1, N]$, $\int_N^\infty e^{\frac{-x}{2}}\,dx = -\frac{1}{2}e^{\frac{-x}{2}}|_N^\infty = \frac{1}{2}e^{\frac{-N}{2}}$ is convergent. Hence, $\int_1^\infty x^{P-1}e^{-x}\,dx < \infty$.

□

**Some Special Values of the Gamma Function**

1.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{2.7}$$

*Proof.* By definition,

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty t^{\frac{1}{2}-1}e^{-t}\,dt, \\
\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty \frac{1}{\sqrt{t}}e^{-t}\,dt.
\end{aligned} \tag{2.8}$$

Set $t = y^2$ in (2.8); then $dt = 2y\,dy$. Now (2.8) becomes

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{1}{y}\,e^{-y^2}\,2y\,dy = 2\int_0^\infty e^{-y^2}\,dy, \tag{2.9}$$

or, with $x$ as the spurious integration variable,

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}dx. \tag{2.10}$$

Multiply the above two results (2.9) and (2.10)

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy. \tag{2.11}$$

Since this integral in the first quadrant, we can evaluate it by using polar coordinates. To do so, we need to remember the following conversion formula

$$r^2 = x^2 + y^2. \tag{2.12}$$

We are now ready to write down a formula for the double integral in terms of polar coordinates

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta = 4.\ \theta\big|_0^{\frac{\pi}{2}}\ .\ \left.\frac{e^{-r^2}}{-2}\right|_0^{\infty} = 4.\ \frac{\pi}{2}.\ \frac{1}{2} = \pi.$$

Therefore,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

□

2.

$$\Gamma(P)\,\Gamma(1-P) = \frac{\pi}{\sin(\pi P)}. \tag{2.13}$$

For the proof, see [24].

## 2.2 Mittag–Leffler Function

### Brief History

The Mittag-Leffler functions appear naturally when solving linear fractional differential equations. Recently, it has been used to study different models. The exponential function, $e^x$, plays a very important role in the theory of integer–order differential equations. Its one-parameter generalization (see Figure 2.2), which is defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \qquad \alpha > 0, \quad z \in \mathbb{C}, \tag{2.14}$$

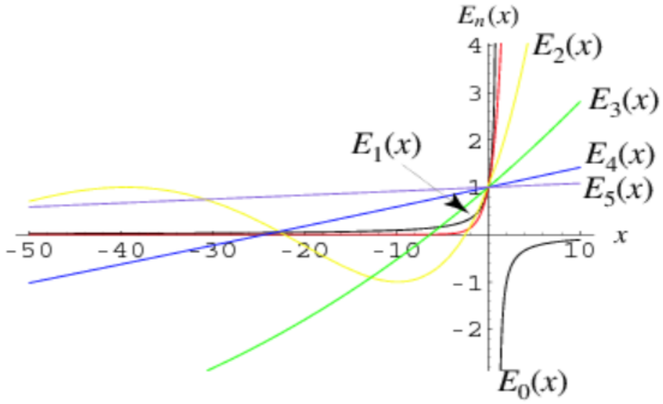


Figure 2.2: The one-parameter Mittag-Leffler function for various integer values of $\alpha$.

was introduced by Mittag-Leffler, G.M. and studied also by Wiman, A. The two-parameter function of the Mittag-Leffler type, which plays a very significant role in the fractional calculus, was in fact introduced by Agarwal. This function could have been called the ”Agarwal function”. However, Humbert and Agarwal generously left the same notation as for the one-parameter Mittag-Leffler function; that is the reason that now the two-parameter function is called the ”Mittag-Leffler function”.

**Definition 2.2.1.** A two-parameter function of the Mittag-Leffler type (see Figure 2.3) is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha\, n\, +\, 1)}, \qquad \alpha,\ \beta > 0, \quad \text{and} \quad z \in \mathbb{C}. \tag{2.15}$$

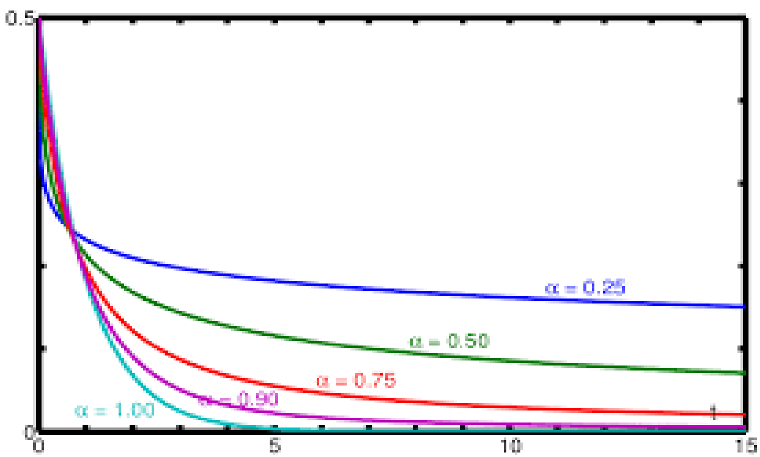


Figure 2.3: The Two-parameter Mittag-Leffler function for various fractional values of $\alpha$.

**Special Cases**

1. For $\beta = 1$, we have

$$E_{\alpha,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} = E_{\alpha}(z). \tag{2.16}$$

2. From (2.15), we obtain

$$\begin{aligned}
E_{1,1}(z) &= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n + 1)} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z, \\
E_{1,2}(z) &= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n + 2)} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)!} = \frac{1}{z}(e^z - 1), \\
E_{1,3}(z) &= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n + 3)} = \sum_{n=0}^{\infty} \frac{z^n}{(n+2)!} = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{z^{n+2}}{(n+2)!} \\
&= \frac{1}{z^2}(e^z - 1 - z),
\end{aligned}$$

In general,

$$E_{1,m}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n + m)} = \frac{1}{z^{m-1}} \left\{ e^z - \sum_{n=0}^{m-2} \frac{z^n}{n!} \right\}. \tag{2.17}$$

3. The hyperbolic sine and cosine are also particular cases of the Mittag-Leffler function

$$E_{2,1}(z^2) = \sum_{n=0}^{\infty} \frac{z^{2n}}{\Gamma(2n + 1)} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = \cosh z. \tag{2.18}$$

$$E_{2,1}(z^2) = \sum_{n=0}^{\infty} \frac{z^{2n}}{\Gamma(2n + 2)} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = \frac{\sinh z}{z}. \tag{2.19}$$

**Definition 2.2.2.** Hyperbolic function of order $k$, which is a generalization of the hyperbolic sine and cosine, are expressed in terms of the Mittag-Leffler function as follows

$$h_r(z, k) = \sum_{n=0}^{\infty} \frac{z^{nk + r - 1}}{(nk + r - 1)!} = z^{r-1} E_{k,r}(z^k), \quad \text{where } r = 1, 2, ..., k. \tag{2.20}$$

**Definition 2.2.3.** Trigonometric functions of order $n$, which are generalizations of the sine and cosine functions:

$$k_r(z, n) = \sum_{n=0}^{\infty} \frac{(-1)^i z^{ni + r - 1}}{(ni + r - 1)!} = z^{r-1} E_{n,r}(-z^n), \quad \text{where } r = 1, 2, ..., n. \tag{2.21}$$

## 2.3 Error Function

The error function, which is also called (the Gauss error function), is a special non-elementary function of sigmoid shape that takes place in statistics, probability and in partial differential equations for describing diffusion.

**Definition 2.3.1.** The error function is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt, \quad \text{for} \quad x \geq 0. \tag{2.22}$$

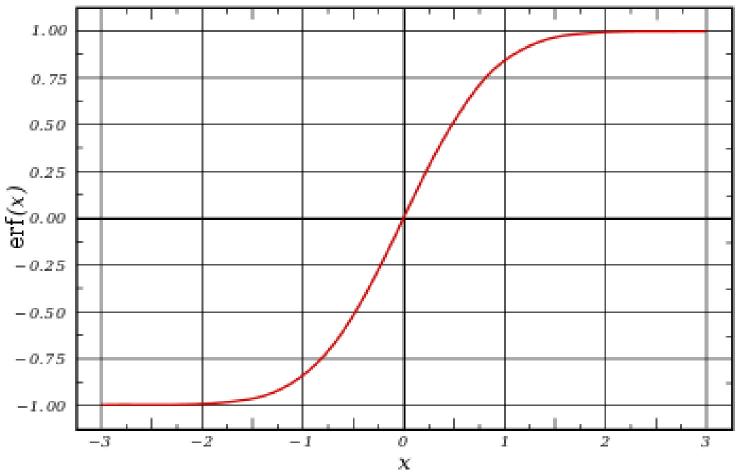


Figure 2.4: The Error function.

**Definition 2.3.2.** The complementary error function is defined as:

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt, \quad \text{for} \quad x \geq 0. \tag{2.23}$$

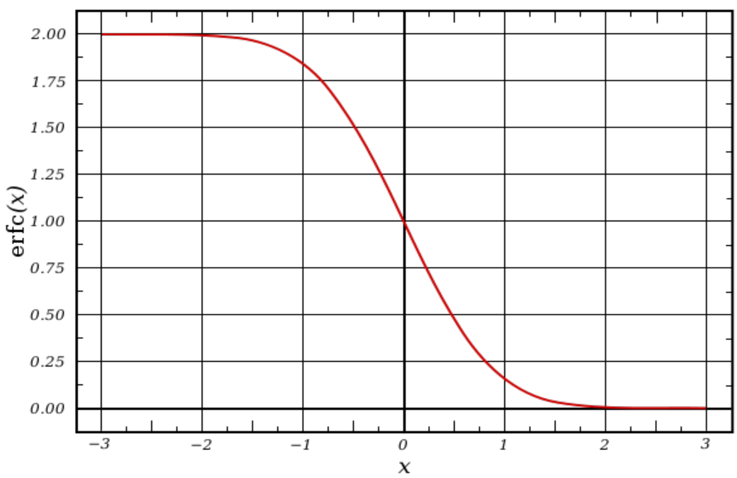


Figure 2.5: The complementary Error function.

### Special Values

1. $erf(0) = 0$
2. $erf(\infty) = 1$
3. $erf(-x) = -erf(x)$
4. $erf(x) = 1 - erfc(x)$

## 2.4 Laplace Transform

### Brief History

The theory of Laplace transform was further developed in the 19th and early 20th centuries by Lerch, Heaviside, and Bromwich. The current widespread use of the transform (mainly in engineering) came about during and soon after World War II replacing the earlier Heaviside operational calculus. The advantages of the Laplace transform had been emphasized by Doetsch to whom the name Laplace Transform is apparently due. The early history of methods having some similarity to Laplace transform is as follows. From 1744, Leonhard Euler investigated integrals of the form

$$\mathcal{L} = \int f(x)\, e^{ax}\, dx, \tag{2.24}$$

as solutions of differential equations but did not pursue the matter very far.
Joseph Louis Lagrange was an admirer of Euler and, in his work on integrating probability density functions, investigated expressions of the form

$$\mathcal{L} = \int f(x)\, e^{-ax} a^x dx, \tag{2.25}$$

which some modern historians have interpreted within modern Laplace transform theory.

These types of integrals seem first to have attracted Laplace's attention in 1782 when he was following in the spirit of Euler in the use of the integrals as solutions of equations. However, in 1785, Laplace took the critical step forward and rather than just looking for a solution in the form of an integral, he started to apply the transforms in the sense that was later to become popular.

Laplace also recognized that Joseph Fourier's method of Fourier series for solving the diffusion equation could only apply to a limited region of space because those solutions were periodic. In 1809, Laplace applied his transform to find solutions that diffused indefinitely in space.

The Laplace Transform turns a differential equation in time $t$ into an algebraic equation in complex variables. The process of applying Laplace transforms is pretty much the same. We apply the transform to a differential equation, and then, turn it into an algebraic equation, thus making it significantly easier to handle. Then, we carry out the simplification of the algebraic expression to the required extent. Finally, the Inverse Laplace Transform is applied to obtain the answer in our actual given domain.

**Definition 2.4.1.** Let $f(t)$ be defined for $t \geq 0$. The Laplace transform of $f(t)$, denoted by $F(s)$ or $\mathcal{L}\{f(t)\}$, is an integral transform given by the Laplace integral:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt. \tag{2.26}$$

**Example 2.2.** *Let* $f(x) = t$.

Then,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st} t\, dt.$$

In fact, integration by parts calling,

$$\begin{aligned} u &= t, & dv &= e^{-st}, \\ du &= dt, & v &= -\tfrac{1}{s}\, e^{-st}, \end{aligned}$$

$$= u\, v - \int_0^\infty v\, du = -\frac{1}{s}\, t\, e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt.$$

The integral is divergent whenever $s \leq 0$. However, when $s > 0$, it converges to

$$-\frac{1}{s}\, \frac{t}{e^{-st}}\Big|_0^\infty - \frac{1}{s^2}\, \frac{t}{e^{-st}}\Big|_0^\infty.$$

By using L'Hospital's rule and after simplification, we obtain,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, dt = \frac{1}{s^2}.$$

**Properties**

- $\mathcal{L}\{0\} = 0$
- Linearity: $\mathcal{L}\{f(t) \pm g(t)\} = \mathcal{L}\{f(t)\} \pm \mathcal{L}\{g(t)\}$.

- Constant Multiple: $\mathcal{L}\{af(t)\} = a.\mathcal{L}\{f(t)\}$.
- Change of Scale: $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$.
- Shifting: $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$.
- Conjugation: $\mathcal{L}\{f^*(t)\} = F^*(s^*)$, where $f^*$ means the complex conjugate.
- Convolution: $\mathcal{L}\{f(t) * g(t)\} = F(s)\, G(s)$.
- Differentiation in s-domain: $\mathcal{L}\{t^n\, f(t)\} = (-1)^n F^{(n)}(s)$.
- Differentiation in Time domain: $\mathcal{L}\{f'(t)\} = s\, F(s)$.
- Integration: $\mathcal{L}\{\int_{-\infty}^{t} f(x)\, dx\} = \frac{1}{s}\, F(s)$.

**Warning:** The Laplace transform, while a linear operator, is not multiplicative. That is, in general

$$\mathcal{L}\{f(t)\, g(t)\} \neq \mathcal{L}\{f(t)\}\, \mathcal{L}\{g(t)\}. \tag{2.27}$$

### 2.4.1 Laplace transform of the Mittag–leffler function

The Laplace transform of the Mittage–Leffler function is given by the following theorem.

**Theorem 2.4.1.**

$$\mathcal{L}^{-1}\left[\frac{s^{-(\beta-\alpha)}}{s^{\alpha} - x}\right] = t^{\beta-1} E_{\alpha,\beta}(x\, t^{\alpha}), \qquad |s^{\alpha} - x| < 1 \tag{2.28}$$

*Proof.* Using the definition of the Laplace transform, we have

$$\begin{aligned} \mathcal{L}[t^{\beta-1} E_{\alpha,\beta}(x\, t^{\alpha})] &= \int_0^{\infty} e^{-s\,t}\, t^{\beta-1} E_{\alpha,\beta}(x\, t^{\alpha})\, dt \qquad (2.29)\\ &= \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha\, n + \beta)} \int_{n=0}^{\infty} e^{-s\,t}\, t^{\alpha\, n+\beta-1}\, dt \\ &= \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha\, n + \beta)} \mathcal{L}(t^{\alpha\, n+\beta-1}) \\ &= \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha\, n + \beta)} \frac{\Gamma(\alpha\, n + \beta)}{s^{\alpha\, n+\beta}} \\ &= \frac{1}{s^{\beta}} \sum_{n=0}^{\infty} \left(\frac{x}{s^{\alpha}}\right)^n \end{aligned}$$

The series above converges for $\left|\frac{x}{s^{\alpha}}\right| < 1$.Hence,

$$\mathcal{L}[t^{\beta-1} E_{\alpha,\beta}(x\, t^{\alpha})] = \frac{s^{-\beta}}{1 - \frac{x}{s^{\alpha}}} = \left[\frac{s^{-(\beta-\alpha)}}{s^{\alpha} - x}\right].$$

□

### 2.4.2 On the solution of differential equation problems

In general, exact solutions for differential equations that arise in various fields in sciences and engineering can not be obtained. So, researchers seek approximate solutions by various numerical methods. Most widely used analytical and collocation methods include but are not limited to the variational iteration method (VIM) [25, 26, 27, 28], Homotopy perturbation method [29], homotopy analysis method [30], Taylor collocation method [31], Chebyshev collocation method [32], spline collocation methods [33, 34], wavelet-based methods [35] and Laplace and Adomian decomposition methods [36].

Most of the numerical methods applied to ordinary differential equations have been modified to provide approximate solutions to fractional differential equations. In this thesis, the Laplace decomposition method and Green's function based methods are modified to solve Fractional differential equations.

We begin by reviewing Laplace transforms for a system of linear differential equations.

**Theorem 2.4.2. Laplace transform of derivatives for integer order**

*Suppose $f$ is of integer order, and that $f$ is continuous and $f'$ is piecewise continuous on any interval $0 \leq t \leq A$. Then*

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \tag{2.30}$$

Applying the theorem multiple times yields

$$\begin{aligned}
\mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0), \\
\mathcal{L}\{f'''(t)\} &= s^3\mathcal{L}\{f(t)\} - s^2f(0) - f'(0) - f''(0), \\
&. \\
&. \\
&. \\
\mathcal{L}\{f^{(n)}(t)\} &= s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) \\
&- \ ... - s^2f^{(n-3)}(0) - sf^{(n-2)}(0) - f^{(n-1)}(0).
\end{aligned} \tag{2.31}$$

Equally importantly, it says that the Laplace transform, when applied to a differential equation, would change derivatives into algebraic expressions in terms of s and dependent variable $t$. Thus, it can transform a differential equation into an algebraic equation.

### 2.4.3 Laplace Transform of Fractional Differential Operators

**Definition 2.4.2. [Caputo Fractional Derivative]**

Let $f \in C^n[a,b], a \geq 0$, and $n-1 < a \leq n$. Then

$$\begin{aligned} D^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} \ \frac{d^n f(x)}{dx^n}\, dx, \quad a \leq t < b, \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(x)}{(t-x)^{\alpha+1-n}} \ dx. \end{aligned} \tag{2.32}$$

**Definition 2.4.3. [Riemann-Liouville Fractional Derivative]**

Let $f \in C^n[a,b], a \geq 0$, and $n-1 < a \leq n$. Then

$$\begin{aligned} D_L^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} \ f(x)\, dx, \quad a \leq t < b, \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(x)}{(t-x)^{\alpha+n-1}} \ dx. \end{aligned} \tag{2.33}$$

**Lemma 2.4.1.** *The Laplace transform of Riemann-Liouville fractional integral operator of order $\alpha > 0$ can be obtained in the form of:*

$$\mathcal{L}[J^\alpha f(t)] = \frac{F(s)}{s^\alpha}, \textit{where } J^\alpha \textit{ is the } \alpha \textit{ integral.} \tag{2.34}$$

*Proof.* The Laplace transform of Riemann-Liouville fractional integral operator of order $\alpha > 0$ is

$$\mathcal{L}[J^\alpha f(t)] = \mathcal{L}\left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)\, dx\right] = \frac{1}{\Gamma(\alpha)} F(s)\, G(s),$$

where,

$$G(s) = \mathcal{L}[t^{\alpha-1}] = \frac{\Gamma(\alpha)}{s^\alpha}.$$

Hence

$$\mathcal{L}[J^\alpha f(t)] = \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{s^\alpha} F(s) = \frac{F(s)}{s^\alpha}.$$

□

**Lemma 2.4.2.** *The Laplace transform of Caputo fractional derivative for $m-1 < \alpha \leq m, m \in \mathbb{N}$, can be obtained in the form of (see [37, 38]):*

$$\mathcal{L}[D^\alpha f(t)] = \frac{s^m f(s) - s^{m-1} f(0) - s^{m-2} f'(0) - ... - f^{(m-1)}(0)}{s^{m-\alpha}}. \tag{2.35}$$

*Proof.* The Laplace transform of Caputo fractional derivative of order $\alpha > 0$ is

$$\mathcal{L}[D^{\alpha} f(t)] = \mathcal{L}[J^{m-\alpha} f^{m}(t)] = \frac{\mathcal{L}[f^{m}(t)]}{s^{m-\alpha}}. \tag{2.36}$$

$\square$

We are now ready to see how the Laplace transform can be is used differentiation equations.

**Solving differential equation problems using the method of Laplace transform:**

To solve a linear differential equation using Laplace transforms, there are only 3 basic steps:

1. Take the Laplace transforms of both sides of an equation.
2. Simplify algebraically the result to solve for $\mathcal{L}\{f(t)\} = F(s)$ in terms of s.
3. Find the inverse transform of $F(s)$. This inverse transform, $f(t)$, is the solution of the given differential equation.

The method will also solve a nonhomogeneous linear differential equation directly, using the exact same three basic steps, without having to separately solve for the complementary and particular solutions.

