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Path Independence of Exotic Options and Convergence of Binomial Approximations

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Abstract

The analysis of the convergence of tree methods for pricing barrier and lookback options has been the subject of numerous publications aiming at describing, quantifying, and improving the slow and oscillatory convergence in such methods. For barrier and lookback options, we find *path-independent options* whose price is exactly that of the original path-dependent option. The usual binomial models converge at a speed of order $1/\sqrt{n}$ to the Black-Scholes price. Our new path-independent approach yields convergence of order $1/n$. Furthermore, we derive a closed form formula for the coefficient of $1/n$ in the expansion of the error of our *path-independent pricing* when the underlying is approximated by the Cox, Ross, and Rubinstein (CRR) model. Using this we obtain a corrected model with a convergence of order $n^{-3/2}$ to the price of barrier and lookback options in the Black-Scholes model. Our results are supported and illustrated by numerical examples.

Key words. Black-Scholes, exotic, barrier, lookback, binomial, path dependence, convergence

1 Introduction

In this paper, we study barrier and lookback options. We assume the stock price evolves as in the Black-Scholes model and use the following notations: S_0 as the initial stock price, K as the strike price, r as the continuously compounded interest rate, σ as the volatility, T as the time to maturity and B as the barrier. We show that, for barrier and lookback options, there are *path-independent options* whose price is exactly that of the original path-dependent option. Certainly for barrier options this is not a new observation. According to page 188 in Björk (2009) (see also Kennedy (2016)), the price at time $t = 0$ of an up and out option with payoff $\phi(S_T)$ and barrier B is

$$e^{-rT}E(G(S_T)),$$

where

$$G(S_T) = \begin{cases} \phi(S_T) & (S_0^2/B \le S_T < B) \\ \phi(S_T) - (B/S_0)^{2r/\sigma^2-1}\max\{B^2S_T/S_0^2 - K, 0\} & (S_T < S_0^2/B) \end{cases}. \tag{1}$$

and 0 otherwise. Figure 1 shows a graph of the function G for an up and out call option with strike $K = 105$, barrier $B = 120$, when $S_0 = 100$, $r = 0.05$, $\sigma = 0.2$, $T = 1$. Note that the function is discontinuous and can be negative.

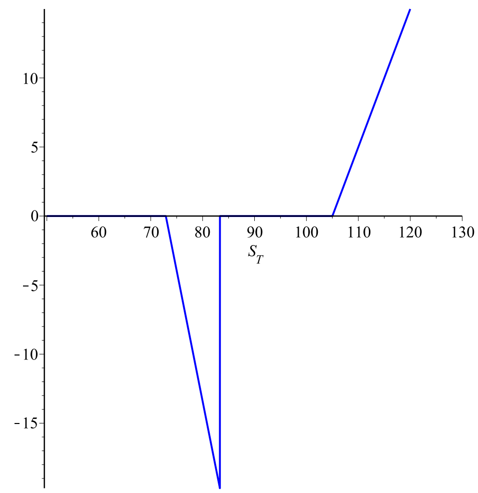

Figure 1: This is Björk's payoff function $G(S_T)$ for an up and out call option with strike $K = 105$, barrier $B = 120$, when $S_0 = 100$, $r = 0.05$, $\sigma = 0.2$, $T = 1$. Note that this function is discontinuous and takes negative values.

However, what we show here is that there are better choices for $G(S_T)$, better in the sense that computing $e^{-rT}E(G(S_T))$ using the binomial method leads to a faster rate of convergence to the Black-Scholes price than the usual method.

In Lin and Palmer (2013), it was shown that the difference in the binomial price and Black-Scholes price for a barrier option is $c_n/\sqrt{n} + O(1/n)$, n being the number of periods. Note in general here c_n depends on n, that is, the convergence is not smooth and the binomial price oscillates as $n \to \infty$. In Section 2 we show that $G(S_T)$ can be chosen in such a way that when the barrier option price is calculated as $e^{-rT}E(G(S_T))$ using the CRR binomial model, then the difference is $c_n/n + O(1/n^{3/2})$. Using the results in Leduc (2013), we are also able to calculate c_n. In this case the convergence is not smooth because $G(S_T)$ is only continuous, not differentiable.