**Example 2.3.** *Consider the following differential equation:*

$$y' + 2y = 4t\, e^{-2t},$$

*with*

$$y(0) = -3.$$

Transform both sides

$$\mathcal{L}\{y'\} + \mathcal{L}\{2y\} = \mathcal{L}\{4te^{-2t}\},$$

$$s\mathcal{L}\{y\} - y(0) + 2\mathcal{L}\{y\} = \frac{4}{(s+2)^2}.$$

Simplify to find $F(s) = \mathcal{L}\{y\}$

$$(s\mathcal{L}\{y\} - (-3)) + 2\mathcal{L}\{y\} = \frac{4}{(s+2)^2},$$

$$\mathcal{L}\{y\}(s+2) + 3 = \frac{4}{(s+2)^2},$$

$$\mathcal{L}\{y\}(s+2) = \frac{4}{(s+2)^2} - 3,$$

$$\mathcal{L}\{y\} = \frac{4}{(s+2)^3} - \frac{3}{(s+2)} = \frac{4-3(s+2)^2}{(s+2)^3} = \frac{-3s^2-12s-8}{(s+2)^3},$$

By partial fractions

$$\begin{aligned}
\mathcal{L}\{y\} &= \frac{-3s^2-12s-8}{(s+2)^3} &=& \frac{a}{(s+2)^3} + \frac{b}{(s+2)^2} + \frac{c}{(s+2)} \\
&= \frac{-3s^2-12s-8}{(s+2)^3} &=& \frac{a}{(s+2)^3} + \frac{b(s+2)}{(s+2)^2} + \frac{c(s+2)^2}{(s+2)} \\
&= \frac{-3s^2-12s-8}{(s+2)^3} &=& \frac{a+bs+2b+cs^2+4cs+4c}{(s+2)^3} \\
&= \frac{-3s^2-12s-8}{(s+2)^3} &=& \frac{cs^2+(b+4c)s+(a+2b+4c)}{(s+2)^3}.
\end{aligned}$$

By equating the nominator of both fractions, we obtain

$$\begin{aligned}
c &= -3 \\
(b+4c) &= -12 \\
(a+2b+4c) &= -8
\end{aligned}$$

Solving the above system, we obtain

$$c=-3,\ a=4,\ b=0.$$

Now, by substituting the values in the expression of $\mathcal{L}\{y\}$, we obtain

$$\mathcal{L}\{y\} = \frac{-3s^2-12s-8}{(s+2)^3} = \frac{4}{(s+2)^3} - \frac{3}{(s+2)}.$$

Hence

$$\begin{aligned}
y(t) &= 4\mathcal{L}^{-1}\left(\frac{1}{(s+2)^3}\right) - 3\mathcal{L}^{-1}\left(\frac{1}{(s+2)}\right) \\
&= 2t^2e^{-2t} - 3e^{-2t}.
\end{aligned}$$

**Example 2.4.** *Consider the following differential equation:*

$$D^{\alpha}y = 2x,$$

*with*

$$y(0)=0,$$

*and*

$$\alpha = 0.5.$$

Using equation (2.35), we have

$$\frac{s\,\mathcal{L}[y]-y(0)}{s^{1-\alpha}} \;=\; 2\,\mathcal{L}[x].$$

After simplifications, we have

$$\mathcal{L}[y] \;=\; \frac{2}{s^{\alpha+2}}.$$

Apply the laplace inverse on both sides to find $y$, we obtain

$$y \;=\; 1.504505556\,x^{\frac{3}{2}}.$$

We will discuss later how we solve differential equation problems of nonlinear fractional (non-integer) order.

# Chapter 3: Fractional Calculus

Standard calculus involves differentiation and integration of integer order, while fractional calculus involves differentiation and integration of arbitrary real numbers and complex numbers. Fractional calculus is a branch of mathematical analysis that studies the several different possibilities of defining real number powers or complex number powers of the differentiation operator $D$

$$D\,f(x) \;=\; \frac{d}{dx}\,f(x), \tag{3.1}$$

and of the integration operator $J$

$$J\,f(x) \;=\; \int_0^x f(s)\,ds. \tag{3.2}$$

Moreover, one can look at the question of defining a linear functional $D^\alpha$, for every real-number $\alpha$. Fractional differential equations, also known as extraordinary differential equations, which are a generalization of differential equations through the application of fractional calculus.

## 3.1 Fractional Derivative

**Definition 3.1.1.** For any function $f(x)$ consider the following strategy:
The first derivative is:

$$f'(x) \;=\; \frac{d}{dx}f(x) \;=\; Df(x).$$

The second derivative is:

$$f''(x) \;=\; \frac{d^2}{dx^2}f(x) \;=\; D^2f(x).$$

Continuing in this fashion gives more general result that,

$$f^{(n)}(x) \;=\; \frac{d^n}{dx^n}f(x) \;=\; D^nf(x). \tag{3.3}$$

Note that, $n$ can be integer or any real number. To be more specific, we will go through each one of them independently.

1. **Fractional derivative of basic power function**

Let

$$\begin{aligned} f(x) &= x^n, \\ f'(x) &= Df(x) = nx^{n-1}, \\ f''(x) &= D^2 f(x) = n(n-1)x^{n-2} = \frac{n!}{(n-2)!}\, x^{n-2}, \\ f'''(x) &= D^3 f(x) = n(n-1)(n-2)x^{n-3} = \frac{n!}{(n-3)!}\, x^{n-3}, \end{aligned}$$

Essentially, it could be generalized as follows:

$$f^{(\alpha)}(x) = D^\alpha f(x) = \frac{n!}{(n-\alpha)!}\, x^{n-\alpha}. \tag{3.4}$$

The above formula true for any integer $n$. Here, we extend the factorial to any real number by using the fact that $\Gamma(n+1) = n!$, Eq.(3.1) can be written in its general form as follows

$$f^{(\alpha)}(x) = D^\alpha f(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}\, x^{n-\alpha}. \tag{3.5}$$

In general, if $f(x) = x^\beta$ then $D^\alpha f(x) = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\, x^{\beta-\alpha}$.

**Example 3.1.** *Find the quarter derivative of $f(x) = x^2$.*

From Eq.(3.5), for $n = 2$ and $\alpha = \frac{1}{4}$, we obtain

$$f^{(\frac{1}{4})}(x) = D^{\frac{1}{4}} f(x) = \frac{\Gamma(2+1)}{\Gamma(2-\frac{1}{4}+1)}\, x^{2-\frac{1}{4}} = \frac{2}{\Gamma(\frac{11}{4})}\, x^{\frac{7}{4}},$$

From the following special identity that was mentioned in the section on Gamma function

$$\Gamma(p)\,\Gamma(p-1) = \frac{\pi}{\sin(\pi p)},$$

and by computing $\Gamma(\frac{11}{4})$ using (3.5), we obtain

$$\begin{aligned} p &= \frac{11}{4} \\ \Gamma(\tfrac{11}{4})\Gamma(-\tfrac{7}{4}) &= \sin\left(\frac{11\pi}{4}\right) \\ \Gamma(\tfrac{11}{4}) &= \frac{\pi}{\sin\left(\frac{11\pi}{4}\right)\Gamma\left(-\frac{7}{4}\right)} \\ &= \frac{\pi}{\frac{1}{\sqrt{2}}\Gamma(-\frac{7}{4})} \\ &= \frac{\frac{\pi}{\sqrt{2}}}{\Gamma(\frac{-7}{4})}. \end{aligned}$$

Thus,

$$D^{\frac{1}{4}}\left(x^2\right) = \frac{2}{\frac{\frac{\pi}{\sqrt{2}}}{\Gamma\left(-\frac{7}{4}\right)}}\, x^{\frac{7}{4}} = \frac{\sqrt{2}\Gamma\left(-\frac{7}{4}\right)}{\pi}\, \sqrt[4]{x^7}.$$

**2. Fractional derivative of exponential function**

Let

$$f(x) = e^{ax}.$$

The first derivative is:

$$f'(x) = \frac{d}{dx} f(x) = Df(x) = ae^{ax}.$$

The second derivative is:

$$f''(x) = \frac{d^2}{dx^2} f(x) = D^2 f(x) = a^2 e^{ax},$$

$$\dots$$

The $n$th order derivative of $f(x) = e^{ax}$ is given as:

$$f^n(x) = \frac{d^n}{dx^n} f(x) = D^n f(x) = a^n e^{ax}.$$

In other words,

$$D^\alpha e^{ax} = a^\alpha e^{ax}. \tag{3.6}$$

Note that, the above result is true for any real number $n$. This clarifies the generalization of $n$th derivative of the exponential to fractional orders.

**Example 3.2.** *Find the half derivative of:*

$[i]$ $f(x) = e^{ax}$.

From (3.6),

$$\begin{aligned} D^{\frac{1}{2}} f(x) &= a^{\frac{1}{2}} e^{ax}, \\ &= \sqrt{a}\, e^{ax}. \end{aligned}$$

$[ii]$

$$\begin{aligned} D^{\frac{1}{2}}(D^{\frac{1}{2}} f(x)) &= D^{\frac{1}{2}}(\sqrt{a}\, e^{ax}), \\ &= a^{\frac{1}{2}} \sqrt{a}\, e^{ax}, \\ &= \sqrt{a}\sqrt{a}\, e^{ax}, \\ &= ae^{ax} = De^{ax}, \end{aligned}$$

which is the first derivative of $e^{ax}$.

**Example 3.3.** *Calculate the $i^{th}$ derivative of $e^{ix}$.*

Recall Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{3.7}$$

By using Eq.*(3.6),* for $\alpha = i$ and $a = i$, we have

$$\begin{aligned} D^i e^{ix} &= (i)^i e^{ix} \\ &= (e^{\frac{\pi}{2}i})^i e^{ix} \\ &= e^{\frac{\pi}{2}i^2} e^{ix} \\ &= e^{-\frac{\pi}{2}} e^{ix} \\ &= e^{-\frac{\pi}{2}}(\cos x - i\sin x). \end{aligned}$$

**3. Fractional derivative of trigonometric functions** $\sin x$ **and** $\cos x$

We know that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \text{ and } \sin x = \frac{e^{ix} - e^{-ix}}{2}. \tag{3.8}$$

As a result, by using both Eqs.(3.6) and (3.7), for $a = i$ we define the following formulas:

$$\begin{aligned} D^\alpha e^{ix} &= (i)^\alpha e^{ix} \\ &= (e^{\frac{\pi}{2}i})^\alpha e^{ix} \\ &= e^{\frac{\pi}{2}\alpha i} e^{ix} \\ &= e^{i\left(x + \frac{\pi}{2}\alpha\right)}. \end{aligned} \tag{3.9}$$

$$\begin{aligned} D^\alpha e^{-ix} &= (-i)^\alpha e^{-ix} \\ &= (e^{-\frac{\pi}{2}i})^\alpha e^{-ix} \\ &= e^{-\frac{\pi}{2}\alpha i} e^{-ix} \\ &= e^{-i\left(x + \frac{\pi}{2}\alpha\right)}. \end{aligned} \tag{3.10}$$

Now, from the previous formulas we can define

$$D^\alpha \cos x = \frac{e^{i\left(x + \frac{\pi}{2}\alpha\right)} + e^{-i\left(x + \frac{\pi}{2}\alpha\right)}}{2} = \cos\left(x + \frac{\pi}{2}\alpha\right) \tag{3.11}$$

and

$$D^{\alpha} \sin x = \frac{e^{i\left(x+\frac{\pi}{2}\alpha\right)} - e^{-i\left(x+\frac{\pi}{2}\,\alpha\right)}}{2!} = \sin\left(x + \frac{\pi}{2}\alpha\right). \tag{3.12}$$

We know that:

$$D \cos x = -\sin x. \tag{3.13}$$

*Proof.* To prove Eq.(3.13), we use (3.11) with $\alpha = 1$

$$\begin{aligned} D\cos x &= \cos\left(x + \frac{\pi}{2}\,\alpha\right) \\ &= \cos(x)\cos\left(\frac{\pi}{2}\right) - \sin(x)\sin\left(\frac{\pi}{2}\right). \\ &= -\sin(x). \end{aligned}$$

□

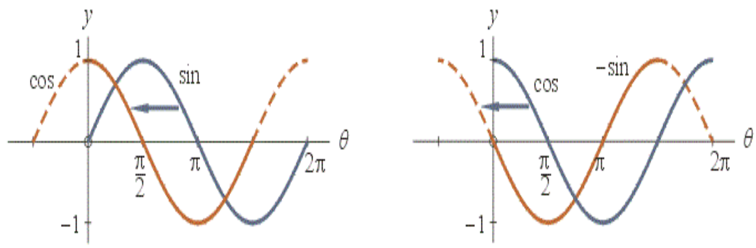


Figure 3.1: The fractional derivative of sine and cosine functions.

Similarily,

**Example 3.4.** $D^{\frac{1}{2}} \cos x \; = \; \cos\left(x + \frac{\pi}{2} \cdot \frac{1}{2}\right) = \cos\left(x + \frac{\pi}{4}\right).$

### 3.1.1 The Caputo Fractional Differential Operator

**Definition 3.1.2. (Caputo Fractional Derivative)**

Let $f \in C^n[a, b]$, $a \geq 0$, and, $n - 1 < \alpha \leq n$, then,

$$
\begin{aligned}
D^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} \, \frac{d^n f(x)}{dx^n} \, dx, \quad a \le t < b, \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(x)}{(t-x)^{\alpha+1-n}} \, dx.
\end{aligned} \tag{3.14}
$$

The benefit of using the Caputo definition is that it does not only allow for the consideration of easily interpreted initial conditions, but it is also bounded, meaning that the derivative of a constant is equal to $0$.

**Differentiability**

Consider the following well-known theorem:

**Theorem 3.1.1. Fundamental Theorem of Calculus (FTC)**

**[Part 1]**

Let $f(x)$ be a continuous real-valued function defined on a closed interval $[a,b]$, and

$$F(x) = \int_a^x f(x)\, dx, \qquad \text{for all } \ x \in [a,b]. \tag{3.15}$$

Then, $F(x)$ is uniformly continuous on $[a,b]$, differentiable on the open interval $(a,b)$, and

$$F'(x) = f(x), \qquad \text{for all } \ x \in (a,b). \tag{3.16}$$

**[Part 2]**

Let $f$ be a real-valued function on a closed interval $[a,b]$ and $F$ an antiderivative of $f$ in $[a,b]$, *i.e.*,

$$F'(x) = f(x). \tag{3.17}$$

If $f$ is Riemann integrable on $[a,b]$ then

$$\int_a^b f(x)\, dx = F(b) - F(a). \tag{3.18}$$

$f(x)$ is $n$th differentiable on $[a,b]$, then it is continuous since $f \in C^n[a,b]$ and $(t-x)^{n-\alpha-1}$ is continuous on the interval $[0,t)$. Since

$$f^{(n)}(x) \text{ is bounded on } [a,b], \text{ and } -1 < n-\alpha-1 \le 0, \text{ then}$$

$$\frac{f^{(n)}(x)}{(t-x)^{\alpha+1-n}}$$

is integrable over $[0,t]$, where $a \le t \le b$. Thus, by FTC, $\dfrac{f^{(n)}(x)}{(t-x)^{\alpha+1-n}}$ is differentiable and then it is continuous.

**Example 3.5.** *Find the second derivative of $f(x) = x^3$ using Caputo definition.*

From (3.14), we have

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} \; \frac{d^n f(x)}{dx^n} \, dx, \quad 0 \le t < b$$

for $n = 3$ and $\alpha = 2$

$$D^2 f(t) = \frac{1}{\Gamma(3-2)} \int_0^t (t-x)^{3-2-1} \; \frac{d^3 f(x)}{dx^3} \, dx$$

for $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$, and $f'''(x) = 6$.

Thus

$$\begin{aligned} D^2 f(t) &= \frac{1}{\Gamma(1)} \int_0^t (t-x)^0 \;\; 6 \, dx \\ &= \int_0^t (6) \, dx = 6t. \end{aligned}$$

Note that $\Gamma(1) = 1$.

**Example 3.6.** *Find the half derivative of $f(x) = x^3$ using Caputo definition.*

For $n = 3$ and $\alpha = \frac{1}{2}$,

$$D^{\frac{1}{2}} f(t) = \frac{1}{\Gamma(3-\frac{1}{2})} \int_0^t (t-x)^{3-\frac{1}{2}-1} \; \frac{d^3 f(x)}{dx^3} \, dx.$$

From the previous example, we have $f^{(3)}(x) = f'''(x) = 6$. Then, we have

$$\begin{aligned} D^{\frac{1}{2}} f(t) &= \frac{1}{\Gamma\left(\frac{5}{2}\right)} \int_0^t (t-x)^{\frac{3}{2}} \;\; 6 \, dx \\ &= \frac{6}{\Gamma\left(\frac{5}{2}\right)} \int_0^t (t-x)^{\frac{3}{2}} \, dx. \end{aligned}$$

Using the properties of the Gamma function, the latter integral becomes

$$\begin{aligned} D^{\frac{1}{2}} f(t) &= \frac{6 \cdot 2}{5\,\Gamma\left(2+\frac{1}{2}\right)}\,(t-x)^{\frac{3}{2}+1}\Big|_0^t \\ &= \frac{12}{5 \cdot \frac{3}{4}\,\Gamma\left(\frac{1}{2}\right)}(-t)^{\frac{5}{2}} = \frac{8}{15\sqrt{\pi}}\,(-t)^{\frac{5}{2}}. \end{aligned}$$

### 3.1.2 The Riemann-Liouville Fractional Differential Operator

**Definition 3.1.3. Riemann-Liouville Fractional Derivative**

Let $f \in C^n[a,b], a \geq 0$, and, $n-1 < a \leq n$, then,

$$\begin{aligned} D_L^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} \;\; f(x)\,dx \quad a < t < b, \\ &= \frac{1}{\Gamma(n-\alpha)}\,\frac{d^n}{dt^n} \int_0^t \frac{f(x)}{(t-x)^{\alpha+n-1}}\,dx. \end{aligned} \tag{3.19}$$

**Example 3.7.** *Find the half derivative of $f(x) = x$ using the Riemann-Liouville definition.*

From (3.19), we have

$$D_L^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\,\frac{d^n}{dt^n} \int_0^t (t-x)^{n-\alpha-1} \;\; f(x)\,dx,$$

for $n = 1$ and $\alpha = \frac{1}{2}$.

$$D_L^{\frac{1}{2}} f(t) = \frac{1}{\Gamma(1-\frac{1}{2})}\,\frac{d}{dt} \int_0^t (t-x)^{1-\frac{1}{2}-1} \;\; x\,dx.$$

Using simple substitution, we obtain

$$\begin{aligned} D_L^{\frac{1}{2}} f(t) &= \frac{1}{\Gamma(\frac{1}{2})}\,\frac{d}{dt}\left(\frac{4t^{\frac{3}{2}}}{3}\right) \\ &= \frac{2\sqrt{t}}{\sqrt{\pi}}. \end{aligned}$$

### 3.1.3 Properties for the Caputo and Riemann-Liouville Fraction Differential Operators

**Proposition:** In general, the two operators, Riemann-Liouville and Caputo, do not coincide, i.e.

$$D^{\alpha} f(t) \neq D_L^{\alpha} f(t). \tag{3.20}$$

1. **Interpolation**

**Lemma 3.1.1.** *Let* $n-1 < \alpha \leq n,\ n \in \mathbb{N}, \alpha \in \mathbb{R}$ *and* $f(t)$ *be such that* $D^{\alpha} f(t)$ *exists. Then, the following properties for the Caputo operator hold*

$$\begin{aligned} \lim_{\alpha \to n} D^{\alpha} f(t) &= f^{(n)}(t), \\ \lim_{\alpha \to n-1} D^{\alpha} f(t) &= f^{(n-1)}(t) - f^{(n-1)}(0). \end{aligned} \tag{3.21}$$

*Proof.* In this proof, we use integration by parts.

$$\begin{aligned} D^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} \frac{d^n f(x)}{dx^n}\, dx \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(x)}{(t-x)^{\alpha+1-n}}\, dx \\ &= \frac{1}{\Gamma(n-\alpha)} \left( -f^{(n)}(x)\, \frac{(t-x)^{n-\alpha}}{(n-\alpha)} \Big|_0^t - \int_0^t -f^{(n+1)}(x)\, \frac{(t-x)^{n-\alpha}}{(n-\alpha)}\, dx \right) \\ &= \frac{1}{\Gamma(n-\alpha+1)} \left( f^{(n)}(0)\, t^{n-\alpha} + \int_0^t f^{(n+1)}(x)(t-x)^{n-\alpha}\, dx \right). \end{aligned}$$

Now, by taking the limit as $\alpha \to n$ and $\alpha \to n-1$, respectively, it follows that,

$$\begin{aligned} \lim_{\alpha \to n} D^{\alpha} f(t) &= f^{(n)}(0) + f^{(n)}(x)\big|_0^t = f^{(n)}(t), \text{and} \\ \lim_{\alpha \to n-1} D^{\alpha} f(t) &= f^{(n)}(0) + f^{(n)}(x)(t-x)\big|_0^t - \int_0^t \left(-f^{(n)}(x)\right)\, dx \\ &= f^{(n-1)}(x)\big|_0^t = f^{n}(0) - f^{n}(0)\, t + f^{(n-1)}(t) - f^{(n-1)}(0). \end{aligned}$$

□

Similarly, the Riemann-Liouville fractional differential operator has the following correspondence interpolation property:

$$\begin{aligned} \lim_{\alpha\to n} D_L^{\alpha} f(t) &= f^{(n)}(t), \text{and} \\ \lim_{\alpha\to n-1} D_L^{\alpha} f(t) &= f^{(n-1)}(t). \end{aligned} \tag{3.22}$$

2. **Linearity**

**Lemma 3.1.2.** *Let $n-1<\alpha\leq n, n\in\mathbb{N}, \alpha,\lambda\in\mathbb{C}$ and the functions $f(t)$ and $g(t)$ be such that $D^{\alpha}f(t)$ and $D^{\alpha}g(t)$ exist. The Caputo fractional derivative is a linear operator, i.e.*

$$D^{\alpha}(\lambda\, f(t)+g(t)) = \lambda\, D^{\alpha}f(t) + D^{\alpha}g(t). \tag{3.23}$$

*Proof.* The linearity of these operators follow from the linearity of the integer order derivatives by which they are defined.

□

The Riemann-Liouville operator also satisfies the following linearity property:

$$D_L^{\alpha}(\lambda\, f(t)+g(t)) = \lambda\, D_L^{\alpha}f(t) + D^{\alpha}g(t). \tag{3.24}$$

3. **Non-commutation**

**Lemma 3.1.3.** *Let $n-1<\alpha\leq n, m,n\in\mathbb{N}, \alpha\in\mathbb{R}$ and the function $f(t)$ is such that $D^{\alpha}f(t)$ exists. Then, in general,*

$$D^{\alpha}D^{m}f(t) = D^{\alpha+m}f(t) \neq D^{m}D^{\alpha}f(t). \tag{3.25}$$

THE RIEMANN-LIOUVILLE OPERATOR IS ALSO NON-COMMUTATIVE AND STATISTICS.

$$D^{\alpha}D^{m}f(t) = D^{\alpha+m}f(t) \neq D^{\alpha}D^{m}f(t). \tag{3.26}$$

## 3.2 Fractional Integration

In this section, we introduce the fractional integral, which is a generalization of the $n$-tuple iterated integral to any real order. We start by expressing any $n^{th}$ iterated integral as a single integral, using Cauchy's formula for repeated integration.