In Heuwelyckx (2014), it was shown that the difference in the binomial price and Black-Scholes price for a lookback option with floating strike is $c/\sqrt{n} + O(1/n)$, n being the number of periods. Here c is a constant and so the convergence is smooth. In Section 3 we show that $G(S_T)$ can be chosen in such a way that when the lookback option (with floating or fixed strike) price is calculated as $e^{-rT}E(G(S_T))$ using the CRR binomial model, then the difference is $c/n + O(1/n^{3/2})$. Using the results in Leduc (2013), we are also able to calculate c. In this case the convergence is smooth because $G(S_T)$ is differentiable.

Our general approach to finding G is as follows. Let V_T be the payoff to an option, which may be path dependent. Then, taking the expected value with respect to the risk neutral measure, its price at time $t = 0$ is

$$e^{-rT}E(V_T) = e^{-rT}E(E(V_T|S_T)) = e^{-rT}E(G(S_T))$$

where

$$G(S_T) = E(V_T|S_T).$$

We call this the *conditional expectation approach*. In Sections 2 and 3 we also describe an *alternative approach* which follows more directly from Carbone (2004) and leads to a different G.

The analysis of the convergence of tree methods for barrier and lookback options has been the subject of numerous scholarly works. Early papers include Boyle and Lau (1994) and Derman et al. (1995) where techniques to enhance the convergence for barrier options evaluated with the binomial model are discussed. Cheuk and Vorst (1997) (see also Babbs (1992, 2000)) describe a simple way to calculate the binomial price of lookback options. Embedding binomial trees in a Brownian motion with drift, Rogers and Stapleton (1997) develop an accelerated binomial method. Lyuu (1998) uses combinatorial methods to develop efficient pricing for barrier options in the binomial and trinomial models. Broadie et al. (1999) suggest the *enhanced* trinomial method to obtain a *faster* convergence to discretely and continuously monitored barrier and lookback options. Dai (2000) suggests a modified version of the classical binomial tree scheme to ameliorate its slow convergence. Carbone (2004) studies the rate of convergence of the binomial method for barrier and lookback options. Gaudenzi and Lepellere (2006) improve the efficiency of the standard binomial method using interpolations of the lattice values. The bino-trinomial trees of Dai and Lyuu (2010) improve convergence of double barrier options by insuring that barriers fall exactly on nodes of the tree. Lin and Palmer (2013) provide an explicit formula for the coefficient of $1/\sqrt{n}$ and $1/n$ in the expansion of the error of barrier options in the Cox, Ross and Rubinstein (CRR) model. Heuwelyckx (2014) extends the result to lookback options. Grosse-Erdmann and Heuwelyckx (2016) generalize Heuwelyckx (2014) to any given time after emission. Appolloni et al. (2014) introduced the binomial interpolated lattice approach to deal with the 'near barrier' problem and improve the convergence of American option prices. Bock and Korn (2016) use Edgeworth expansions to construct a fast converging binomial tree for vanilla and barrier options.