**Theorem 3.2.1. (Cauchy formula for repeated integration)**

*Let $f$ be some continuous function on the interval $[a, b]$. The $n^{th}$ repeated integral of $f$ based at $a$,*

$$Jf(x) = \int_a^x f(t)\, dt, \tag{3.27}$$

$$J^n f(x) = \int_a^x \int_a^{t_1} \int_a^{t_2} \int_a^{t_3} \dots \int_a^{t_{n-1}} f(t_n)\, dt_n\, dt_{n-1} \dots dt_1,$$

*which is given by single integration*

$$J^n f(x) = D^{-n} f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt, \quad t > 0, \quad \textit{and} \quad n \in \mathbb{R}. \tag{3.28}$$

*Proof.* The proof is using induction argument. Considering the case $n = 1$, we have

$$(x-t)^{n-1} = (x-t)^{1-1} = (x-t)^0 = 1.$$

Then

$$\int_a^x f(t_1)\, dt_1 = \frac{1}{0!} \int_a^x (x-t)^0 f(t)\, dt = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt.$$

Thus, the statement holds for $n = 1$.

For $n = 2$, we have

$$\int_a^x \int_a^{t_1} f(t_2)\, dt_2\, dt_1.$$

By switching the order of integration as follows, we obtain

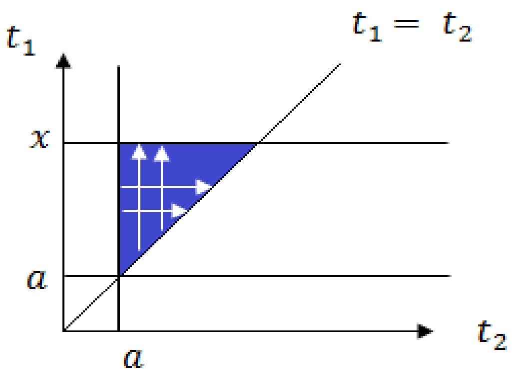


Figure 3.2: Integration of order $n = 2$.

$$\int_a^x \int_{t_2}^x f(t_2)\, dt_1\, dt_2 \;=\; \int_a^x f(t_2)\,(x-t_2)\, dt_2.$$

Since $t_2$ is a dummy variable, the above integral can be written as follows

$$\int_a^x f(t)(x-t)\, dt = \frac{1}{(2-1)!}\int_a^x (x-t)^{2-1} f(t)\; dt = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\; dt.$$

Thus, the statement also holds for $n = 2$.

Now suppose the statement holds for some arbitrary $n$. We will prove it for $n+1$ by switching the order of integration as follows:

$$\begin{aligned}
J^{n+1} f(x) &= \int_a^x \int_a^{t_1} \int_a^{t_2} \int_a^{t_3} \cdots \int_a^{t_n} f(t_{n+1})\, dt_{n+1}\, dt_n \ldots dt_1 \\
&= \frac{1}{(n-1)!}\int_a^x \int_a^{t_n} (x-t)^{n-1}\; f(t)\; dt\, dt_n \\
&= \frac{1}{(n-1)!}\int_a^x \int_t^x (x-t)^{n-1}\; f(t)\; dt_n\, dt \\
&= \frac{1}{n(n-1)!}\int_a^x (x-t)^{n}\; f(t)\; dt \\
&= \frac{1}{n!}\int_a^x (x-t)^{n}\; f(t)\; dt.
\end{aligned}$$

The result follows by induction. □

In a natural way, one is led to extend the above formula from positive integer values of the index to any positive real values by using the Gamma function. Indeed, noting that $(n-1)! = \Gamma(n)$, and introducing the arbitrary positive real number $\alpha$, one defines the fractional Integral of order $\alpha > 0$ as follows

**Definition 3.2.1. Riemann-Liouville operator**

Let $f$ be a continuous function, $0 < \alpha \leq 1$, and $t, x \in \mathbb{R}^+$. The fractional integral of order $\alpha$ is defined as:

$$J^{\alpha} f(x) = D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\; dt. \tag{3.29}$$

**Example 3.8.** *Given* $f(x) = (x-a)^{\beta-1},\ find\ the\ value\ of\ J^{\alpha} f(x).$

By definition, we have,

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}(t-a)^{\beta-1}\; dt.$$

Substituting $t = a + y(x-a)$ in the above integral, it reduces to

$$J^\alpha f(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,(x-a)^{\alpha+\beta-1}, \qquad \text{where} \qquad \beta \in \mathbb{R}^+.$$

Thus,

$$J^\alpha f(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,(x-a)^{\alpha+\beta-1}.$$

**Basic properties of Fractional Integration**

1. **The zero rule**

$$J^0\, f(t) = f(t), \tag{3.30}$$

$i.e.$, $J^0 = I$, which is called the identity operator.

2. **Linearity**

**Lemma 3.2.1.** *Let* $n-1 < \alpha \leq n,\ C,\ K \in \mathbb{R}$ *, and let f and g be functions such that their fractional derivatives and integrals exist. Then,*

$$J^\alpha(Cf(t) + Kg(t)) = CJ^\alpha f(t) + KJ^\alpha g(t). \tag{3.31}$$

*Proof.* The linearity of these operators follows from the linearity of the integer order derivatives and integrals by which they are defined. □

**Example 3.9.** *We know that*

$$\int_0^x (x-t)^{\alpha-1}dt \;=\; \frac{t^\alpha}{\alpha}, \qquad \text{when} \qquad f(t) = 1.$$

Thus the fractional integral of order $\alpha$ of $1$ is given by:

$$J^\alpha\, 1 = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}dt \;=\; \frac{t^\alpha}{\alpha\Gamma(\alpha)} \;=\; \frac{t^\alpha}{\Gamma(\alpha+1)}.$$

Hence, the $n^{th}$ fractional integral of order $\alpha$ of $1$ is then given by:

$$J^{\alpha n}\, 1 \;=\; \frac{t^{\alpha n}}{\Gamma(\alpha n+1)}.$$

# Chapter 4: Laplace Decomposition Method (LDM)

## 4.1 Method Description

The Laplace Decomposition Method is a semi-analytical method, that is a combination of two strategies, Laplace transform and decomposition method. This method was introduced by Khuri [39] and implemented by many others to solve differential equation problems of different types (see for example [40, 41, 42, 43, 44, 45, 46, 47]). To describe the method, consider the following general nonlinear equation:

$$D^{\alpha} y + f(y) = g(x), \quad where \ 1 < \alpha \leq 2, \tag{4.1}$$

supplemented with the initial conditions

$$y(0) = a, \quad y'(0) = b, \tag{4.2}$$

or the boundary conditions

$$y(0) = c, \quad y(1) = d, \tag{4.3}$$

where $a, b, c, d$ are real numbers and $\alpha$ lies between two consecutive integer numbers. The strategy consists first of applying the Laplace transform integral operator (which is denoted by $\mathcal{L}$) on both of equation (4.1)

$$\mathcal{L}[D^{\alpha} y] + \mathcal{L}[f(y)] = \mathcal{L}[g(x)]. \tag{4.4}$$

Applying lemma (2.4.2) for the Laplace transform of the fractional derivative, based on the Caputo fractional derivative, we obtain

$$\frac{s^2\mathcal{L}[y] - sy(0) - y'(0)}{s^{2-\alpha}} + \mathcal{L}[f(y)] = \mathcal{L}[g(x)]. \tag{4.5}$$

By substituting the initial conditions, given in (4.2), into (4.5), it follows that

$$\frac{s^2\mathcal{L}[y] - as - b}{s^{2-\alpha}} + \mathcal{L}[f(y)] = \mathcal{L}[g(x)]. \tag{4.6}$$

Likewise, using the first boundary condition given in (4.3), while setting $y'(0) = k$, it follows that

$$\frac{s^2\mathcal{L}[y] - cs - k}{s^{2-\alpha}} + \mathcal{L}[f(y)] = \mathcal{L}[g(x)]. \tag{4.7}$$

Eqs. (4.6) & (4.7) can be rearranged respectively as follows

$$\mathcal{L}[y] \;=\; \frac{a}{s} \;+\; \frac{b}{s^2} \;-\; \frac{1}{s^\alpha}\mathcal{L}[f(y)] \;+\; \frac{1}{s^\alpha}\mathcal{L}[g(x)], \tag{4.8}$$

and

$$\mathcal{L}[y] \;=\; \frac{c}{s} \;+\; \frac{k}{s^2} \;-\; \frac{1}{s^\alpha}\mathcal{L}[f(y)] \;+\; \frac{1}{s^\alpha}\mathcal{L}[g(x)]. \tag{4.9}$$

The decomposition iterative technique consists of seeking a solution as an infinite series of the form

$$y = \sum_{n=0}^{\infty} y_n, \tag{4.10}$$

or equivalently

$$y \;=\; y_0 + y_1 + y_2 + y_3 + y_4 + ...., \tag{4.11}$$

where the components $y_n$ will be computed recursively. More precisely, the decomposition method assumes that the nonlinear term $f(y)$ can be decomposed by an infinite series of polynomials

$$f(y) = \sum_{n=0}^{\infty} A_n, \tag{4.12}$$

where $A_n = A_n(y_0, y_1, y_2, y_3, .., y_n)$ are the so-called *Adomian polynomials* and are determined by the formula

$$A_n = \frac{1}{n!}\,\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, \qquad n = 0, 1, 2, .... \tag{4.13}$$

Without loss of generality, we will construct the iterative scheme for the nonlinear equation (4.1) subject to the BCs (4.3). Substituting Eqs. (4.10) and (4.12) into equation (4.9) results

$$\mathcal{L}\left[\sum_{n=0}^{\infty} y_n\right] = \frac{c}{s} + \frac{k}{s^2} - \frac{1}{s^\alpha}\,\mathcal{L}\left[\sum_{n=0}^{\infty} A_n\right] + \frac{1}{s^\alpha}\mathcal{L}[g(x)]. \tag{4.14}$$

Using the linearity of Laplace transform, we obtain

$$\sum_{n=0}^{\infty}\mathcal{L}\left[y_n\right] = \frac{c}{s} + \frac{k}{s^2} - \frac{1}{s^\alpha}\sum_{n=0}^{\infty}\mathcal{L}\left[A_n\right] + \frac{1}{s^\alpha}\mathcal{L}[g(x)]. \tag{4.15}$$

Therefore, the formal recurrence scheme is defined by

$$\begin{aligned} \mathcal{L}[y_0] &= \frac{k}{s^2} + \frac{1}{s^\alpha}\mathcal{L}[g(x)], \\ \mathcal{L}[y_1] &= -\frac{1}{s^\alpha}\,\mathcal{L}\left[A_0\right], \\ &\;\vdots \\ \mathcal{L}[y_{n+1}] &= -\frac{1}{s^\alpha}\,\mathcal{L}\left[A_n\right]. \end{aligned} \tag{4.16}$$

In order to find the first iterate $y_0$, the inverse Laplace transform was applied to the first term in the algorithm (4.16):

$$\begin{aligned} y_0 &= \mathcal{L}^{-1}\left[\frac{k}{s^2} + \frac{1}{s^\alpha}\mathcal{L}[g(x)]\right], \\ y_1 &= \mathcal{L}^{-1}\left[-\frac{1}{s^\alpha}\mathcal{L}\left[A_0\right]\right], \\ &\vdots \end{aligned} \tag{4.17}$$

The higher iterates are found iteratively in a similar fashion. However, the approximate solution has yet to satisfy the second boundary condition in (4.3), which was not used to obtain the approximate solution. Applying this condition, namely setting $y(0) = d$, and by solving the approximate results for the unknown constant $k$, we eventually obtain the numerical solution.

Now, consider the nonlinear function $f(y)$. Then, the infinite series produced by applying the Taylor's series expansion of $f$ about the initial function $y_0$ is given by

$$f(y) = f(y_0) + f'(y_0)(y - y_0) + \frac{1}{2!}f''(y_0)(y - y_0)^2 + \frac{1}{3!}f'''(y_0)(y - y_0)^3 + ... \tag{4.18}$$

By substituting equation (4.11) into equation (4.18), we obtain

$$\begin{aligned} f(y) &= f(y_0) + f'(y_0)(y_1 + y_2 + ...) + \tfrac{1}{2!}f''(y_0)(y_1 + y_2 + ...)^2 \\ &+ \tfrac{1}{3!}f'''(y_0)(y_1 + y_2 + ...)^3 + ... \end{aligned} \tag{4.19}$$

Now, we expand equation (4.19). In order to obtain Adomian polynomials, we need first to reorder and rearrange the terms in such a way so that the sum of the subscripts of $y_i$ terms add up to same numbers. Indeed, it is necessary to determine the order of each term in (4.19), which in fact depends on both the subscripts and the $y_n's$ exponents. For example, $y_1$ is of order $1$; $y_2$ is of order $2$; ${y_1}^2$ is of order $2$; ${y_2}^2$ is of order $4$; ${y_1}^3$ is of order $3$; ${y_2}^3$ is of order $6$; and so on. In general, it could be defined as ${y_n}^k$ is of order $kn$. In case a particular term includes the multiplication of $y_n's$ components, then its order is determined by computing the sum of the terms of the $y_n's$ in each component. For instance, ${y_1}^5{y_2}^3$ is of order $11$ because $(1)(5) + (2)(3) = 11$.

As an outcome, of reordering the terms in the expansion (4.19) based on their order, we have

$$
\begin{aligned}
f(y) &= f(y_0) + f'(y_0)y_1 + f'(y_0)y_2 + \tfrac{1}{2!}f''(y_0)y_1^2 + f'(y_0)y_3 \\
&+ \tfrac{2}{2!}f''(y_0)y_1y_2 + \tfrac{1}{3!}f'''(y_0){y_1}^3 + f'(y_0)y_4 + \tfrac{1}{2!}f''(y_0)y_2^2 \\
&+ \tfrac{2}{2!}f''(y_0)y_1y_3 + \tfrac{3}{3!}f'''(y_0)y_1^2y_2 + \tfrac{1}{4!}f''''(y_0)y_1^4 + f'(y_0)y_4 \\
&+ \tfrac{2}{2!}f''(y_0)y_2y_3 + \tfrac{2}{2!}f''(y_0)y_1y_4 + \tfrac{3}{3!}f'''(y_0)y_1y_2^2 \\
&+ \tfrac{3}{3!}f'''(y_0)y_1^2y_3 + \tfrac{4}{4!}f''''(y_0)y_1^3y_2 + \tfrac{1}{5!}f'(y_0)y_1^5 + ...
\end{aligned} \tag{4.20}
$$

The Adomain polynomial are constructed in a certain manner, so that $A_1$ involves all terms of order 1 in the expansion (4.20); $A_2$ involves all terms of order 2 in the expansion (4.20); $A_3$ involves all terms of order 3 in the expansion (4.20), and so on. Generally, $A_n$ involves all terms of order $n$ in the expansion (4.20). Subsequently, the first eight terms of Adomian polynomials are listed below:

$A_0 = f(y_0),$

$A_1 = y_1\, f^{(1)}(y_0),$

$A_2 = y_2\, f^{(1)}(y_0) + \frac{1}{2!}y_1^2\, f^{(2)}(y_0),$

$A_3 = y_3\, f^{(1)}(y_0) + y_1y_2\, f^{(2)}(y_0) + \frac{1}{3!}y_1^3\, f^{(3)}(y_0),$

$A_4 = y_4\, f^{(1)}(y_0) + \frac{1}{2!}y_2^2\, f^{(2)}(y_0) + y_1y_3\, f^{(2)}(y_0) + \frac{1}{2!}y_1^2y_2\, f^{(3)}(y_0) + \frac{1}{4!}y_1^4\, f^{(4)}(y_0),$

$A_5 = y_5\, f^{(1)}(y_0) + y_2y_3\, f^{(2)}(y_0) + y_1y_4\, f^{(2)}(y_0) + \frac{1}{2!}y_1y_2^2\, f^{(3)}(y_0)$
$+ \frac{1}{2!}y_1^2y_3\, f^{(3)}(y_0) + \frac{1}{3!}y_1^3y_2\, f^{(4)}(y_0) + \frac{1}{5!}y_1^5\, f^{(5)}(y_0),$

$A_6 = y_6\, f^{(1)}(y_0) + y_1y_5\, f^{(2)}(y_0) + y_2y_4\, f^{(2)}(y_0) + \frac{1}{3!}y_1^3y_3\, f^{(4)}(y_0) + \frac{1}{4!}y_1^4y_2\, f^{(5)}(y_0)$
$+ \frac{1}{2!}y_1^2y_4\, f^{(3)}(y_0) + \frac{1}{3!}y_2^3\, f^{(3)}(y_0) + \frac{1}{6!}y_1^6\, f^{(6)}(y_0),$

$A_7 = y_7\, f^{(1)}(y_0) + y_3y_4\, f^{(2)}(y_0) + y_2y_5\, f^{(2)}(y_0) + y_1y_6\, f^{(2)}(y_0) + \frac{1}{2}y_2^2y_3\, f^{(3)}(y_0) + \frac{1}{2}y_1y_3^2\, f^{(3)}(y_0)$
$+ \frac{1}{2}y_1^2y_2y_3\, f^{(4)}(y_0) + \frac{1}{6}y_4y_1^3\, f^{(4)}(y_0) + y_1y_2y_4\, f^{(3)}(y_0) + \frac{1}{2}y_1^2y_5\, f^{(3)}(y_0) + \frac{1}{6}y_2^3y_1\, f^{(4)}(y_0)$
$+ \frac{1}{12}y_1^3y_2^2\, f^{(5)}(y_0) + \frac{1}{24}y_1^4y_3\, f^{(5)}(y_0) + \frac{1}{6!}y_1^5y_2\, f^{(6)}(y_0) + \frac{1}{7!}y_1^7\, f^{(7)}(y_0).$

In the subsequent sections, the Laplace decomposition method will be used to approximate the solution of several interesting linear and nonlinear well-known problems with integer and fractional order derivatives.

## 4.2 Patching Algorithm

In this section, we present a patching approach which is based on a domain decomposition method (DDM) scheme. The proposed strategy aims to minimize and/or overcome the setback resulting from the application of the Laplace decomposition method. Usually, the LDM results in a numerical series solution that is highly accurate locally, in our case, in the neighborhood of the left endpoint of the interval. The only disadvantage of this approach is that the error deteriorates when we move away from the left endpoint towards the right endpoint of the interval. This drawback can be prevented by splitting the interval into a union of smaller subintervals and then applying the LDM to each of them [48].

The idea of this domain decomposition is to partition the large universal computational domain $[0, 1]$ into two or more non-overlapping sub-domains. In this study, we split the domain $[0, 1]$ into two non-overlapping sub-domains $[0, k] \cup [k, 1]$. The LDM is applied on the first sub-domain $[0, k]$ while the decomposition strategy is performed on the outer sub-domain $[k, 1]$, making use of the LDM solution to assess the initial conditions needed at $x = k$, namely $y(k)$ and/or $y'(k)$. More precisely, the LDM gives a series solution defined on $[0, 1]$, that is anticipated to converge to the exact solution. However, this approximate solution is used to estimate the value(s) of $y(k)$ and/or $y'(k)$ which will be used as representing the new initial condition(s) on the interval $[k, 1]$. With these initial conditions, the decomposition method is applied in order to obtain an approximate solution on $[k, 1]$.

The major aim is to control the accuracy of the numerical solution that is acquired by Adomian decomposition method. Therefore, the break point $k$ is selected in a way to guarantee that the computational value of the boundary/or initial condition(s) at $x = k$ stabilizes. More particularly, the turning point $x = k$ is to be chosen in a way to satisfy the condition:

$$|y_n(k) - y_{n-1}(k)| < Tolerance, \tag{4.21}$$

where $y_n$ is the $n^{th}$ iterate obtained by LDM.

The solution to problem (4.1)-(4.2) on $[k, 1]$ is constructed as follows. Without loss of generality we will show the solution when $\alpha = 2$. Operating with the integral operator $L^{-1}$ defined by

$$L^{-1}[.] = \int_k^x \int_k^x [.] \, dz \, dz \tag{4.22}$$

to both side of equation (4.1) we obtain

$$y(x) - y(k) - y'(k)(x - k) \;=\; -L^{-1}[f(y)] + L^{-1}[g(x)]. \tag{4.23}$$

By expanding the nonlinear term $f(y)$ in terms of Adomian polynomial we have:

$$y(x) = \sum_{n=0}^{\infty} y_n(s) = y(k) + y'(k)(x - k) + L^{-1}[g(x)] - L^{-1}\left[\sum_{n=0}^{\infty} A_n\right]. \tag{4.24}$$

This produces the following iterative scheme:

$$\begin{aligned} y_0 &= y(k) + y'(k)(x - k) + L^{-1}[g(x)], \\ y_1 &= -L^{-1}[A_0], \\ \dots & \\ y_n &= -L^{-1}[A_{n-1}]. \end{aligned} \tag{4.25}$$

The solution for problem (4.1)-(4.3) is constructed in a similar fashion.

## 4.3 Numerical Results

In this section, we apply the Laplace decomposition method to well-known differential equation problems and obtain the numerical solution in order to illustrate the efficiency and applicability of the iterative LDM. Moreover, we implement the proposed method for the fractional-order nonlinear /and linear initial-value problems (IVPs) and boundary-value problems (BVPs). The solution for the integer-order will be also included.

### 4.3.1 Bratu's Problem

The classical and fractional Bratu's problem have gained the attention of researchers due to its simplicity and its appearance in variety physical and engineering models. Several methods for approximating the solution of Bratu's problem are realized. For instance, Reproducing Kernel Hilbert space Method (RKM) [49, 50], Bezier curve method (BCM) [51], homotopy perturbation method [52] and many others. Here, we use the LDM to approximate the solution of the fractional-order nonlinear initial-value problems (IVPs) and boundary-value problems (BVPs) for Bratu equation.