As can be seen from the literature survey above, many methods have been used to improve the convergence of the binomial method. The work done in this article differs significantly from these papers. In particular, the conditional expectation approach, to our knowledge, has never been used before for any path-dependent option. It is simple and could in principle work for any path-dependent option (without the need of any deep understanding of the option) because it is a purely mathematical phenomenon. Our new *path-independent approach* yields convergence of order $1/n$, whereas the usual binomial models converge at a speed of only order $1/\sqrt{n}$ to the Black-Scholes price. Moreover we derive a closed form formula for the coefficient of $1/n$ in the expansion of the error of our *path-independent pricing*, which has never been done before, and using this, we can obtain a corrected model with a convergence of order $1/n^{3/2}$ to the price of barrier and lookback options in the Black-Scholes model. Note that for barrier options the error expansion for the CRR binomial price is known but for lookbacks only in the floating strike case which, as we point out below, is just a special case of the fixed strike case. The CRR binomial prices of lookback options with fixed strike appear to converge at a rate of $1/\sqrt{n}$ to their Black-Scholes limits. However, to the best of our knowledge this has never been proved. This paper is the first to develop an error expansion for a binomial price of *fixed strike* lookback options. For our path independent pricing, we establish a convergence of order $1/n$ and indeed of order $1/n^{3/2}$ in our corrected model. Moreover, our results remain valid for a broad family of tree models, while papers published so far were focussed on the CRR and a couple of other binomial models.

2 Barrier Options

In this section we study barrier options. First we calculate a function G using the conditional expectation approach. Next we describe an alternative approach. Then, by way of an example, we compare the prices given by the standard binomial method for barrier options with those given by using $G(S_T)$, where G is as in (1), then as in the conditional expectation approach and finally as in the alternative approach. Finally we consider G found by the conditional expectation approach and use the method given in Leduc (2013) to calculate the coefficient of $1/n$ in the error.

2.1 Conditional Expectation Approach

Consider an up and out option with payoff $\phi(S_T)$ and barrier B where $B > S_0$. Then, following the treatment in Shreve (2004), the payoff to this barrier option is

$$V_T = \phi(S_T)\mathbf{1}_{\{\max_{0\le t\le T} S_t < B\}} = \phi(S_T)\mathbf{1}_{\{\widehat{M}_T < b\}},$$

where

$$\alpha = r/\sigma - \sigma/2, \quad \widehat{W}_t = \alpha t + W_t, \quad \widehat{M}_T = \max_{0\le t\le T} \widehat{W}_t, \quad S_T = S_0 e^{\sigma \widehat{W}_T}, \quad b = \log(B/S_0)/\sigma > 0,$$

and W_t is a Brownian motion. We calculate the conditional expectation $G(S_T) = E(V_T|S_T)$. Note that

$$G(y) = E\left(\phi(S_T)\mathbf{1}_{\{\widehat{M}_T < b\}} \middle| S_T = y\right) = E\left(\phi(S_T)\mathbf{1}_{\{\widehat{M}_T < b\}} \middle| \widehat{W}_T = \log(y/S_0)/\sigma\right) = \phi(y)g(x),$$

where $x = \log(y/S_0)/\sigma$ and

$$g(x) = E\left(\mathbf{1}_{\{\widehat{M}_T < b\}} \middle| \widehat{W}_T = x\right).$$

Now $g(x) = 0$ when $x \geq b$ since $x \geq b$ implies that $\widehat{M}_T \geq \widehat{W}_T = x \geq b$.

Now note that the conditional density of $\widehat{M}_T$ is

$$f_{\widehat{M}_T}(a|\widehat{W}_T = x) = \begin{cases} \dfrac{2(2a-x)}{T} e^{-2(a^2-ax)/T} & (a \geq x^+) \\ 0 & (a < x^+), \end{cases} \tag{2}$$

since $\widehat{W}_T$ is a normal random variable with expectation αT and variance T and, according to (Shreve, 2004, Th. 7.2.1), the joint density of $\left(\widehat{M}_T, \widehat{W}_T\right)$ is

$$f_{\widehat{M}_T, \widehat{W}_T}(a, x) = \frac{2(2a-x)}{T\sqrt{2\pi T}} e^{\alpha x - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2a-x)^2}, \tag{3}$$

when $a \geq 0$, $x \leq a$ and 0 otherwise. Then if $x < b$,

$$E(\mathbf{1}_{\{\widehat{M}_T < b\}}|\widehat{W}_T = x) = \int_{x^+}^{b} \frac{2(2a-x)}{T} e^{-2(a^2-ax)/T} da = 1 - e^{-2b(b-x)/T}.$$