**Example 4.1.** *Consider the BVP:*

$$-D^{\alpha} y = \lambda e^{y}, \quad \textit{for} \quad 1 < \alpha \leq 2, \tag{4.26}$$

*and subject to*

$$y(0) = 0, \quad y(1) = 0. \tag{4.27}$$

Applying the Laplace transform integral operator to both sides of equation (4.26) and using the formula of Laplace transform of Caputo fractional derivative gives

$$\frac{s^2\mathcal{L}[y] - sy(0) - y'(0)}{s^{2-\alpha}} \;=\; -\lambda\,\mathcal{L}[e^y]. \tag{4.28}$$

Using the first boundary condition in (4.27), while setting $y'(0) = k$, implies

$$\mathcal{L}[y] \;=\; \frac{k}{s^2} - \frac{\lambda}{s^\alpha}\,\mathcal{L}[e^y]. \tag{4.29}$$

It is worth nothing that the solution has yet to satisfy the second boundary condition in (4.27), namely $y(1) = 0$. Using Adomian polynomial representation for the nonlinear term $e^y$, gives

$$\sum_{n=0}^{\infty}\mathcal{L}\left[y_n\right] = \frac{k}{s^2} - \frac{\lambda}{s^\alpha}\sum_{n=0}^{\infty}\mathcal{L}\left[A_n\right]. \tag{4.30}$$

This, in turn, gives the first few polynomials as follows:

$$\begin{aligned}
\mathcal{L}[y_0] &= \frac{k}{s^2},\\
\mathcal{L}[y_1] &= -\frac{\lambda}{s^\alpha}\,\mathcal{L}\left[e^{y0}\right],\\
\mathcal{L}[y_2] &= -\frac{\lambda}{s^\alpha}\,\mathcal{L}\left[y_1\,e^{y0}\right],\\
\mathcal{L}[y_3] &= -\frac{\lambda}{s^\alpha}\,\mathcal{L}\left[\left(y_2 + \frac{1}{2}y_1^2\right)\,e^{y0}\right],\\
\mathcal{L}[y_4] &= -\frac{\lambda}{s^\alpha}\,\mathcal{L}\left[\left(y_3 + y_1y_2 + \frac{1}{3!}y_1^3\right)\,e^{y0}\right],\\
&\cdots
\end{aligned} \tag{4.31}$$

Next, we apply the Laplace inverse to both sides of equation (4.31) in order to obtain $y_0, y_1, y_2, \cdots$

This problem has been solved, with $n = 10$ and $\lambda = 1,\ \alpha = 1.5$, to obtain the approximate value of the solution $y_{approx}(x) = \sum_{n=0}^{10} y_n$. The initial iterate is set first to be $y_0 = kx$, then the second condition in (4.27), which is $y(1) = 0$, is used to estimate the value of $k$ using $y_{approx}$. This yields

$$y_0 = 0.78420149988749964837x.$$

Table 4.1: Approximate solution and residual error for Example 4.1 obtained by LDM with $\lambda = 1$ and $\alpha = 1.5$.

| *X* | *LDM* | *LDM Residual error* |
|---|---|---|
| 0.1 | 0.054045402441250074909948640076 | 1.342504612969271373019592397729(−12) |
| 0.2 | 0.086627434257815044103627691913 | 2.918390298286447479829168873545(−9) |
| 0.3 | 0.104448147599214043657045972420 | 2.804210905569436814239292113004(−7) |
| 0.4 | 0.110158588199913254607022754743 | 6.241959162779448972411585474371(−6) |
| 0.5 | 0.105562671299589656476907597702 | 6.668787164915876596239057649025(−5) |
| 0.6 | 0.092113360128763033088628181148 | 4.544539728380399488829563127091(−4) |
| 0.7 | 0.071077576093815492486804716186 | 2.282507129632693750574386788130(−3) |
| 0.8 | 0.043638793285588149720232670651 | 9.190490983189129666434538770463(−3) |
| 0.9 | 0.011096054918988845652523532176 | 3.129403202804738538774775806976(−2) |
| 1.0 | 0.024488479799988564520678527558 | 9.342081180929936198172430954584(−2) |

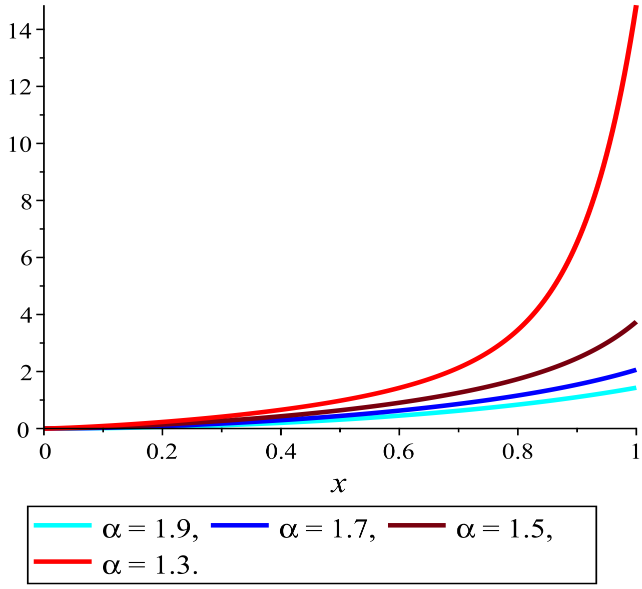


Figure 4.1: The numerical solution of Example 4.1 for different values of $\alpha$ ranging between 1 and 2.

Table 4.1 presents the numerical solution along with the residual error. It can be seen that the LDM is highly accurate close to $x = 0$, but the error deteriorates as we approach $x = 1$. Figure 4.1 depicts the numerical solution for different values of $\alpha$ ranging between 1 and 2. As noticed from the graph that as the value of $\alpha$ increases to 2, the corresponding approximate solution of fractional order starts decaying, while for $\alpha = 1$, the graph attains the greatest approximate solution.

**Example 4.2.** *Consider the IVP:*

$$D^{\alpha} y \;-\; e^{2y} \;=\; 0, \quad for \quad 1 < \alpha \leq 2, \tag{4.32}$$

$$y(0) = 0, \quad y'(0) = 0. \tag{4.33}$$

By applying the formulas of fractional derivative, operating with Laplace transform to both sides of equation (4.32), and substituting the initial conditions given in (4.33), we obtain

$$\mathcal{L}[y] \;=\; -\frac{\lambda}{s^{\alpha}}\,\mathcal{L}[e^{2y}]. \tag{4.34}$$

Using Adomian polynomial representation to decompose the nonlinear term, we have the first few polynomials are listed below

$$\begin{aligned}
\mathcal{L}[y_0] &= 0, \\
\mathcal{L}[y_1] &= -\frac{\lambda}{s^{\alpha}}\mathcal{L}\left[e^{2y0}\right], \\
\mathcal{L}[y_2] &= -\frac{\lambda}{s^{\alpha}}\,\mathcal{L}\left[2y_1 e^{2y0}\right], \\
\mathcal{L}[y_3] &= -\frac{\lambda}{s^{\alpha}}\,\mathcal{L}\left[\left(2y_2 + 2y_1^3\right) e^{2y0}\right], \\
\mathcal{L}[y_4] &= -\frac{\lambda}{s^{\alpha}}\,\mathcal{L}\left[\left(2y_3 + 4y_1 y_2 + \frac{4}{3}y_1^3\right)\; e^{2y0}\right], \\
&\ldots.
\end{aligned} \tag{4.35}$$

In this problem, we set $n = 12$ and $\lambda = -1,\ \alpha = 1.9$ to obtain $y_{approx}(x)$. The first few iterates are found to be:

$$\begin{aligned}
y_0 &= 0, \\
y_1 &= 0.54723901807770337612\,x^{\frac{19}{10}}, \\
y_2 &= 0.11212106036245137092\,x^{\frac{19}{5}}, \\
y_3 &= 0.03551901653571660363\,8\,x^{\frac{57}{10}}.
\end{aligned} \tag{4.36}$$

The numerical results for LDM along with a comparison with the (RKM) [49] and Bezier Curve Method (BCM) [51] are reported in Table 4.2 for various values of $x$ ranging between $0$ and $1$. From the obtained results, we can observe that the LDM provides better results than the approximate solution obtained by BCM and RKM. In addition, the residual error resulting from the LDM is included. However, the numerical experiment demonstrates that the LDM is highly accurate locally but worsens as we move close to $x = 1$. This problem could be resolved by applying the DDM strategy that is outlined in section 4.2 to overcome this deficiency.

Table 4.2: Comparison between the approximate solution for Example 4.2 obtained by LDM and that of BCM and RKM with $\lambda = -1$ and $\alpha = 1.9$.

| $X$ | *LDM* | *LDM Residual error* | *RKM* | *BCM* |
|---|---|---|---|---|
| 0.1 | 0.00690717224498257596 | $7.11180(-14)$ | $6.5411(-3)$ | 0.006541208907 |
| 0.2 | 0.02596315507415622557 | $1.92733160600(-10)$ | $2.5712(-2)$ | 0.02335095133 |
| 0.3 | 0.05674700322267323603 | $1.969676647640(-8)$ | $5.6625(-2)$ | 0.04844045146 |
| 0.4 | 0.09961224904499771618 | $5.266897218715(-7)$ | $9.9615(-2)$ | 0.08222901871 |
| 0.5 | 0.15543544423034500263 | $6.766283602000(-6)$ | $1.5563(-1)$ | 0.1268340698 |
| 0.6 | 0.22566536573782200661 | $5.482019082231(-5)$ | $2.2552(-1)$ | 0.1853611509 |
| 0.7 | 0.31250355464795531555 | $3.249249666093(-4)$ | $3.1162(-1)$ | 0.2611939599 |
| 0.8 | 0.41925490500596343556 | $1.552559823337(-3)$ | $4.1889(-1)$ | 0.3572843682 |
| 0.9 | 0.55098410042474816513 | $6.509670720713(-3)$ | $5.5255(-1)$ | 0.4754424432 |
| 1.0 | 0.71580807838463783364 | $2.666111772373(-2)$ | $7.1494(-1)$ | 0.6156264703 |

In the following example, we consider two cases for an IVP of Bratu type: The first case is direct implementation of LDM, while in the second case the patching approach will be performed to overcome the deterioration of the error as we move away from the origin.

**Example 4.3.** *Consider the following IVP:*

$$D^{\alpha}y - 2e^{y} = 0, \quad for \quad 1 < \alpha \leq 2, \tag{4.37}$$

*subject to the initial conditions*

$$y(0) = 0, \quad y'(0) = 0. \tag{4.38}$$

**Case 1.**

The iterative LDM has been applied successfully to this latter problem with $n = 12$ and $\lambda = -2,\ \alpha = 1.9$ in order to obtain $y_{approx}(x)$. The first few iterates that were computed numerically are given by

$$
\begin{aligned}
y_0 &= 0, \\
y_1 &= 1.094478036\, x^{\frac{19}{10}}, \\
y_2 &= 0.22424212072490274185\, x^{\frac{19}{5}}, \\
y_3 &= 0.07103803307143320728 4\, x^{\frac{57}{10}}.
\end{aligned}
\tag{4.39}
$$

The approximate solution acquired by the LDM is compared with that resulting from the RKM and BCM strategies. Clearly, the outcome of our approach is better and more precise. The comparisons between the three techniques are reported in Table 4.3. The table also contains the residual error generated by the LDM.

Table 4.3: Comparison between the approximate solution for Example 4.3 case 1 obtained by LDM and that of BCM and RKM with $\lambda = -2$ and $\alpha = 1.9$.

| *X* | *LDM* | *LDM Residual error* | *RKM* | *BCM* |
|---|---|---|---|---|
| 0.1 | 0.0138143444899651521 | 1.399905(−13) | 1.3082(−2) | 0.01031561203 |
| 0.2 | 0.0519263101483149530 | 3.783342520(−10) | 5.1424(−2) | 0.04026954618 |
| 0.3 | 0.1134940064459998643 | 3.85011037823(−8) | 1.1325(−1) | 0.09121393595 |
| 0.4 | 0.1992244981262296286 | 1.02388590844(−6) | 1.9923(−1) | 0.1643473163 |
| 0.5 | 0.3108708893132952162 | 1.30683094490(−5) | 3.1127(−1) | 0.2611684808 |
| 0.6 | 0.4513307430618472894 | 1.05102884069(−4) | 4.5103(−1) | 0.3839303388 |
| 0.7 | 0.6250072167932513407 | 6.17946383914(−4) | 6.2324(−1) | 0.5360937724 |
| 0.8 | 0.8385105659972154615 | 2.927506061653(−3) | 8.3787(−1) | 0.7228144938 |
| 0.9 | 1.1019725383523112885 | 1.217824756636(−2) | 1.1051 | 0.9512319022 |
| 1.0 | 1.4316376658611168728 | 4.973428828397(−2) | 1.4299 | 1.231252941 |

As noticed from the three tables, the main disadvantage of the LDM is that the error worsens as we move away from the origin towards $x = 1$. We apply the patching approach based on domain decomposition method (DDM) to resolve this shortcoming as detailed in Section 4.2.

**Case 2.**

First, the LDM was applied to the IVP (4.37) subject to the initial conditions given in(4.38) for $\lambda = -2$ and $\alpha = 2$ on $[0, 1]$ using $n = 12$. The results are recorded in the first column of Table 4. The exact solution for this IVP is $y_{exact}(x) = -2\ln(\cos x)$.

Using the domain decomposition method (DDM), we divide the domain $[0, 1]$ into two non-overlapping sub-domains $[0, 0.5]$ and $[0.5, 1]$. Second, the LDM was applied on the subinterval $[0, 0.5]$ using $n = 12$. Then, the approximate solution $y_{approx} = \sum_{n=1}^{12} y_n$ was utilized to estimate the values of initial conditions $y(0.5)$ and $y'(0.5)$, which are found to be

$$\begin{aligned} y(0.5) &= 0.26116847554656590 49, \\ y'(0.5) &= 1.09260483001069969 21. \end{aligned} \tag{4.40}$$

The DD iterative scheme given in (4.25) is applied to (4.37) subject to the initial conditions (4.38). This leads to the absolute error given in the second column of Table 4.4. It is clear that the error has been improved as we move to the right end of the interval. Figure 4.2 shows the numerical solution obtained by LDM for $\alpha = 2$.

Table 4.4: Comparison between the absolute error resulting from the LDM before and after applying the patching approach for Example 4.3 case 2 with $\lambda = -2$ and $\alpha = 2$.

| $X$ | *Absolute Error using LDM* | *Absolute Error using Patching Strategy* |
|---|---|---|
| 0.1 | $8.62(-19)$ | $8.62(-19)$ |
| 0.2 | $1.428654800(-14)$ | $1.428654800(-14)$ |
| 0.3 | $4.173053689(-12)$ | $4.173053689(-12)$ |
| 0.4 | $2.3445211341(-10)$ | $2.3445211341(-10)$ |
| 0.5 | $5.3408765676(-9)$ | $5.3408791336(-9)$ |
| 0.6 | $6.8801690916(-8)$ | $2.0449656739(-8)$ |
| 0.7 | $5.9900039160(-7)$ | $3.6173340501(-8)$ |
| 0.8 | $3.9262828323(-6)$ | $5.3245979901(-8)$ |
| 0.9 | $2.0845224959(-5)$ | $8.0552071270(-8)$ |
| 1.0 | $9.5015072808(-5)$ | $4.2287454192(-7)$ |

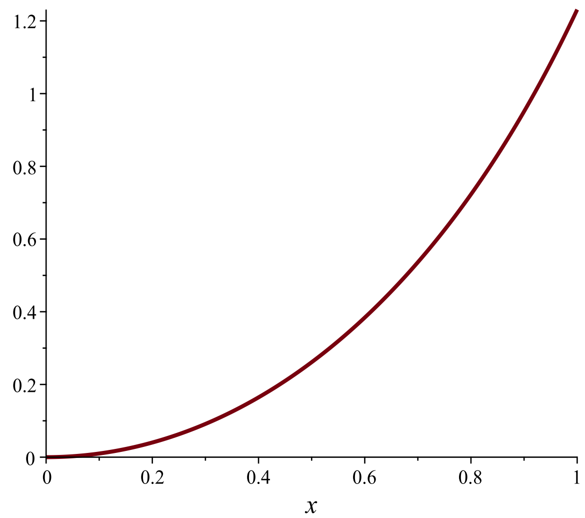


Figure 4.2: The numerical solution of Example 4.3 case 2 for $\alpha = 2$.

### 4.3.2 Lienard's Equation

The Lienard equation is a second order differential equation, named after the scientist French physicist Alfred-Marie Lienard. Various methods are utilized to obtain the solution of Lienard's equation, such as differential transform method (DTM) [53], hybrid heuristic computation [54], residual power series method [55], variational iteration method (VIM) [56] and many others. Here, we consider the generalized form of Lienard's equation, given by:

$$D^{\alpha} y + \lambda y + \mu y^3 + \nu y^5 = 0, \qquad for \quad 1 < \alpha \leq 2, \tag{4.41}$$

to obtain the approximate solution for the standard Lienard's order as well as the fractional order, for a specific parameters chosen in the model equation, via the iterative LDM.

**Example 4.4.**

<u>**Case 1**</u>**.** In this case, we consider the standard Lienard's equation subject to the initial conditions

$$y(0) = c_1 = \sqrt{\frac{-2\lambda}{\mu}}, \quad \text{and} \quad y'(0) = c_2 = -\frac{\lambda\sqrt{-\lambda}}{\mu\sqrt{\frac{-2\lambda}{\mu}}}. \tag{4.42}$$

The exact solution of the Lienard's equation of order $\alpha = 2$, is given by:

$$y(x) = \sqrt{\frac{-2\lambda}{\mu}(1 + \tanh(\sqrt{-\lambda}x))}. \tag{4.43}$$

Applying the Laplace transform integral operator with the usage of Caputo fractional derivative and substituting the initial conditions given in (4.41), we get

$$\mathcal{L}[y] = \frac{\sqrt{\frac{-2\lambda}{\mu}}}{s(1+\frac{\lambda}{s^\alpha})} + \frac{-\frac{\lambda\sqrt{-\lambda}}{\mu\sqrt{\frac{-2\lambda}{\mu}}}}{s^2(1+\frac{\lambda}{s^\alpha})} - \frac{\mu\,\mathcal{L}[y^3]}{s^\alpha(1+\frac{\lambda}{s^\alpha})} - \frac{\nu\,\mathcal{L}[y^5]}{s^\alpha(1+\frac{\lambda}{s^\alpha})}. \tag{4.44}$$

Employing Adomian polynomial representation for the nonlinear terms $y^3$ and $y^5$ with the usage of the linearity property of Laplace transform we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{L}\left[y_n\right] &= \frac{\sqrt{\frac{-2\lambda}{\mu}}}{s(1+\frac{\lambda}{s^\alpha})} + \frac{-\frac{\lambda\sqrt{-\lambda}}{\mu\sqrt{\frac{-2\lambda}{\mu}}}}{s^2(1+\frac{\lambda}{s^\alpha})} - \frac{\mu}{s^\alpha(1+\frac{\lambda}{s^\alpha})}\sum_{n=0}^{\infty}\mathcal{L}\left[A_n\right] \\ &- \frac{\nu}{s^\alpha(1+\frac{\lambda}{s^\alpha})}\sum_{n=0}^{\infty}\mathcal{L}\left[B_n\right].\end{aligned} \tag{4.45}$$

The first few iterates of the iterative algorithm are given by:

$$\begin{aligned}\mathcal{L}[y_0] &= \frac{\sqrt{\frac{-2\lambda}{\mu}}}{s(1+\frac{\lambda}{s^\alpha})} + \frac{-\frac{\lambda\sqrt{-\lambda}}{\mu\sqrt{\frac{-2\lambda}{\mu}}}}{s^2(1+\frac{\lambda}{s^\alpha})}, \\ \mathcal{L}[y_1] &= \frac{\mu}{s^\alpha(1+\frac{\lambda}{s^\alpha})}\mathcal{L}\left[y_0^3\right] + \frac{\mu}{s^\alpha(1+\frac{\lambda}{s^\alpha})}\mathcal{L}\left[y_0^5\right], \\ \mathcal{L}[y_1] &= \frac{\mu}{s^\alpha(1+\frac{\lambda}{s^\alpha})}\mathcal{L}\left[3y_0^2 y_1\right] + \frac{\mu}{s^\alpha(1+\frac{\lambda}{s^\alpha})}\mathcal{L}\left[5y_0^5 y_1\right], \\ &\cdots\end{aligned} \tag{4.46}$$

In order to find the first iterate $y_0$ the inverse Laplace transform is applied to the first term in the algorithm (4.46). The higher iterates are found iteratively in similar fashion iteratively. The LDM has been applied to the standard Lienard's equation with $n = 10,\ \lambda = -1,\ \nu = -3,\ \mu = 4,$ and $\alpha = 2$ to obtain $y_{approx}(x)$. The first few terms are found to be:

$$\begin{aligned} y_0 &= 0.70710678118654752440\cosh(x) + 0.35355339059327376220\sinh(x), \\ y_1 &= -0.050150029280930677685\cosh(3x) - 0.046567884332292772147\sinh(3x) \\ &+ 0.0052653214907689696004\cosh(5x) + 0.0052221631178938141152\sinh(5x) \\ &- 2.6041666666666666667\times10^{-22}\cosh(x)(5.8468891969362648419\times10^{20}x \\ &- 1.7235727791422095904\times10^{20}) - 4.3402777777777777778\times10^{-22}\sinh(x) \\ &\quad (7.0162670363235178110\times10^{20}x - 6.1253124920284679299\times10^{20}).\end{aligned} \tag{4.47}$$

In the following table, we consider the comparison between the numerical solution using LDM with the fractional homotopy analysis transform method (FHATM) solution [57].

Table 4.5: Comparison between the LDM and the FHATM for Example 4.4 case 1 with $\lambda = -1, \mu = 4, \nu = -3$, and $\alpha = 2$.

| *X* | *Exact* | *LDM* | *Err LDM* | *FHATM* | *Err FHATM* |
|---|---|---|---|---|---|
| 0.00 | 0.7071067811 | 0.70710678118654752436 | 1.0(-20) | 0.7071067810 | 0 |
| 0.02 | 0.7141419094 | 0.71414190948295806186 | 3.0(-20) | 0.7141419094 | 1.8669(-6) |
| 0.04 | 0.7211028637 | 0.72110286378267961029 | 9.0(-20) | 0.7211028634 | 6.2706(-6) |
| 0.06 | 0.7279862991 | 0.72798629915993044796 | 5.0(-20) | 0.7279862988 | 4.94502(-5) |
| 0.08 | 0.7347890068 | 0.73478900682819532816 | 3.5(-19) | 0.7347890065 | 1.161249(-4) |
| 0.10 | 0.7415079212 | 0.74150792127426251882 | 4.90(-18) | 0.7415079207 | 2.24737(-4) |

As seen in Table 4.5, the numerical results achieved by the LDM yield better numerical values that converge relatively faster than the approximate solution obtained by FHATM.

The approximate solutions using the LDM [58] with a comparison with the Hybrid Genetic Algorithm [54] are given in Table 4.6. Obviously, the absolute errors confirm that our strategy gives better results and is more accurate. Figure 4.3 depicts the numerical solution obtained by LDM for $\alpha = 2$.

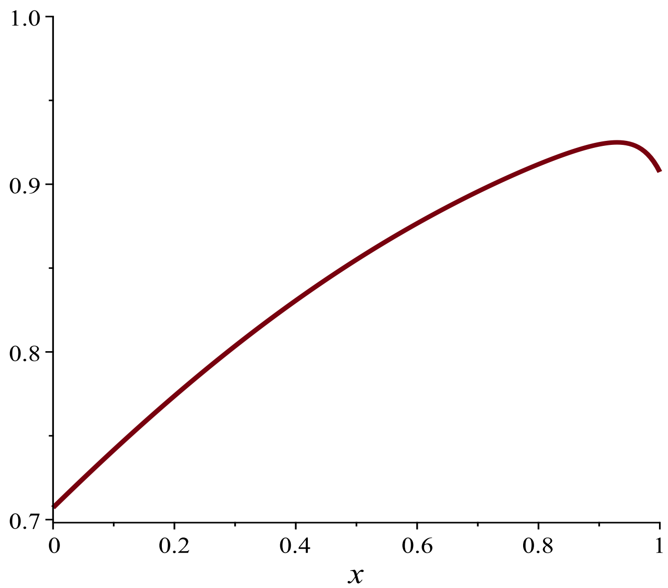


Figure 4.3: The numerical solution of Example 4.4 case 1 for $\alpha = 2$.

Table 4.6: Comparison between the proposed LDM and that of Hybrid Genetic Algorithm (HGA) for Example 4.4 case 1 with $\lambda = -1, \mu = 4, \nu = -3$, and $\alpha = 2$.

| $x$ | *Exact* | *LDM* | *Err LDM* | *IPA* | *Err IPA* |
|---|---|---|---|---|---|
| 0.1 | 0.7415079212 | 0.7415079212 | $4.936043(-18)$ | 0.74150168 | $6.24(-6)$ |
| 0.2 | 0.7737490937 | 0.7737490937 | $8.270675(-14)$ | 0.77374349 | $5.60(-6)$ |
| 0.3 | 0.8035274147 | 0.8035274147 | $2.392399(-11)$ | 0.80352226 | $5.15(-6)$ |
| 0.4 | 0.8306470255 | 0.8306470242 | $1.334015(-09)$ | 0.83064195 | $5.08(-6)$ |
| 0.5 | 0.8550196364 | 0.8550196011 | $3.525332(-08)$ | 0.85501464 | $5.00(-6)$ |
| 0.6 | 0.8766554531 | 0.8766546785 | $7.745691(-07)$ | 0.87665085 | $4.60(-6)$ |
| 0.7 | 0.8956471897 | 0.8956317811 | $1.540863(-05)$ | 0.89564323 | $3.96(-6)$ |
| 0.8 | 0.9121504180 | 0.9119205555 | $2.298624(-04)$ | 0.91214703 | $3.39(-6)$ |
| 0.9 | 0.9263632846 | 0.9238055088 | $2.557775(-03)$ | 0.92636013 | $3.15(-6)$ |
| 1.0 | 0.9385078997 | 0.9073294623 | $3.117843(-02)$ | 0.93850472 | $3.18(-6)$ |

Table 4.6 shows that the absolute error resulting from our technique deteriorates as we move toward $x = 1$. However, we can apply the patching algorithm outlined in Section $4.2$ to overcome this deficiency. First, the LDM was applied to the latter problem (4.41) using the initial conditions (4.42) with $\lambda = -1, \mu = 4, \nu = -3$, and $\alpha = 2$ on $[0, 1]$ using $n = 10$. By using the domain decomposition method (DDM) technique, we split the domain $[0, 1]$ into two non-overlapping sub-domains $[0, 0.5]$ and $[0.5, 1]$. Second, the LDM was applied on the subinterval [0,0.5] using $n = 10$; the approximate solution $y_{approx} = \sum_{n=1}^{10} y_n$ was used to estimate the values of $y(0.5)$ and $y'(0.5)$ which are used to update the initial conditions on the subinterval $[0.5, 1]$ and are found to be

$$\begin{aligned} y(0.5) &= 0.6045901580184063737371489105 84\sqrt{2}, \\ y'(0.5) &= 0.16259856577285318857016749051 4\sqrt{2}. \end{aligned} \tag{4.48}$$

The DDM iterative method is finally applied successfully to the Lienard's equation (4.41) subject to the updated initial conditions given in (4.48). Table 4.7 presents the comparison between the absolute error before and after applying the patching strategy [58]. It is clear the error has been improvement as we move to the right end of the interval. By subdividing the interval to more than two sub-intervals, further improvement can be achieved.

Table 4.7: Comparison between the absolute errors resulting from the LDM before and after applying the patching approach for Example 4.4 case 1 with $\lambda = -1, \mu = 4, \nu = -3$, and $\alpha = 2$.

| $X$ | *Absolute Error using LDM* | *Absolute Error using Patching Strategy* |
|---|---|---|
| 0.0 | $4.0(-20)$ | $4.0(-20)$ |
| 0.1 | $4.936043004823(-18)$ | $4.936043004823(-18)$ |
| 0.2 | $8.270675938781(-14)$ | $8.270675938781(-14)$ |
| 0.3 | $2.392399608248(-11)$ | $2.392399608248(-11)$ |
| 0.4 | $1.334015014125(-09)$ | $1.334015014125(-09)$ |
| 0.5 | $3.525332072970(-08)$ | $3.525332072970(-08)$ |
| 0.6 | $7.745691949085(-07)$ | $1.453375334519(-07)$ |
| 0.7 | $1.540863151286(-05)$ | $2.564200239815(-07)$ |
| 0.8 | $2.298624430588(-04)$ | $3.702046024846(-07)$ |
| 0.9 | $2.557775812591(-03)$ | $4.892503795428(-07)$ |
| 1.0 | $3.117843742068(-02)$ | $6.172185576795(-07)$ |

**Case 2.**

Here, we consider the Lienard's equation (4.41) for fractional order with $n = 10$. The problem has been solved using LDM for $\lambda = -1$, $\mu = 4$, $\nu = -3$, and $\alpha = 1.25$, with the initial conditions given in (4.48).

The numerical results are reported in Table 4.8 [58]. Moreover, for this case ( fractional case ), the analytical solution is unknown. Therefore, we will examine the accuracy by finding the residual error. The first two iterates using the iterative method are found to be:

$$\begin{aligned}
y_0 &= 0.70710678118654752440 + 0.35355339059327376220x, \\
y_1 &= 5.9166088222334158263 \times 10^{-25} x^{5/4}(2.9092393283103669576 \times 10^{21} x^5 \\
&+ 3.6365491603879586970 \times 10^{22} x^4 + 1.4000714267493640983 \times 10^{23} x^3 \\
&+ 1.0818733752154177123 \times 10^{23} x^2 - 2.9300737245417563039 \times 10^{23} x \\
&- 2.6370663520875806738 \times 10^{23}).
\end{aligned} \tag{4.49}$$

Table 4.8: Approximate solution for Example 4.4 case 2 obtained by LDM and its residual error with $\lambda = -1, \mu = 4, \nu = -3$, and $\alpha = 1.25$.

| *X* | *LDM* | *LDM Residual Error* |
|---|---|---|
| 0.0 | 0.707106781186547524400844362105 | 0.0 |
| 0.1 | 0.732958139553049454121768629713 | 2.62210854546515428362(−10) |
| 0.2 | 0.753822575338418152757227413567 | 5.42662333027240839556(−8) |
| 0.3 | 0.771424594859425817708345576845 | 1.49640118381205435781(−6) |
| 0.4 | 0.786421667445439116352391815243 | 2.22656677145558000110(−5) |
| 0.5 | 0.799262368176075205362685044168 | 2.23752615849312568499(−4) |
| 0.6 | 0.810270324942293925996382426007 | 1.50301302631548190917(−3) |
| 0.7 | 0.819595130163062819432724338259 | 7.98658781229298206948(−3) |
| 0.8 | 0.826755751633567809111209047446 | 4.68943082880602411792(−2) |
| 0.9 | 0.827555254203689356410501366571 | 2.65540397855468943864(−1) |
| 1.0 | 0.806516855152622440713257122053 | 7.41785292637895536757(−1) |

Table 4.9 provides an approximate solution for various values of $\alpha$. Figure 4.4 depicts the numerical solution obtained by LDM for various values of $\alpha$.

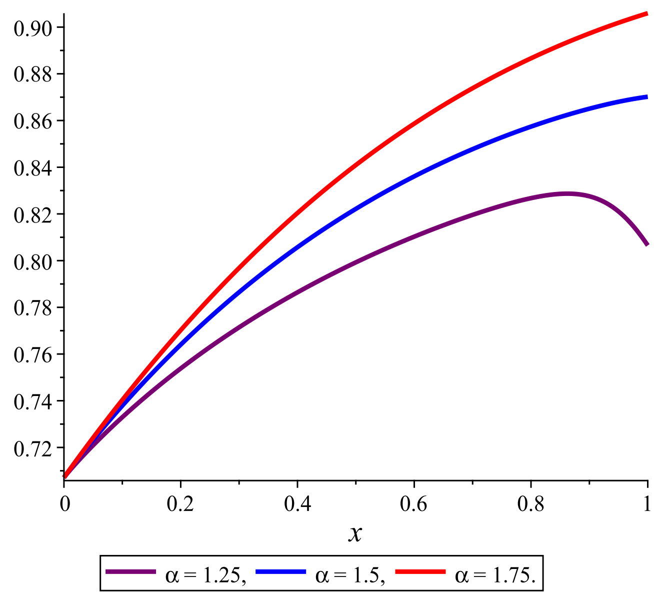


Figure 4.4: The numerical solution of Example 4.4 case 2 for various values of $\alpha$.

Table 4.9: Approximate solution for Example 4.4 case 2 obtained by LDM [58] and its residual error with $\lambda = -1, \mu = 4, \nu = -3$, and $\alpha = 1.5$.

| $X$ | *LDM for* $\alpha = 1.5$ | *LDM for* $\alpha = 1.75$ |
|---|---|---|
| 0.0 | 0.70710678118654752440 | 0.70710678118654752440 |
| 0.1 | 0.73788820230199637066 | 0.74034205018394961825 |
| 0.2 | 0.76401255438945728450 | 0.77020307329868743576 |
| 0.3 | 0.78648155743991699586 | 0.79685603341329815005 |
| 0.4 | 0.80574450957846447132 | 0.82038653995069625368 |
| 0.5 | 0.82216661191334990764 | 0.84093157196385339128 |
| 0.6 | 0.83607224394895173241 | 0.85867696610376466621 |
| 0.7 | 0.84774393686739328929 | 0.87383843552092334391 |
| 0.8 | 0.85739545656770376000 | 0.88664046808724416370 |
| 0.9 | 0.86506456243103595976 | 0.89729312886882853304 |
| 1.0 | 0.87017349345503663330 | 0.90595348328151955880 |

### 4.3.3 Boundary Value Problems

In this section, we apply the LDM to approximate the solution for various fractional boundary value problems (FBVP). In the following, we consider several examples to demonstrate and illustrate the technique and to confirm its applicability and performance.

**Example 4.5.** *Consider the following inhomogeneous linear fractional differential equation*

$$D^{\alpha}y(x) \;+\; cy(x) = g(x), \qquad x \in [0,1], \tag{4.50}$$

*and subject to the boundary conditions*

$$y(0) = 0, \quad y(1) = \frac{1}{\Gamma(\alpha+2)}, \tag{4.51}$$

where $1 < \alpha \leq 2, c \in \mathbb{R}$ and $g(x) = x + \dfrac{cx^{\alpha+1}}{\Gamma(\alpha+2)}$.

Applying the Laplace transform integral operator to both sides of the equation and using the Caputo fractional derivative gives:

$$\frac{s^2\mathcal{L}[y] - sy(0) - y'(0)}{s^{2-\alpha}} \;=\; -c\,\mathcal{L}[y] + \mathcal{L}[g(x)]. \tag{4.52}$$

Using the first BC in (4.51), while setting $y'(0) = k$, and after simplifying the results we get

$$\mathcal{L}[y] = \frac{k}{s^2} - \frac{c}{s^\alpha}\mathcal{L}[y] + \frac{1}{s^\alpha}\mathcal{L}[g(x)]. \tag{4.53}$$

Now, we use the decomposition series to represent the solution $y$ as an infinite series, namely,

$$y = \sum_{n=0}^{\infty} y_n = y_0 + y_1 + y_2 + y_3 + .... \tag{4.54}$$

Based on substituting equation (4.54) into (4.53) and using the linearity of Laplace transform, results in:

$$\sum_{n=0}^{\infty}\mathcal{L}[y_n] = \frac{k}{s^2} - \frac{c}{s^\alpha}\sum_{n=0}^{\infty}\mathcal{L}[A_n] + \frac{1}{s^\alpha}\mathcal{L}[g(x)], \tag{4.55}$$

where the $A_n{}'s$ are the Adomian polynomials. Based on matching both sides of the latter equation, leads to the following recursive relation

$$\begin{aligned}
y_0 &= kx + \frac{1}{s^\alpha}\mathcal{L}^{-1}[\mathcal{L}(g(x))],\\
y_1 &= -\frac{c}{s^\alpha}\mathcal{L}^{-1}[\mathcal{L}(y_0)],\\
&\cdots\\
y_{n+1} &= -\frac{c}{s^\alpha}\mathcal{L}^{-1}[\mathcal{L}(y_n)].
\end{aligned} \tag{4.56}$$

The exact solution of this fractional boundary value problem is $y_{exact}(x) = \dfrac{x^{\alpha+1}}{\Gamma(\alpha+1)}$. We use $n = 10, c = \frac{3}{57}$, and $\alpha = 1.2$, to obtain the approximate solution $y_{approx}(x) = \sum_{n=0}^{10} y_n$, where the second boundary condition in (6.46) is used to estimate the value of the parameter $k$ which appears in the numerical solution $y_{approx}$. The result of the first iterate is given by:

$$\begin{aligned}
y_0 &= 2.0438238157441767468 \times 10^{-19}x + 0.41254712918876375296 \times x^{11/5}\\
&+ 0.0051924871829667351270 \times x^{17/5},
\end{aligned} \tag{4.57}$$

where the value of the parameter $k$ was found to be

$$k = 2.0438238157441767468 \times 10^{-19}.$$

In the following table, the numerical solution and the absolute error using LDM [59] together with a comparison with the Haar wavelet method (HWM) [60], are reported.

Table 4.10: Comparison between the approximate solution obtained by LDM and HWM for Example 4.5, with $c = \frac{3}{57}$ and $\alpha = 1.2$.

| $x$ | *Exact* | *LDM* | *LDM Abs Err* | *HWM Abs Err* |
|---|---|---|---|---|
| 0.1 | 0.002602996411058695 | 0.002602996411058696000 | 2.0(−20) | 1.53063(-6) |
| 0.2 | 0.011960230781785240 | 0.011960230781785240013 | 4.1(−20) | 1.52699(-7) |
| 0.3 | 0.029183698484675718 | 0.029183698484675718060 | 6.1(−20) | 8.07661(-7) |
| 0.4 | 0.054954789697686429 | 0.054954789697686429080 | 8.1(−20) | 6.31371(-7) |
| 0.5 | 0.089785783925369295 | 0.089785783925369295057 | 1.0(−19) | 5.19845(-7) |
| 0.6 | 0.134093065768305844 | 0.134093065768305844063 | 1.2(−19) | 1.82879(-6) |
| 0.7 | 0.188230179915200837 | 0.188230179915200837091 | 1.4(−19) | 2.43150(-6) |
| 0.8 | 0.252505906099761630 | 0.252505906099761631060 | 1.6(−19) | 3.11752(-6) |
| 0.9 | 0.327195325212939288 | 0.327195325212939288037 | 1.8(−19) | 3.96605(-6) |

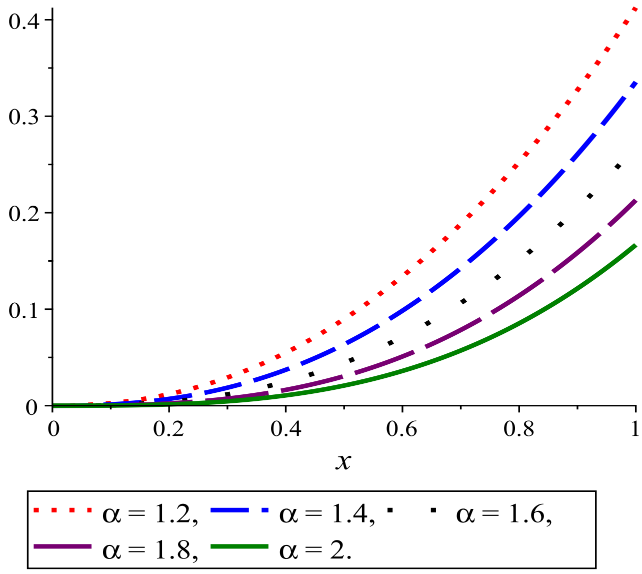


Figure 4.5: The numerical solution of Example 4.5 for values of $\alpha$ ranging between 1 and 2.

Table 4.10 shows the comparison of the approximate solution obtained by our method (LDM) and that by HWM. Obviously, we can see that our scheme yields better and more accurate results. In order to approximate the solution using LDM, only 10 iterations were needed, while 32 iterations were required to estimate the solution using HWM with appropriate precision. Figure 4.5 shows the solution for different $\alpha$ values. It is obvious from the graph that as the value of $\alpha$ increases to 2, the corresponding approximate solution of fractional order starts decaying.

**Example 4.6.** *Consider the following fractional boundary value problem:*

$$D^{\alpha}y(x) \;+\; cy(x) = g(x), \quad 1 < \alpha \leq 2, \tag{4.58}$$

*which complimented with the following boundary conditions*

$$y(0) = 0, \quad y(1) = -\frac{1}{40}. \tag{4.59}$$

By using the formulas on Laplace transform of Caputo fractional derivative and operating with Laplace transform on both sides of equation (4.58) by using the first BC given in (4.59), while setting $y'(0) = k$, we get

$$\mathcal{L}[y] \;=\; \frac{k}{s^2} - \frac{c}{s^{\alpha}}\,\mathcal{L}[y] \;+\; \frac{1}{s^{\alpha}}\,\mathcal{L}[g(x)]. \tag{4.60}$$

The Laplace transform expresses the solution $y$ as an infinite series solution of the form:

$$y = \sum_{n=0}^{\infty} y_n \;=\; y_0 + y_1 + y_2 + y_3 + ...., \tag{4.61}$$

By substituting equation (4.61) into (4.60) and using the linearity of Laplace transform, we get:

$$\sum_{n=0}^{\infty} \mathcal{L}\left[y_n\right] = \frac{k}{s^2} - \frac{c}{s^{\alpha}} \sum_{n=0}^{\infty} \mathcal{L}\left[A_n\right] + \frac{1}{s^{\alpha}}\,\mathcal{L}[g(x)], \tag{4.62}$$

where the $A_n{}'s$ are the Adomian polynomials. Matching both sides of the equation (4.62) the components of $y$ can be elegantly determined by utilizing the iterative relation:

$$\begin{aligned}
y_0 &= kx \;+\; \frac{1}{s^{\alpha}}\,\mathcal{L}^{-1}\left[\mathcal{L}(g(x))\right], \\
y_1 &= -\frac{c}{s^{\alpha}}\,\mathcal{L}^{-1}\left[\mathcal{L}(y_0)\right], \\
&\cdots \\
y_{n+1} &= -\frac{c}{s^{\alpha}}\,\mathcal{L}^{-1}\left[\mathcal{L}(y_n)\right].
\end{aligned} \tag{4.63}$$

We find the approximate solution $y_{approx}(x) = \sum_{n=0}^{7} y_n$, where $n = 7$, with $c = \frac{e^{-3\pi}}{\sqrt{\pi}}, \alpha = 3/2$, and

$$g(x) = \frac{-4144x^{\frac{3}{2}} + 5120x^{\frac{7}{2}} + (280x^5 - 518x^3 + 231x^2)x^{-3\pi} + 924\sqrt{x}}{280\sqrt{\pi}}.$$

The exact solution for this problem is given by $y_{exact}(x) = \left(x^3 - \frac{37x}{20} + \frac{33}{40}\right)x^2$. The second boundary condition in (4.59) has been used to determine the value of $k$ using $y_{approx}$. This yields

$$\begin{aligned} y_0 &= 3.4998165058029539531\,10^{-46}\,x + 6.4585599264410330 85\,10^{-6}\,x^{7/2} \\ &- 9.6552209001340696620\,10^{-6}\,x^{9/2} + 2.9197417199422626081\,10^{-6}\,x^{13/2} \\ &+ 0.825\,x^2 - 1.85\,x^3 + x^5, \end{aligned} \tag{4.64}$$

where $k$ are found to be

$$k = 3.49981650580295395310077190090986 \times 10^{-46}.$$

The numerical results using the LDM [59] and the absolute errors are recorded in Table 4.11 with a comparison with the HWM [60] for various values of $x \in [0, 1]$.

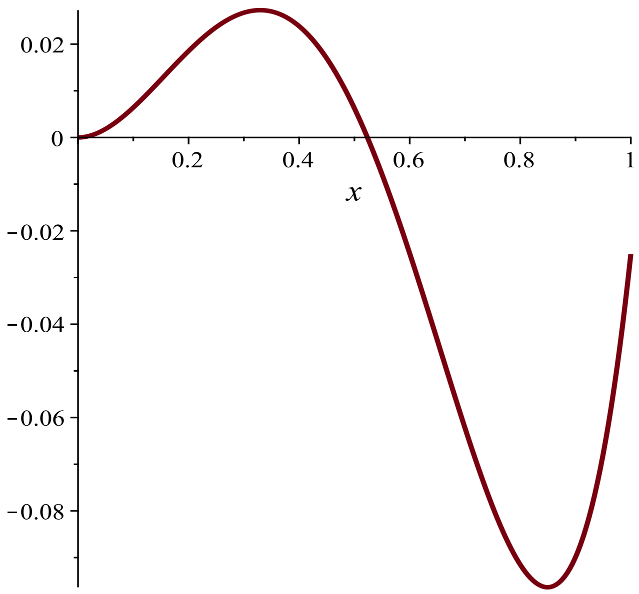


Figure 4.6: The numerical solution of Example 4.6 for $\alpha = 1.5$.

Table 4.11: Comparison between the absolute errors for Example 4.6 determined by LDM and HWM for $\alpha = \frac{3}{2}$.

| $x$ | *Exact* | *LDM* | *LDM Abs Err* | *HWM Abs Err* |
|---|---|---|---|---|
| 0.1 | 0.00641000000 | 0.00641000000 | 0.0 | 2.41789(-10) |
| 0.2 | 0.01852000000 | 0.01852000000 | 0.0 | 3.59522(-10) |
| 0.3 | 0.02673000000 | 0.02673000000 | 0.0 | 8.06354(-10) |
| 0.4 | 0.02384000000 | 0.02384000000 | 0.0 | 6.64016(-10) |
| 0.5 | 0.00625000000 | 0.00625000000 | 0.0 | 6.69882(-10) |
| 0.6 | $-0.024840000$ | $-0.024840000$ | 0.0 | 1.41570(-9) |
| 0.7 | $-0.062230000$ | $-0.062230000$ | 0.0 | 7.94582(-10) |
| 0.8 | $-0.091520000$ | $-0.091520000$ | 0.0 | 8.23084(-10) |
| 0.9 | $-0.089910000$ | $-0.089910000$ | 0.0 | 5.19118(-10) |

Table 4.11 confirms that the LDM is rapidly convergent using only few iterates. In this example, we used 7 iterations to achieve a highly accurate solution while for HWM, 256 iterations were used to produce even less accurate results. Figure 4.6 depicts the numerical solution of Example 4.6 for $\alpha = 1.5$.

**Example 4.7.** *Consider the following multi–term BVP with variable coefficients:*

$$D^{\alpha}y(x) + \phi(x)\, D^{\beta}y(x) + \psi(x)y(x) = g(x), \tag{4.65}$$

$$y(0) = 0, \quad y(1) = 0, \tag{4.66}$$

where $1 < \alpha \leq 2, \quad 0 < \beta \leq 1.$

By applying the Laplace transform integral operator to both sides of equation (4.65) and using the formulas on Laplace transform of Caputo fractional derivative, we get

$$\frac{s^2\mathcal{L}[y] - sy(0) - y'(0)}{s^{2-\alpha}} = -\mathcal{L}[\phi(x)\, D^{\beta}y] - \mathcal{L}[\psi(x)y] + \mathcal{L}[g(x)]. \tag{4.67}$$

Using the first boundary condition given in (4.66) while setting $y'(0) = k$, results

$$\mathcal{L}[y] = \frac{k}{s^2} - \frac{1}{s^{\alpha}}\mathcal{L}[\phi(x)\, D^{\beta}y] - \frac{1}{s^{\alpha}}\mathcal{L}[\psi(x)y] + \frac{1}{s^{\alpha}}\mathcal{L}[g(x)]. \tag{4.68}$$

Using the decomposition series for $y$, gives:

$$\sum_{n=0}^{\infty}\mathcal{L}\left[y_n\right] = \frac{k}{s^2} - \frac{1}{s^{\alpha}}\sum_{n=0}^{\infty}\mathcal{L}\left[\phi(x)\, D^{\beta}A_n\right] - \frac{1}{s^{\alpha}}\sum_{n=0}^{\infty}\mathcal{L}\left[\psi(x)B_n\right] + \frac{1}{s^{\alpha}}\mathcal{L}[g(x)]. \tag{4.69}$$

where the $A_n's, B_n's$ are the Adomian polynomials for $\phi(x), \psi(x)$, respectively. Matching both sides of the last equation leads to the iterative relation

$$\begin{aligned}
y_0 &= \frac{k}{s^2} + \frac{1}{s^\alpha}\,\mathcal{L}^{-1}\left[\mathcal{L}(g(x))\right], \\
y_1 &= -\frac{1}{s^\alpha}\,\mathcal{L}^{-1}\left[\phi(x)\,D^\beta y_0\right] - \frac{1}{s^\alpha}\left[\psi(x) y_0\right], \\
&\cdots \\
y_{n+1} &= -\frac{1}{s^\alpha}\,\mathcal{L}^{-1}\left[\phi(x)\,D^\beta y_n\right] - \frac{1}{s^\alpha}\left[\psi(x) y_n\right].
\end{aligned} \tag{4.70}$$

The LDM is implemented to this problem with $n = 10,\ \alpha = \frac{7}{4},\ \beta = \frac{1}{4},\ \phi(x) = -\dfrac{1}{\sqrt{x}}$, $\psi(x) = \dfrac{4x^{-\frac{3}{4}}}{3\Gamma(\frac{3}{4})}$, and $g(x) = -\dfrac{4\,x^{\frac{1}{4}}\,\sqrt{2}\,\Gamma(\frac{3}{4})}{\pi} + \dfrac{4\,x^{\frac{5}{4}}}{21\,\Gamma(\frac{3}{4})}$. We obtain the approximate value of the solution $y_{approx}(x) = \displaystyle\sum_{n=0}^{10} y_n$. Table 4.12 contains a comparison between the approximate solution along with the absolute error obtained by the [59] with that results from HWM [60]. The second BC given in (4.66) is utilized to estimate the value of $k$ using $y_{approx}$. The first iterate is found to be

$$\begin{aligned}
y_0 &= 0.02935193570623649694752798965999721792140\,x^3 - x^2 \\
&+ 0.9999999999999999999748773463215118572653\,x,
\end{aligned} \tag{4.71}$$

where $k = 0.9999999999999999997$.

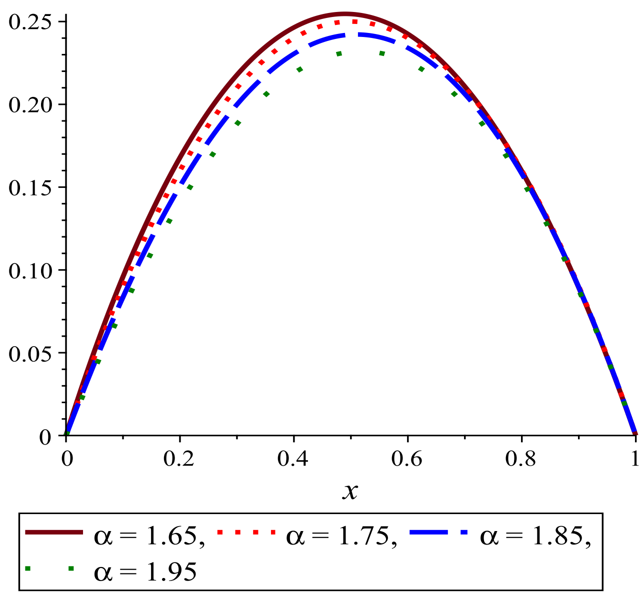


Figure 4.7: The numerical solution of Example 4.7 for $\alpha$ ranging between 1 and 2.

Table 4.12: Approximate solution for Example 4.7 and comparison in absolute error between LDM and HWM.

| $x$ | *Exact* | *LDM* | *LDM Err* | *HWM Err* |
|---|---|---|---|---|
| 0.1 | 0.09 | 0.0899999999999999999997487734 | $2.5122653678463020(-21)$ | 2.17578(-5) |
| 0.2 | 0.16 | 0.1599999999999999999994975469 | $5.0245307151171506(-21)$ | 1.80519(-5) |
| 0.3 | 0.21 | 0.2099999999999999999992463207 | $7.5367920981839847(-21)$ | 1.54805(-5) |
| 0.4 | 0.24 | 0.2399999999999999999989951107 | $1.0048892876120354(-20)$ | 1.32721(-5) |
| 0.5 | 0.25 | 0.2499999999999999999987441739 | $1.2558260109058709(-20)$ | 1.12113(-5) |
| 0.6 | 0.24 | 0.2399999999999999999984959219 | $1.5040780277836418(-20)$ | 9.21482(-6) |
| 0.7 | 0.21 | 0.2099999999999999999982657553 | $1.7342446669571050(-20)$ | 7.24277(-6) |
| 0.8 | 0.16 | 0.1599999999999999999981283009 | $1.8716990450786153(-20)$ | 5.27084(-6) |
| 0.9 | 0.09 | 0.0899999999999999999983775453 | $1.6224546820410042(-20)$ | 3.28365(-6) |

The above table assures that the LDM outperforms the HWM strategy. In Table 4.12 we record the estimated solution using 10 LDM iterations and then compare the absolute error with the results of HWM using 256 iterations. Figure 4.7 depicts the approximate solution of Example 4.7 for different values of $\alpha$ on the interval $[1, 2]$.

**Example 4.8.** *Consider the following Bagley–Torvik equation [61]:*

$$a_1\, y'' \,+\, a_2\, D^{\alpha} y(x) \,+\, a_3\, y(x) = g(x), \quad x \in [0,1], \tag{4.72}$$

*with the boundary conditions*

$$y(0) = 0, \quad y(1) = 0, \tag{4.73}$$

where $1 < \alpha \leq 2$.

The LDM is applied to this latter problem, the formulas on Laplace transform have been utilized with the decomposition series for $y$. By using 20 iterations we obtain the approximate value of the solution $y_{approx}(x) = \sum_{n=0}^{10} y_n$, with $a_1 = 1,\ a_2 = \frac{8}{17},\ a_3 = \frac{13}{51},\ \alpha = \frac{3}{2}$, and

$$\begin{aligned} g(x) \;=\; & \frac{3648}{425\, x^{\frac{5}{2}}} - \frac{2784}{85\, x^{\frac{7}{2}}} + \frac{960}{17\, x^{\frac{9}{2}}} + \frac{520}{17\, x^{6}} - \frac{1508}{85\, x^{5}} \\ & + \frac{52976}{425\, x^{4}} - \frac{149369}{2125\, x^{3}} + \frac{38877}{2125\, x^{2}} - \frac{339}{125\, x} - \frac{2712}{2125\, x^{\frac{3}{2}}}. \end{aligned}$$

The exact solution is given by $y(x) = x^5 - \dfrac{29x^4}{10} + \dfrac{76x^3}{25} - \dfrac{339x^2}{250} + \dfrac{27x}{125}$. The numerical results with the absolute error obtained by the LDM [59] are reported in Table 4.13, which also includes a comparison with the HWM [60] using $256$ iterations. The initial iterate is set first to be

$$y_0 = kx + \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\,\mathcal{L}(g(x))\right],$$

then the second boundary condition in (4.73) is used to estimate the value of $k$ using $y_{approx}$, and is found to be

$$k = 0.2159999999999998919.$$

The numerical result for the first iterate is given by

$$\begin{aligned}
y_0 \;=\;& 0.2159999999999998919\,x \;+\; 0.0764642823820017550\,x^{3/2},\\
-\;& 1.356\,x^2 \;+\; 3.0491764705882352941\,x^3 \;-\; 2.9288039215686274510\,x^4\\
-\;& 0.3840206181851645616\,x^{5/2} \;+\; 1.0387450980392156863\,x^5\\
-\;& 0.0246405228758169935\,x^6 \;+\; 0.7379410109247157189\,x^{7/2}\\
+\;& 0.0060690943043884220\,x^7 \;-\; 0.6257394537080922763\,x^{9/2}\\
+\;& 0.1961565685605304941\,x^{11/2}.
\end{aligned} \tag{4.74}$$

Table 4.13: Comparison of the absolute error between LDM and HWM for Example 4.8 with $\alpha = \frac{3}{2}$.

| $x$ | *Exact* | *LDM* | *LDM Err* | *HWM Err* |
|---|---|---|---|---|
| 0.1 | 0.010800000 | 0.0107999999999999998919138305 | 1.0808616942760(−17) | 3.89916(-6) |
| 0.2 | 0.008960000 | 0.0089599999999999997840735431 | 2.1592645682051(−17) | 4.17115(-6) |
| 0.3 | 0.003780000 | 0.0037799999999999996767077893 | 3.2329221067702(−17) | 3.94253(-6) |
| 0.4 | 0.0 | −0.0000000000000000042993412 | 4.2993412053826(−17) | 3.37304(-6) |
| 0.5 | −0.0010000 | −0.0010000000000000053527920 | 5.3527920662387(−17) | 2.60917(-6) |
| 0.6 | 0.0 | −0.0000000000000000063678143 | 6.3678143562432(−17) | 1.78827(-6) |
| 0.7 | 0.001260000 | 0.0012599999999999992764713748 | 7.2352862518744(−17) | 1.04097(-6) |
| 0.8 | 0.001280000 | 0.0012799999999999992433028793 | 7.5669712060674(−17) | 4.92027(-7) |
| 0.9 | 0.0 | −0.0000000000000000061908479 | 6.1908479841833(−17) | 2.61094(-7) |

**Example 4.9.** *Consider the following FBVP:*

$$D^\alpha y(x) \;=\; D^\beta y(x) \;-\; e^{x-1} \;-1, \tag{4.75}$$

*subject to*

$$y(0) = 0, \quad y(1) = 0, \tag{4.76}$$

where $1 < \alpha \leq 2$, and $0 < \beta \leq 1$.

In this example, the exact solution is known for integer order only. However, we will consider here two special cases for integer and non-integer order, respectively.

**Case 1.** $\alpha = 2$ and $\beta = 1$.

We implement the LDM for $n = 10$ iterations with the exact solution $y_{exact}(x) = x(1 - e^{x-1})$ to find $y_{approx}(x) = \sum_{n=0}^{10} y_n$. The initial iterate is set to be

$$y_0 = kx - \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\,\mathcal{L}(e^{x-1}+1)\right],$$

and, the second boundary condition in (4.76) is used to estimate the value of the parameter $k$. The first two iterates of the numerical results are given as follows:

$$\begin{aligned} y_0 &= -0.36787944117144232160\,e^x - 0.50000000000000000000\,x^2 \\ &+ 1.0000000006911715380\,x + 0.36787944117144232160, \\ y_1 &= -0.36787944117144232160\,e^x + 0.50000000034558576900\,x^2 \\ &+ 0.36787944117144232160\,x + 0.36787944117144232160 \\ &- 0.16666666666666666667\,x^3, \end{aligned} \tag{4.77}$$

where $k$ was found to be $k = 0.63212055951972921640$.

Table 4.14: Comparison of the approximate solution of the LDM, HWM and Fourth order HPM for Example 4.9 case 1, with $\alpha = 2$ and $\beta = 1$.

| $x$ | *Exact* | *LDM* | *LDM Abs Err* | *HWM* | *4th order HPM* |
|---|---|---|---|---|---|
| 0.1 | 0.05934303402 | 0.05934303409863 | 7.26911451(−11) | 0.05934300 | 0.05934820 |
| 0.2 | 0.11013420717 | 0.11013420732958 | 1.53027279(−10) | 0.11013418 | 0.11014318 |
| 0.3 | 0.15102440886 | 0.15102440910438 | 2.41811768(−10) | 0.15102438 | 0.15103441 |
| 0.4 | 0.18047534556 | 0.18047534590230 | 3.39913920(−10) | 0.18047531 | 0.18048329 |
| 0.5 | 0.19673467014 | 0.19673467059175 | 4.48070727(−10) | 0.19673463 | 0.19673826 |
| 0.6 | 0.19780797237 | 0.19780797294413 | 5.65517292(−10) | 0.19780792 | 0.19780653 |
| 0.7 | 0.18142724552 | 0.18142724620644 | 6.83647497(−10) | 0.18142718 | 0.18142196 |
| 0.8 | 0.14501539753 | 0.14501539830108 | 7.63467150(−10) | 0.14501532 | 0.14500893 |
| 0.9 | 0.08564632376 | 0.08564632443693 | 6.69303200(−10) | 0.08564623 | 0.08564186 |

Table 4.14 includes the approximate solution resulting from the LDM [59] with a comparison with that resulting from the Haar wavelet method [60] and the Fourth order homotopy method (HPM) [62]. Clearly, the LDM achieves better results and is more accurate.

**Case 2.** $\alpha = 1.9,$ and $\beta = 1.$

The LDM is applied to this latter problem, with $n = 12$ iterations. In Table 4.15 we report the numerical results obtained by the proposed approach [59]. Since the exact solution is unknown for this case, we have also included the residual error of the LDM.

The initial iterate is set first to be:

$$y_0 = kx - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\,\mathcal{L}(e^{x-1}+1)\right],$$

then the second boundary condition in (4.76) is utilized to estimate the value of the parameter $k$ which was found to be:

$$k = 0.694432541008310219889666986730068018338 5.$$

Table 4.15: The approximate solution and the residual error obtained by LDM for Example 4.9 case 2, with $\beta = 1$ and $\alpha = 1.9$.

| *x* | *LDM* | *LDM Residual Error* |
|---|---|---|
| 0.1 | 0.06447598491054325004046048459 | 4.81992514769655025590981065 42(−3) |
| 0.2 | 0.11913155933009484768816622660 | 1.82861491336015732365146783 18(−3) |
| 0.3 | 0.16290420092042056927874008236 | 1.21148181353583121631954279 61(−3) |
| 0.4 | 0.19429005918243805936814538829 | 9.19773011513686920483264622 71(−4) |
| 0.5 | 0.21149103445945060509148731641 | 7.45532898904421309033022467 34(−4) |
| 0.6 | 0.21241894928861222338411740632 | 6.28989856028426155326659567 22(−4) |
| 0.7 | 0.19467418130753968654451297997 | 5.45245977065391162609467833 13(−4) |
| 0.8 | 0.15551313733481943646867614837 | 4.78655069131545980883603465 90(−4) |
| 0.9 | 0.09180807502462913677604721234 | 3.02190899178862538631223940 18(−4) |
| 1.0 | 0.0 | 1.77688760443443925100054395 50(−3) |

### 4.3.4 Ray Tracing Equation

In this section, we will employ the Laplace decomposition method to obtain the approximate solution for tracing light rays through the crystalline lens that have the following form:

$$\frac{d^2r}{dt^2} = \Lambda_1 \, e^{\beta \, r} \; - \; \Lambda_2 \, e^{-2 \, \beta \, r}, \tag{4.78}$$

where $\beta$ is the exponential coefficient and $r$ is the distance from the optic axis. We will implement the method on both integer order and fractional order. Here, we generalize equation (4.78) to an arbitrary order $\alpha$, which leads to the following IVP:

**Example 4.10.**

$$D^\alpha r \; = \; \Lambda_1 \, e^{-\beta \, r} - \Lambda_2 \, e^{-2\beta \, r}, \tag{4.79}$$

*with the initial conditions*

$$r(0) = 0.001, \quad r'(0) = 1.976, \tag{4.80}$$

where $1 < \alpha \leq 2$.

By applying the formulas on Laplace transform for fractional derivative, and operating with Laplace transform on both sides of equation (4.79), we get

$$\frac{s^2 \mathcal{L}[r] - sr(0) - r'(0)}{s^{2-\alpha}} \; = \; \Lambda_1 \, \mathcal{L}[e^{-\beta r}] \; - \Lambda_2 \, \mathcal{L}[e^{-2\beta r}]. \tag{4.81}$$

Using the initial conditions given in (4.80), and simplifying the results, it follows that

$$\mathcal{L}[r] \; = \; \frac{0.001}{s} + \frac{1.976}{s^2} + \frac{\Lambda_1}{s^\alpha} \, \mathcal{L}[e^{-\beta r}] - \frac{\Lambda_2}{s^\alpha} \, \mathcal{L}[e^{-2\beta r}]. \tag{4.82}$$

Utilizing the decomposition series for $r$ and the Adomian polynomial representation for the nonlinear terms, namely, $N(r) = e^{-\beta r}$, and $M(r) = e^{-2\beta r}$, gives

$$\sum_{n=0}^{\infty} \mathcal{L}[r_n] \; = \; \frac{0.001}{s} + \frac{1.976}{s^2} + \frac{\Lambda_1}{s^\alpha} \sum_{n=0}^{\infty} \mathcal{L}[N_n] - \frac{\Lambda_2}{s^\alpha} \sum_{n=0}^{\infty} \mathcal{L}[M_n]. \tag{4.83}$$

Matching both sides of Eq. (4.83) gives the following iterative algorithm:

$$\begin{aligned}
\mathcal{L}[r_0] \quad &= \quad \frac{a}{s} + \frac{b}{s^2}, \\
\mathcal{L}[r_1] \quad &= \quad \frac{\Lambda_1}{s^\alpha} \, \mathcal{L}\,[N_0] \; - \frac{\Lambda_2}{s^\alpha} \, \mathcal{L}\,[M_0], \\
&\dots \\
\mathcal{L}[r_{n+1}] \quad &= \quad \frac{\Lambda_1}{s^\alpha} \, \mathcal{L}\,[N_n] \; - \frac{\Lambda_2}{s^\alpha} \, \mathcal{L}\,[M_n].
\end{aligned} \tag{4.84}$$

The first few Adomain polynomials for the nonlinear terms $N(r)\,\&\,M(r)$ are given respectively as follows:

$$\begin{cases} N_0 &= e^{-\beta\, r_0} \\ N_1 &= -\beta\, r_1\, e^{-\beta\, r_0} \\ N_2 &= \left(-\beta\, r_2 + \frac{\beta^2}{2} r_1^2\right) e^{-\beta\, r_0} \\ N_3 &= \left(-\beta\, r_3 + \beta^2\, r_1 r_2 - \frac{\beta^3}{3!} r_1^3\right) e^{-\beta\, r_0} \\ \dots \end{cases} \tag{4.85}$$

and

$$\begin{cases} M_0 &= -\,e^{-2\beta\, r_0} \\ M_1 &= 2\beta\, r_1\, e^{-2\beta\, r_0} \\ M_2 &= (2\beta\, r_2 - 2\beta^2 r_1^2)\, e^{-2\beta\, r_0} \\ M_3 &= \left(2\beta\, r_3 - 4\beta^2\, r_1 r_2 + \frac{8\beta^3}{3!} r_1^3\right) e^{-2\beta\, r_0} \\ \dots \end{cases} \tag{4.86}$$

In order to find the first iterate $r_0$, the inverse Laplace transform is applied to the first term in the algorithm (4.84). The higher iterates are found in a similar fashion.
Here, we will consider two special cases for integer order and fractional order, respectively.

**Case 1.** $\alpha = 2$.

Upon using the described method with $n = 7$ iterations, $\beta = 0.91, k = 2, \Lambda_1 = -0.0254072,$ and $\Lambda_2 = -0.000091,$ we obtain the approximate value of the solution $y_{approx}(x) = \sum_{n=0}^{7} y_n.$ The numerical solution and the residual error together with a comparison with the 5-terms multi-step differential transform method (MsDTM) [63] are reported in Table 4.16. However, the numerical result for the first iterate is given by:

$$\begin{aligned} y_0 &= 0.0010000000000000000000000000000000000 \\ &+ 1.976000000000000000000000000000000\, x. \end{aligned} \tag{4.87}$$

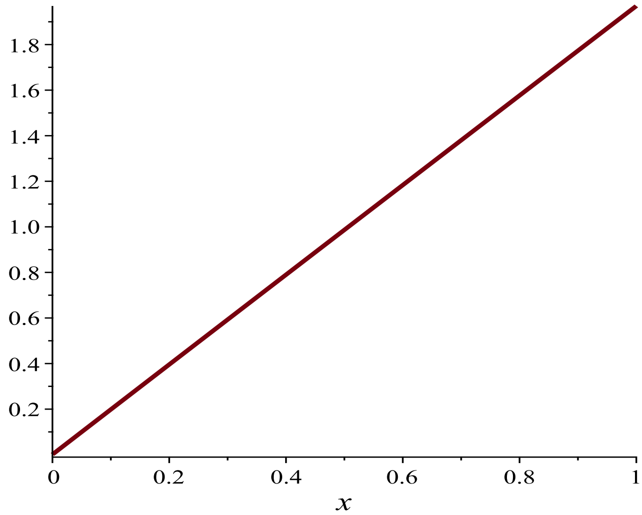


Figure 4.8: The numerical solution of Example 4.10 case 1, for $\alpha = 2$ .

Table 4.16: The approximate solution obtained by the LDM with a comparison with that attained by MsDTM for Example 4.10 case 1, with $\beta = 0.91, \Lambda_1 = -0.0254072, \Lambda_2 = -0.000091$ and $\alpha = 2$.

| $x$ | *LDM* | *LDM Err* | *MsDTM Err* |
|---|---|---|---|
| 1.00 | 1.969445607645401968521494 | 5.223135161405920796334275(−12) | 2.00000000(−9) |
| 2.00 | 3.932407650186922573978171 | 6.510342519760909528712327(−14) | 1.00000000(−9) |
| 3.00 | 5.894450780930431182014999 | 1.706978904729296811253414(−16) | 3.00000000(−9) |
| 4.00 | 7.856339792503087818005026 | 2.133308325115731300166462(−16) | 1.30000000(−8) |
| 5.00 | 9.818202951601105848515534 | 1.087947396050170567143762(−16) | 4.20000000(−8) |
| 6.00 | 11.78006177401202388873132 | 4.338472918616162900377010(−17) | 7.00000000(−8) |
| 7.00 | 13.74191986895171822599585 | 3.630705170240612792596598(−17) | 6.00000000(−8) |
| 8.00 | 15.70377784185937901732542 | 2.962097559870027209776538(−14) | 6.00000000(−8) |
| 9.00 | 17.66563579429623350280118 | 2.634263832332057541580414(−12) | 5.00000000(−8) |
| 10.0 | 19.62749374329218218418472 | 8.425031131808244983822462(−11) | 5.00000000(−8) |

Table 4.16 assures that the LDM is rapidly convergent using only few iterates. Figure 4.8 depicts the numerical solution of Example 4.10 (case 1), for $\alpha = 2$ for different values of $x$ ranging between 0 and 1.

**Case 2.** $\alpha = 1.8$.

The LDM is applied to this problem, using $n = 7$ iterates, with $\beta = 0.91, k = 2, \Lambda_1 = -0.0254072$, and $\Lambda_2 = -0.000091$, we obtain the approximate value of the solution

$y_{approx}(x) = \sum_{n=0}^{7} y_n$. The numerical results are reported in Table 4.17. In addition, we have included the residual error of the LDM. The first two iterations are given by

$$
\begin{aligned}
y_0 &= 0.0010000000000000000000 + 1.9760000000000000000\,x, \\
y_1 &= 1.0303746610008685233\,10^{-16}\,x^{\frac{84}{5}} - 9.6249480137717500882\,10^{-15}\,x^{\frac{79}{5}} \\
&+ 4.2704532921599768238\,10^{-13}\,x^{\frac{74}{5}} - 1.1971153232035579395\,10^{-11}\,x^{\frac{69}{5}} \\
&+ 2.3791098421694005549\,10^{-10}\,x^{\frac{64}{5}} - 3.5652474211235348 04\,10^{-9}\,x^{\frac{59}{5}} \\
&+ 4.1828190481047235047\,10^{-8}\,x^{\frac{54}{5}} - 3.9365482974472878776\,10^{-7}\,x^{\frac{49}{5}} \\
&+ 3.0194861331995299 47\,10^{-6}\,x^{\frac{44}{5}} - 0.000019059802177651722916\,x^{\frac{39}{5}} \\
&+ 0.000099423603622814136752\,x^{\frac{34}{5}} - 0.00042790745651021344376\,x^{\frac{29}{5}} \\
&+ 0.0015061689529771150911\,x^{\frac{24}{5}} - 0.0042541152457735710560\,x^{\frac{19}{5}} \\
&+ 0.0093088016422719769198\,x^{\frac{14}{5}} - 0.014794646828018427988\,x^{\frac{9}{5}}.
\end{aligned} \tag{4.88}
$$

Table 4.17: The approximate solution and the residual error obtained by LDM for Example 4.10 case 2, with $\beta = 0.91, \Lambda_1 = -0.0254072, \Lambda_2 = -0.000091$ and $\alpha = 1.8$.

| $x$ | *LDM* | *LDM Residual Err* |
|---|---|---|
| 1.00 | 1.9684103159184399840 | 1.2317041630132388006(−05) |
| 2.00 | 3.9330164081958655137 | 1.0918741643232629033(−07) |
| 3.00 | 5.8984657955899349659 | 1.4783824522248488330(−10) |
| 4.00 | 7.8647146429237204960 | 2.0972956643999870 00(−15) |
| 5.00 | 9.8315335808149142152 | 3.7625641810009000000(−18) |
| 6.00 | 11.798770630418340720 | 1.3561652830904754750(−15) |
| 7.00 | 13.766331665432795294 | 1.0083360755354191619(−10) |
| 8.00 | 15.734155135812481314 | 6.1317035388545998133(−08) |
| 9.00 | 17.702198562268168252 | 5.6963717882063610128(−06) |
| 10.0 | 19.670449965878745316 | 1.8938454665192535823(−04) |

Table 4.17 presents the approximate solution and the residual error for $\alpha = 1.8$ using only 7 iterations. It is obvious from the results that the LDM provides highly accurate solution using only few iterates.

Table 4.18: Approximate solution for Example 4.10 case 2, using LDM with $\alpha = 1.25$ and $\alpha = 1.5$.

| $x$ | *LDM for* $\alpha = 1.25$ | *LDM for* $\alpha = 1.5$ |
|---|---|---|
| 1.00 | 1.9658026941316956440 | 1.9668175572592734768 |
| 2.00 | 3.9365827397956513083 | 3.9344972156022543925 |
| 3.00 | 5.9098426378039396134 | 5.9045376266462732242 |
| 4.00 | 7.8840034042573431718 | 7.8758002070176216909 |
| 5.00 | 9.8585786491700744803 | 9.8477591265525358587 |
| 6.00 | 11.833392607269193284 | 11.820171344061794837 |
| 7.00 | 13.808365785708932780 | 13.792908906514318298 |
| 8.00 | 15.783454785967921864 | 15.765895416598778743 |
| 9.00 | 17.758633580863748286 | 17.739081387895577188 |
| 10.0 | 19.733920766431980279 | 19.712459161260865732 |

Table 4.18 shows the approximate solution for different values of $1 < \alpha \leq 2$.

## 4.4 Convergence Analysis

**Theorem 4.4.1** (Banach Fixed Point Theorem)**.**
*Let $(X, d)$ be a complete metric space and let $T : X \to X$ be a contraction mapping. Then $T$ has a unique fixed point $\bar{x}$ and for any $x \in X$ the sequence $(T^n(x))_{n=1}^{\infty}$ converges to $\bar{x}$.*
This theorem is also called the Contraction Mapping Theorem.
It is clear from (4.13) that the $A_n{}'s$ are indeed polynomials and hence the $y_{n+1}$ term is obtained from (4.17). In this section, we discuss a new approach for the convergence of the Adomian Decomposition Method [64]. The sufficient condition that guarantees existence of a unique solution is inserted in Theorem 4.4.2, convergence of the series solution $y = \sum_{n=0}^{\infty} y_n$ is proved in Theorem 4.4.3.
Consider the following general equation

$$L\,y \,+\, R\,y \,+\, N\,y \,=\, g, \tag{4.89}$$

with

$$y(0) \,=\, 0, \quad y'(0) \,=\, 0, \tag{4.90}$$

where $N(y)$ represents the nonlinear term. Assume here that $y(t)$ is bounded for all $t \in J = [0, K]$ and the nonlinear term $N(y)$ is Lipschitzian such that $|N(y_2) - N(y_1)| \leq$

$L_1\,|y_2 \;-\; y_1|$, where $L_1$ is Lipschitz constant.

The decomposition method assumes an infinite series solution for $y$ which is given by

$$y = \sum_{n=0}^{\infty} y_n, \tag{4.91}$$

then, the nonlinear term $N(y)$ decomposed as a sum of the infinite series

$$N(y) \;=\; \sum_{n=0}^{\infty} A_n, \tag{4.92}$$

where $A_n's$ are the Adomian polynomials obtained by

$$A_n = \frac{1}{n!}\,\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, \qquad n = 0, 1, 2, .... \tag{4.93}$$

Operating both sides of equation (4.89) by the inverse operator we get:

$$L^{-1}\,L\,y \,+\, L^{-1}\,R\,y \,+\, L^{-1}\,N\,y \,=\, L^{-1}\,g, \tag{4.94}$$

which is equivalent to the following expression:

$$y \;=\; -L^{-1}\,R\,y \;-\; L^{-1}\,N\,y \;+\; L^{-1}\,g \tag{4.95}$$

Substituting equations (4.92) and (4.93) into (4.95), we obtain:

$$\sum_{n=0}^{\infty} y_n \;=\; -L^{-1}R\sum_{n=0}^{\infty} y_n \;-\; L^{-1}\sum_{n=0}^{\infty} A_n \;+\; L^{-1}\,g. \tag{4.96}$$

The method consists of the following recursive relation

$$\begin{cases} y_0 &= \; L^{-1}\,g, \\ y_n &= \; -L^{-1}\,R\,y_{n-1} - L^{-1}\,A_{n-1}. \end{cases} \tag{4.97}$$

**Theorem 4.4.2** (Uniqueness Theorem)**.**

*The equation* $L\,y \;+\; Ry \;+\; Ny \;=\; g$ *has a unique solution when* $0 \;<\; \alpha \;<\; 1$, *where* $\alpha \;=\; \frac{(L_1+L_2)t^m}{m!}$.

*Proof.* Let $T \;=\; (C[J], \|.\|)$ be the complex Banach space that contains all continuous functions on the interval $J \;=\; [0, K]$ with the norm $\|y(t)\| \;=\; \max_{t\in J}|y(t)|$. Now, define a mapping $X \,:\, T \,\to\, T$ where, $X(y(t)) \;=\; -L^{-1}\,R\,y(t) \;-\; L^{-1}\,N\,y(t) \;+\; L^{-1}\,g(t)$. Let

$y, \bar{y} \in T$, then, we have:

$$
\begin{aligned}
\|Xy - X\bar{y}\| &= \max_{t\in J}|-L^{-1}Ry - L^{-1}Ny + L^{-1}g - (-L^{-1}R\bar{y} - L^{-1}N\bar{y} + L^{-1}g)| \\
&= \max_{t\in J}| - L^{-1}Ry - L^{-1}Ny + L^{-1}g + L^{-1}R\bar{y} + L^{-1}N\bar{y} - L^{-1}g)| \\
&= \max_{t\in J}| - L^{-1}Ry - L^{-1}Ny + L^{-1}R\bar{y} + L^{-1}N\bar{y}|
\end{aligned}
$$

By taking $L^{-1}$ as a common factor, we get:

$$
\begin{aligned}
&= \max_{t\in J}| - L^{-1}(Ry - R\bar{y}) - L^{-1}(Ny - N\bar{y})| \\
&= \max_{t\in J}|L^{-1}(Ry - R\bar{y}) + L^{-1}(Ny - N\bar{y})| \\
&\leq \max_{t\in J}(|L^{-1}(Ry - R\bar{y})| + |L^{-1}(Ny - N\bar{y})|)
\end{aligned}
$$

Now, suppose that $Ry$ is also Lipschitzian with $|Ry - R\bar{y}| \leq L_2|y - \bar{y}|$, where $L_2$ is Lipschitz constant. Thus,

$$
\|Xy - X\bar{y}\| \leq \max_{t\in J}(|L^{-1}(Ry - R\bar{y})| + |L^{-1}(Ny - N\bar{y})|)
$$

Since $|Ry - R\bar{y}| \leq L_2|y - \bar{y}|$ and $|Ny - N\bar{y}| \leq L_1|y - \bar{y}|$, then,

$$
\leq \max_{t\in J}(L^{-1}L_2|(y - \bar{y})| + L^{-1}L_1|(y - \bar{y})|)
$$

$L_1, L_2$ are constants, so we take them out, then,

$$
\begin{aligned}
&= (L_1 + L_2)\max_{t\in J}(L^{-1}(|y - \bar{y}|)) \\
&\leq (L_1 + L_2)\, L^{-1}\, (\max_{t\in J}|y - \bar{y}|)
\end{aligned}
$$

We know that $\max_{t\in J}|y_n(t)| = \|y(t)\|$, thus,

$$
= (L_1 + L_2)\ \|y - \bar{y}\|\ L^{-1}(1)
$$

Since $L^{-1} = \int_0^t \dots \int_0^t [.]\, dt$, then,

$$
\begin{aligned}
&= (L_1 + L_2)\ \|y - \bar{y}\|\ \frac{t^m}{m!} \\
&= \alpha\, \|y - \bar{y}\|
\end{aligned}
$$

where, $\alpha = \dfrac{(L_1 + L_2)t^m}{m!}$.

Therefore, there exists a unique solution for the equation $L\,y + Ry + Ny = g$ by Banach fixed point theorem for contraction. $\square$

**Theorem 4.4.3** (Convergence Theorem)**.**
*The infinite series solution* $y = \sum_{n=0}^{\infty} y_n$ *of equation* $L\,y + Ry + Ny = g$ *using Adomian decomposition method converges if* $0 < \alpha < 1$ *and* $|y_1| < \infty$.

*Proof.* Let $B_n = \sum_{i=0}^{n} y_i(t)$. We need to prove here that the sequence $\{B_n\}$ is a Cauchy sequence in Banach space $T$.
$\|B_{n+a} - B_n\| = \max_{t \in J}|B_{n+a} - B_n| = \max_{t \in J}\left|\sum_{i=0}^{n+a} y_i(t) - \sum_{i=0}^{n} y_i(t)\right| = \left|\sum_{i=n+1}^{n+a} y_i(t)\right|$, where, $a = 1, 2, 3, ...$
We know that,
$N(y_0 + y_1 + y_2 + ... + y_n) = \sum_{i=0}^{n} A_i$. Therefore, it could be arranged as follows:

$$\begin{array}{ccccc} A_0 & = & N(y_0) & = & N(B_0), \\ A_0 + A_1 & = & N(y_0 + y_1) & = & N(B_1), \\ A_0 + A_1 + A_2 & = & N(y_0 + y_1 + y_2) & = & N(B_2), \\ \cdots & & & & \end{array}$$

Thus,

$$A_n + \sum_{i=0}^{n-1} A_i = N(B_n). \tag{4.98}$$

In a similar way, we have:

$$\bar{A}_n + \sum_{r=0}^{n-1} \bar{A}_r = R(B_n). \tag{4.99}$$

Now, since the two operator $Ry$ and $Ny$ are assumed to be Lipschizian operators, and using equation (4.97), we have:

$$\|B_{n+a} - B_n\| = \max_{t \in J}\left|-L^{-1}\sum_{i=n+1}^{n+a} R_{y_{n-1}} - L^{-1}\sum_{i=n+1}^{n+a} A_{n-1}\right|$$

By shifting, we have,

$$\begin{aligned} &= \max_{t \in J}\left|-L^{-1}\sum_{i=n}^{n+a-1} R_{y_n} - L^{-1}\sum_{i=n}^{n+a-1} A_n\right| \\ &= \max_{t \in J}\left|L^{-1}\sum_{i=n}^{n+a-1} R_{y_n} + L^{-1}\sum_{i=n}^{n+a-1} A_n\right| \\ &= \max_{t \in J}\left|L^{-1}\sum_{i=n}^{n+a-1} R_{y_n} + L^{-1}\sum_{i=n}^{n+a-1} A_n\right| \end{aligned}$$

$$
\begin{aligned}
&= \max_{t\in J}\left|L^{-1}(R(B_{n+a-1})-R(B_{n-1}))+L^{-1}(N(B_{n+a-1})-N(B_{n-1}))\right| \\
&\leq \max_{t\in J}\left|L^{-1}(R(B_{n+a-1})-R(B_{n-1}))\right|+\max_{t\in J}\left|L^{-1}(N(B_{n+a-1})-N(B_{n-1}))\right| \\
&\leq \max_{t\in J}L^{-1}\left|(R(B_{n+a-1})-R(B_{n-1}))\right|+\max_{t\in J}L^{-1}\left|(N(B_{n+a-1})-N(B_{n-1}))\right| \\
&\leq L_2\max_{t\in J}L^{-1}\left|B_{n+a-1}-B_{n-1}\right|+L_1\max_{t\in J}L^{-1}\left|B_{n+a-1}-B_{n-1}\right|
\end{aligned}
$$

$L_1, L_2$ are constants, so we take them out, then,

$$
\begin{aligned}
&= (L_2+L_1)\left(\max_{t\in J}L^{-1}\left|B_{n+a-1}-B_{n-1}\right|+\max_{t\in J}L^{-1}\left|B_{n+a-1}-B_{n-1}\right|\right) \\
&= (L_2+L_1)\ \|B_{n+a-1}-B_{n-1}\|\ \frac{t^m}{m!}
\end{aligned}
$$

where, $\alpha = \dfrac{(L_1+L_2)t^m}{m!}$.

Therefore, where, $\alpha = \dfrac{(L_1+L_2)t^m}{m!}$, we have:

$$
\|B_{n+a}-B_n\| \leq \alpha\|B_{n+a-1}-B_{n-1}\|
$$

Similarily,

$$
\begin{aligned}
\|B_{n+a-1}-B_{n-1}\| &\leq \alpha\|B_{n+a-2}-B_{n-2}\| \\
\|B_{n+a-2}-B_{n-2}\| &\leq \alpha\|B_{n+a-3}-B_{n-3}\|
\end{aligned}
$$

and so on. Thus,

$$
\begin{aligned}
\|B_{n+a}-B_n\| &\leq \alpha^2\|B_{n+a-2}-B_{n-2}\| \\
\|B_{n+a}-B_n\| &\leq \alpha^3\|B_{n+a-3}-B_{n-3}\|
\end{aligned}
$$

and so on. It can be generalized by

$$
\begin{aligned}
\|B_{n+a}-B_n\| &\leq \alpha^n\|B_{n+a-n}-B_{n-n}\| \\
&\leq \alpha^n\|B_a-B_0\|
\end{aligned}
$$

Now, if $a = 1$,

$$\|B_{n+1} - B_n\| \leq \alpha^n \|B_1 - B_0\| \leq \alpha^n \|y_0 + y_1 - y_0\| \leq \|y_1\|$$

Now, let $n > k$, where $n, k \in \mathbb{N}$.

$$\begin{aligned}
\|B_n - B_k\| &\leq \|B_{k+1} - B_k\| + \|B_{k+2} - B_{k+1}\| + \|B_{k+3} - B_{k+2}\| + ... \|B_n - B_{n-1}\| \\
&\leq \left(\alpha^k + \alpha^{k+1} + \alpha^{k+2} + ...\alpha^{n-1}\right) \|y_1\| \\
&\leq \alpha^k \left(1 + \alpha + \alpha^2 + ... + \alpha^{n-1-k}\right) \|y_1\| \\
&\leq \alpha^k \left(\frac{1 - \alpha^{n-k}}{1 - \alpha}\right) \|y_1\|
\end{aligned}$$

Since $0 < \alpha < 1$, so we have also $1 - \alpha^{n-\alpha} < 1$,

then, we get,

$$\|B_n - B_k\| \leq \frac{1 - \alpha^{n-k}}{1 - \alpha} \|y_1\|$$

By assumption, we have $y(t)$ is bounded and $|y_1| < \infty$. Thus, as $n \to \infty$, $\|B_n - B_k\| \to 0$. Hence, the sequence $\{B_n\}$ is a Cauchy sequence in T. Therefore, the series solution $y = \sum_{n=0}^{\infty} y_n$ is convergent. $\square$

# Chapter 5: Green's Function - Fixed Point Iterative Scheme

In the previous chapter, we implemented the Laplace decomposition method to obtain the numerical solution for various well-known problems. However, we faced deficiencies in the method. In particular, the accuracy deteriorates as the applicable domain increases. In this chapter, we suggest an alternate strategy based on applying the Laplace transform integral operator for fractional derivative and then embedding Green's functions into fixed point iterative schemes. The major rationale behind this iterative technique is to surmount the deterioration of the numerical solution obtained by LDM.

Green's function is used to solve inhomogeneous linear differential equations defined in a domain with specified initial conditions or boundary conditions. Green's functions are named for George Green, the British mathematician, who first developed the concept in $1830$. The Green's function methods are studied by many authors to obtain the approximate solution for ordinary differential equations and partial differential equations [65, 66, 67]. A Greens function coupled with fixed point iteration method (GFIM) proposed by Abushammala et al. [68] has been applied to solve various differential equations models that arise in sciences and engineering. Kafri et al. employed the GFIM to solve Toreschs problem [69], Khuri et al. employed the GFIM to solve boundary value problems and partial differential equations [70, 71, 72, 73]. Abukhaled et al used the GFIM to find highly accurate semi-analytical solution for the one dimensional curvature equation [74], a class of strongly nonlinear oscillators [75], amperometric enzymatic reactions [76], and a class of BVPs arising in heat transfer [77].

## 5.1 Important Definitions

**Definition 5.1.1.** The direct delta function is defined as

$$\delta(t-x) = \begin{cases} 0, & t \neq x, \\ \infty, & t = x. \end{cases} \tag{5.1}$$

The delta function satisfies the following properties:

$$
\begin{aligned}
&1.\ \int_{-\infty}^{\infty} \delta(t-x)\, dx \;=1,\\
&2.\ \int_{-\infty}^{\infty} \delta(t-x)\, f(x)\, dt \;= f(t),
\end{aligned}
\tag{5.2}
$$

where the integral can be taken over any interval that includes $t = x$.

**Definition 5.1.2.** The unit step function (or Heaviside function) is defined as

$$
u(t-x) \;=\; \begin{cases} 0, & t < x, \\ 1, & t \geq x. \end{cases} \tag{5.3}
$$

## 5.2 Properties of Green's Function

In this section, we will summarize the properties of Green's function as an instrument for quickly constructing it. Here is a list of the properties based upon the integer order and fractional order derivatives for IVPs and BVPs.

Consider the following third order equation:

$$
K[y] \;=\; y'''(x) \;+\; p_1(x)y''(x) \;+\; p_2(x)\,y'(x) \;+\; p_3(x)y(x) \;=\; f(x). \tag{5.4}
$$

subject to the initial conditions,

$$
\begin{aligned}
y(a) \;&=\; \alpha_1,\\
y'(a) \;&=\; \alpha_2,\\
y'(b) \;&=\; \alpha_3,
\end{aligned}
\tag{5.5}
$$

or boundary conditions,

$$
\begin{aligned}
B_1[y] \;&=\; a_1 y(a) \;+\; a_2 y'(a) \;+\; a_3 y''(a) \;=\; \alpha,\\
B_2[y] \;&=\; b_1 y(b) \;+\; b_2 y'(b) \;+\; b_3 y''(b) \;=\; \beta.
\end{aligned}
\tag{5.6}
$$

The Green's function satisfies

$$
G'''(t|x) \;+\; p_1(t)G''(t|x) \;+\; p_2(t)\,G'(t|x) \;+\; p_3(t)G(t|x) \;=\; \delta(t-x). \tag{5.7}
$$

Green's function for equation (5.4) has the following properties:

1. $G(t|x)$ satisfies the corresponding homogeneous initial condition

$$
G(a|x) \;=\; G'(a|x) \;=\; G(b|x) \;=\; 0, \tag{5.8}
$$

and the corresponding homogeneous boundary conditions

$$
B_1[G(t|x)] \;=\; B_2[G(t|x)] \;=\; 0. \tag{5.9}
$$

2. $G(t|x)$ is continuous at $t = x$, that is

$$\begin{aligned} G(t|x)|_{t\to x^-} &= G(t|x)|_{t\to x^+}, \\ \text{where} & \\ G(t|x)|_{t\to x^-} &= \lim_{t\to x} G(t|x), \quad t < x, \\ G(t|x)|_{t\to x^+} &= \lim_{t\to x} G(t|x), \quad t > x. \end{aligned} \tag{5.10}$$

3. $G'(t|x)$ is continuous at $t = x$, that is

$$\begin{aligned} G'(t|x)|_{t\to x^-} &= G'(t|x)_{t\to x^+}, \\ \text{where} & \\ G'(t|x)|_{t\to x^-} &= \lim_{t\to x} G'(t|x), \quad t < x, \\ G'(t|x)|_{t\to x^+} &= \lim_{t\to x} G'(t|x), \quad t > x. \end{aligned} \tag{5.11}$$

4. $G''(t|x)$ has a unit jump discontinuity at $t = x$, that is

$$G''(t|x)|_{t\to x^+} - G''(t|x)|_{t\to x^-} = 1. \tag{5.12}$$

Let's now consider the following fractional order equation

$$D^{\alpha}y(x) = \Lambda_1 e^{-\beta y(x)} - \Lambda_2 e^{-2\beta y(x)}, \tag{5.13}$$

where $\beta, \Lambda_1, \Lambda_2$, are constants and $1 < \alpha \le 2$, and subject to the following initial conditions:

$$\begin{aligned} y(a) &= \alpha_1, \\ y'(a) &= \alpha_2. \end{aligned} \tag{5.14}$$

Green's function for the fractional equation (5.13) has the following properties:

1. $G(t|x)$ satisfies the homogeneous initial condition

$$G(a|x) = G'(a|x) = 0. \tag{5.15}$$

2. $G(t|x)$ is continuous at $t = x$, that is

$$\begin{aligned} G(t|x)|_{t\to x^-} &= G(t|x)|_{t\to x^+}, \\ \text{where} & \\ G(t|x)|_{t\to x^-} &= \lim_{t\to x} G(t|x), \quad t < x, \\ G(t|x)|_{t\to x^+} &= \lim_{t\to x} G(t|x), \quad t > x. \end{aligned} \tag{5.16}$$

3. $G'(t|x)$ has a unit jump discontinuity at $t = x$, that is

$$G'(t|x)|_{t\to x^+} - G'(t|x)|_{t\to x^-} = \frac{1}{x}. \tag{5.17}$$

## 5.3 Construction of the Green's Function

In this following two subsections, we illustrate how to construct the Green's function for both initial and boundary value problems.

### 5.3.1 Initial Value Problem for Second Order Equations

**Example 5.1.** *Construct a Green's function for the following IVP*

$$K[u] \;=\; y''(t) \;=\; -\lambda\, e^{y(t)}, \tag{5.18}$$

subject to

$$y(0) \;=\; y'(0) \;=\; 0. \tag{5.19}$$

We will seek a particular solution $y_p$, where while $y_p$ is a particular solution which is the solution of inhomogeneous equation subject to the initial conditions $y(0) = y'(0) = 0$. We represent the particular solution as an integral of the Green's function $G(t|x)$

$$G(t|x) \;=\; \int_t^a f(x)\,dx. \tag{5.20}$$

The general solution to the corresponding homogeneous equation of (5.18) is

$$y_h(t) \;=\; at \;+\; b. \tag{5.21}$$

The Green's function satisfies

$$G''(t|x) \;=\; \delta(t-x), \quad G(0|x) \;=\; G'(0|x) \;=\; 0. \tag{5.22}$$

Next, we can construct the Green's function which is given by

$$G(t|x) \;=\; \begin{cases} a_1 t \;+\; b_1, & 0 < t < x, \\ a_2 t \;+\; b_2, & t > x. \end{cases} \tag{5.23}$$

1. Applying the homogeneous initial conditions given in (5.22), we get

$$G(t|x) \;=\; \begin{cases} 0, & 0 < t < x, \\ a_2 t \;+\; b_2, & t > x. \end{cases} \tag{5.24}$$

2. $G(t|x)$ is continuous at $t = x$, then we get

$$G(t|x)_{t\to x^-} \;=\; G(t|x)_{t\to x^+}, \tag{5.25}$$

thus,

$$0 \;=\; a_2 x \;+\; b_2. \tag{5.26}$$

3. Integrating equation (5.23), we have

$$\int_{x^-}^{x^+} [G''(t|x)]\, dt \;=\; \int_{x^-}^{x^+} [\delta(t-x)]\, dt. \tag{5.27}$$

Since $G(t|x)$ is continuous and $G''(t|x)$ has a Dirac delta function type of singularity, thus, $G'(t|x)$ has a jump discontinuity. Hence,

$$\begin{aligned} \textstyle\int_{x^-}^{x^+} G''(t|x)\, ds &= \textstyle\int_{x^-}^{x^+} [\delta(t-x)]\, ds \\ [G'(t|x)]_{x^-}^{x^+} &= [u(t-x)]_{x^-}^{x^+} \\ a_2 &= 1. \end{aligned} \tag{5.28}$$

From (5.26) and (5.28), we obtain

$$a_2 \;=\; 1, \quad b_2 \;=\; -x. \tag{5.29}$$

Therefore,

$$G(t|x) \;=\; \begin{cases} 0, & 0 < t < x, \\ t \,-\, x, & t > x. \end{cases} \tag{5.30}$$

Then for $\lambda \;=\; 1$, we get

$$\begin{aligned} u_p(t) &= \int_0^t G(t|x) f(x) dx \\ &= \int_0^t (t-x)(e^x) dx \\ &= (-x+t+1)e^x. \end{aligned} \tag{5.31}$$

### 5.3.2 Boundary Value Problems for Second Order Equations

**Example 5.2.** *Construct a Green's function for the following BVP*

$$K[u] \;=\; y''(t) \;=\; t^3, \tag{5.32}$$

subject to the BCs

$$y(0) \;=\; y(1) \;=\; 0. \tag{5.33}$$

The solution to the corresponding homogeneous equation of (5.32), $y'' = 0$, is

$$y_h(t) \;=\; At \,+\, B. \tag{5.34}$$

The Green's function satisfies

$$G''(t|x) \;=\; \delta(t-x), \quad G(0|x) \;=\; G(1|x) \;=\; 0. \tag{5.35}$$

Then, the Green's function is given by

$$G(t|x) \;=\; \begin{cases} A_1 t \,+\, B_1, & 0 < t < x, \\ A_2 t \,+\, B_2, & x < t < 1. \end{cases} \tag{5.36}$$

The constructs are found as follows:

1. Applying the first boundary condition $G(0|x)$ for the first interval $0 < t < x < 1$, we get $B_1 \,=\, 0$, then the second boundary condition $G(1|x)$ for the second interval $0 < x < t < 1$, we have

   $$G(1|x) \;=\; A_2 \,+\, B_2 \;=\; 0. \tag{5.37}$$

   Thus,

   $$G(t|x) \;=\; \begin{cases} A_1 t, & 0 < t < x, \\ A_2 t \,+\, b_2, & x < t < 1. \end{cases} \tag{5.38}$$

2. $G(t|x)$ is continuous at $t = x$, then we get

   $$G(t|x)_{t \to x^-} \;=\; G(t|x)_{t \to x^+}, \tag{5.39}$$

   or equivalently,

   $$A_1\, x \;=\; a_2\, x \,+\, b_2. \tag{5.40}$$

3. By the jump discontinuity condition, we have

   $$\begin{aligned} \textstyle\int_{x^-}^{x^+} G''(t|x)\, ds \;&=\; \textstyle\int_{x^-}^{x^+} [\delta(t-x)]\, ds \\ [G'(t|x)]_{x^-}^{x^+} \;&=\; [u(t-x)]_{x^-}^{x^+} \\ A_2 \,-\, A_1 \;&=\; 1. \end{aligned} \tag{5.41}$$

From (5.40) and (5.41), we obtain

$$A_2 \,=\, x, \quad B_2 \,=\, -x, \quad A_1 \,=\, x - 1. \tag{5.42}$$

Thus,

$$G(t|x) \;=\; \begin{cases} (x-1)\, t, & 0 < t < x, \\ x\,(t-1), & x < t < 1. \end{cases} \tag{5.43}$$

Then, the particular solution is found as follows:

$$\begin{aligned} u_p(t) \;&=\; \int_0^1 G(t|x) f(x) dx \\ &=\; \int_0^t (x-1)\, t\, x^3\, dx + \int_t^1 x\,(t-1)\, x^3\, dx \\ &=\; \left( \frac{-1}{20}\, t^5 \,+\, \frac{1}{5}\, t \,-\, \frac{1}{5} \right). \end{aligned} \tag{5.44}$$

### 5.3.3 Solution of Fractional Order Differential Equations

In this subsection, we use the Laplace transform property for fractional derivative to easily construct the Green's function.

We consider in the next example the Ray tracing equation with its generalization to order $\alpha$.

**Example 5.3.**

$$D^{\alpha}y(t) \;=\; \Lambda_1\, e^{-\beta\, y(t)} - \Lambda_2\, e^{-2\beta\, y(t)}, \tag{5.45}$$

with the initial conditions

$$y(0) = 0.001, \quad y'(0) = 1.976. \tag{5.46}$$

where $1 < \alpha \leq 2$.

The Green's function $G(t|x)$ is defined as the solution to

$$D^{\alpha}G(t|x) \;=\; \delta(t-x), \tag{5.47}$$

where $0 < t < 1,$ and

$$G(0|x) \;=\; G'(0|x) \;=\; 0. \tag{5.48}$$

Next, we will use the Laplace transform property for fractional derivative to construct the Green's function. Applying the Laplace transform on both sides of equation (5.47), results:

$$\frac{s^2\mathcal{L}[G(t|x)] - s\mathcal{L}G(0|x) - G'(0|x)}{s^{2-\alpha}} \;=\; e^{-sx}. \tag{5.49}$$

We now solve equation (5.49) for $\mathcal{L}[G(t|x)]$ and upon using the homogeneous initial conditions given in (5.48), we get

$$\begin{aligned} \mathcal{L}[G(t|x)] \;&=\; \tfrac{1}{s^2}\,[s^{2-\alpha}\, e^{-sx}], \\ &=\; s^{-\alpha}\, e^{-sx}. \end{aligned} \tag{5.50}$$

By operating with the inverse Laplace transform results

$$G(t|x) \;=\; \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}\, u(t-x), \tag{5.51}$$

or equivalently,

$$G(t|x) \;=\; \begin{cases} 0, & 0 < t < x, \\ \dfrac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}, & t > x. \end{cases} \tag{5.52}$$

However, the particular solution subject to the initial conditions (5.46) is given by

$$u_p = \int_t^a G(t|x)\, f(x, u_p, u_p', u_p^\alpha)\, dx, \tag{5.53}$$

where the value $a$ can be infinite.

## 5.4 Picard's Green's Method (PGM)

In this section, we present the Picard's fixed point iterative method for finding approximate solutions for fractional Ray tracing problem (5.45) and (5.46) which is given by

$$y_{n+1} = y_n + \int_0^t \left[\frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}\right] \left[D^\alpha y_n(t) - \Lambda_1\, e^{-\beta\, y_n(t)} + \Lambda_2\, e^{-2\beta\, y_n(t)}\right] dx. \tag{5.54}$$

The starting approximate solution $y_0(t)$ is chosen to be the solution of $D^\alpha y(t) = 0$, and subject to the specified initial conditions given in (5.46). It follows that the initial iterate $y_0 = 0.001$, for any value of $\alpha$.

## 5.5 Mann's Green's Method (MGM)

There are various iteration strategies for approximating fixed point equations of different classes. Some of them are Picard's Green's iterative technique, Krasnosel'skii Green's iterative technique and Mann's Green's iterative technique. In this section, instead of Picard's, we will apply Mann's fixed point iteration method, given as

$$y_{n+1} = (1-\alpha)\, y_n + \delta_n\, K[y_n], \qquad n \geq 0, \tag{5.55}$$

where the operator $K[y_n]$ is

$$K[y_n] = y_n + \int_0^t \left[\frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}\right] \left[D^\alpha y_n(t) - \Lambda_1\, e^{-\beta\, y_n(t)} + \Lambda_2\, e^{-2\beta\, y_n(t)}\right] dx. \tag{5.56}$$

Here, $\delta_n$ is a sequence between $0$ and $1$. After a simple manipulation we get the following iterative procedure which we call (MGM):

$$y_{n+1} = y_n + \delta_n \int_0^t \left[\frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}\right] \left[D^\alpha y_n(x) - \Lambda_1\, e^{-\beta\, y_n(x)} + \Lambda_2\, e^{-2\beta\, y_n(x)}\right] dx. \tag{5.57}$$

Note that, in equation (5.57) the sign before the sequence $\alpha_n$ could be either plus or minus, depending on whether the operator $K[y_n]$ is self-adjoint or not. However, The main advantage of Mann's iterative approach is that it can overcome the divergence of Picard's scheme in many cases and, more importantly, speeds the convergence rate if the sequence $\delta_n$ is selected appropriately. Obviously, for the special case $\delta_n = 1$, Mann's reduces to Picard's iterative method.

## 5.6 Numerical Results

In this section, we present the approximate numerical solution using the proposed method for the fractional-order nonlinear initial-value problem (IVP) to illustrate the applicability of the proposed iterative method.

**Example 5.4.**

$$D^{\alpha}y(t) \ = \ \Lambda_1\, e^{-\beta\, y(t)} - \Lambda_2\, e^{-2\beta\, y(t)}, \tag{5.58}$$

*subject to the initial conditions*

$$y(0) = 0.001, \quad y'(0) = 1.976, \tag{5.59}$$

where $\beta, \Lambda_1, \Lambda_2$ are constants, $t \in [0, 10]$ and $1 < \alpha \leq 2$.

We solve this problem for integer and non-integer orders using the Green's function strategy with the fixed point iterative schemes.

**Case 1.** $\alpha = 2$.

Using $n = 7$, we obtain the approximate solution of this latter problem. The numerical solution and the residual error together with a comparison with the 5-terms multi-step differential transform method (MsDTM) [45] are reported in Table $5.1$.

Table 5.1: The approximate solution obtained by the proposed Green's function scheme with a comparison with that obtained by MsDTM for Example 5.4 case 1, with $\beta = 0.91, \Lambda_1 = -0.0254072, \Lambda_2 = -0.000091$ and $\alpha = 2$.

| $t$ | *Present Method* | *Present Method Err* | *MsDTM Err* |
|---|---|---|---|
| 1.00 | 2.11297838455333861418453 | 1.655498444579062279447622(−09) | 2.00000000(−9) |
| 2.00 | 4.41177745758305146668954 | 5.369536577403211523139911(−10) | 1.00000000(−9) |
| 3.00 | 6.74514053341264095148043 | 9.683594133499873540858583(−11) | 3.00000000(−9) |
| 4.00 | 9.08284353389544610661418 | 1.519265421692784113914961(−11) | 1.30000000(−8) |
| 5.00 | 11.4210666779384043864905 | 2.239707123157328089637831(−12) | 4.20000000(−8) |
| 6.00 | 13.7593518144805831737710 | 3.178830529009792069351348(−13) | 7.00000000(−8) |
| 7.00 | 16.0976443345855809751308 | 4.394739783605040672722254(−14) | 6.00000000(−8) |
| 8.00 | 18.4359377340336996393027 | 5.959047772667003375196744(−15) | 6.00000000(−8) |
| 9.00 | 20.7742312382059253430505 | 7.960496197398294732923219(−16) | 5.00000000(−8) |
| 10.0 | 23.1125247547545547903954 | 1.294109600912551070424929(−07) | 5.00000000(−8) |

Table 5.1 confirms that the Green's function method is rapidly convergent using only few iterates. However, we can see that the proposed method can overcome the deficiency problem as we move to the right end point of the interval.

**Case 2.** $\alpha = 1.95$.

We used only 7 iterations to obtain the approximate solution $y_{approx}(t)$ with $\beta = 0.91, \Lambda_1 = -0.0254072, \Lambda_2 = -0.000091$. The numerical solution with the residual error are reported in Table 5.2.

Table 5.2: The approximate solution obtained by the presented Green's function strategy with a comparison with the residual error for Example 5.4 case 2, with $\beta = 0.91, \Lambda_1 = -0.0254072, \Lambda_2 = -0.000091$ and $\alpha = 1.95$.

| $t$ | *Present Method* | *Present Method Err* |
|---|---|---|
| 1.00 | 2.1194028800254012259025686 | 1.97067481507557236167817437(−6) |
| 2.00 | 4.4124052843171405437828668 | 3.82611246944468075414588628(−7) |
| 3.00 | 6.7254912499103336681197205 | 2.73698124947797347646509634(−7) |
| 4.00 | 9.0345565324819456712500206 | 3.26334084140363570562794042(−7) |
| 5.00 | 11.338711991619693077627289 | 2.84623188457417196155669501(−7) |
| 6.00 | 13.638944416738094881932817 | 2.42536747774916750527600879(−7) |
| 7.00 | 15.936035408573295293236597 | 2.098915750556587603974304048(−7) |
| 8.00 | 18.230523803580345930957807 | 1.848817512137924094583563794(−7) |
| 9.00 | 20.522793593488948035283581 | 1.652656020967472081873403123(−7) |
| 10.0 | 22.813131107031926046849895 | 7.678830798884127491050760990(−6) |

# Chapter 6: Conclusions and Future Work

This thesis was divided into five chapters in which we implemented two numerical methods to obtain numerical solutions for a wide class of boundary and initial value problems. In the second chapter, we have discussed some important mathematical definitions to be used in this work, among them: The Gamma function, the Laplace transforms, the Error function and the Mittag-Leffler function.

In the third chapter, we introduced fractional derivatives and integrals with their historical development. We have determined the generalized forms for fractional derivatives and fractional integrals, some basic properties and the important formulas, that have been stimulated by some examples.

In the fourth chapter, we implemented the Laplace decomposition method for fractional DEs, that is a combination of Laplace transform and the decomposition method. This is the first time that this approach is applied successfully to tackle fractional differential equations. This strategy is performed on various well-known differential equation problems with arbitrary order derivative. On the other hand, we observed that the technique is highly accurate locally, namely close to the initial point in the domain. However, the setback is that the error worsens when we move away from the left endpoint towards the right endpoint. In order to overcome this setback, a domain decomposition strategy is applied to improve the error away from the left endpoint of the domain. Moreover, we include the convergence of the method.

Finally, in the fifth chapter, we introduced an efficient iterative algorithm for obtaining numerical solutions for fractional linear and nonlinear boundary and initial value problems by applying well-known fixed point iterative processes, such as Krasnoselskii-Mann's and Picard's, to a customized linear integral operator that is expressed in terms of the Green's functions corresponding to the linear differential term of the equation. The proposed strategy is employed to obtain highly accurate solution for fractional differential equation problems and to overcome the deterioration of the error as we move away from the left endpoint of the interval or as the domain becomes large. The convergence of the proposed technique has been studied.

In future research, we will explore other iterative methods to achieve faster rate of convergence. In our approach we used Mann's and Picard's fixed point iterative schemes; therefore, in future work, we will investigate other techniques such as Ishikawa hybrid iterative method. Moreover, the ultimate aim is to publish all this work in a respected international journal.

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# Vita

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