So

$$g(x) = \begin{cases} 1 - e^{-2b(b-x)/T} & (x < b) \\ 0 & (x \geq b) \end{cases}.$$

Then

$$G(S_T) = \phi(S_T) g(\log(S_T/S_0)/\sigma) = \begin{cases} \phi(S_T)\left[1 - \left(\frac{S_T}{B}\right)^{\beta}\right] & (S_T < B) \\ 0 & (S_T \geq B), \end{cases} \tag{4}$$

where

$$\beta = \frac{2}{\sigma^2 T} \log(B/S_0).$$

Example: Consider an up and out call option with strike $K = 105$, maturity $T = 1$ and barrier $B = 120$. Suppose the interest rate is $r = 0.05$, the volatility $\sigma = 0.2$ and the stock price at $t = 0$ is $S_0 = 100$. Then the function G is

$$G(S_T) = \begin{cases} \max\{S_T - 105, 0\}\left[1 - \left(\dfrac{S_T}{120}\right)^{50\log(1.2)}\right] & (S_T < 120) \\ 0 & (S_T \geq 120). \end{cases}$$

A graph of this is shown in Figure 2.

Remark: For a down and out option with payoff $\phi(S_T)$ and barrier $B < S_0$, the corresponding function is

$$G(S_T) = \begin{cases} \phi(S_T)\left[1 - \left(\frac{S_T}{B}\right)^{\beta}\right] & (S_T > B) \\ 0 & (S_T \leq B), \end{cases}$$

where $\beta = \frac{2}{\sigma^2 T}\log(B/S_0)$ and the price at $t = 0$ is $e^{-rT}E(G(S_T))$. To obtain G for an "in" option, we just subtract the G for the corresponding "out" option from $\phi(S_T)$.

2.2 Alternative approach

Here is an alternative approach along the lines of Carbone (2004). Consider an up and out call with strike K and barrier B where we can assume $K < B$ and $S_0 < B$ since otherwise the value of the option is 0. As in Shreve (2004), the payoff of the option is

$$V_T = (S_0 e^{\sigma \widehat{W}_T} - K)\mathbf{1}_{\{\widehat{W}_T \geq k, \widehat{M}_T \leq b\}},$$

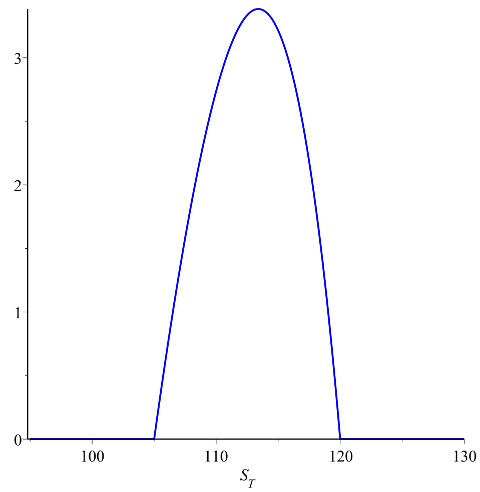

Figure 2: The payoff function under the conditional expectation approach for an up and out call option with strike $K = 105$, maturity $T = 1$ and barrier $B = 120$ when the interest rate is $r = 0.05$, the volatility $\sigma = 0.2$ and the stock price at $t = 0$ is $S_0 = 100$.

where

$$\alpha = r/\sigma - \sigma/2, \quad \widehat{W}_t = \alpha t + W_t, \quad \widehat{M}_T = \max_{0 \le t \le T} \widehat{W}_t, \quad k = \frac{1}{\sigma} \log \frac{K}{S_0}, \quad b = \frac{1}{\sigma} \log \frac{B}{S_0},$$

where W_t is a Brownian motion. Recalling from (3) the joint density of $(\widehat{M}_T, \widehat{W}_T)$, we see that the time $t = 0$ price of the option is

$$\begin{aligned} V_0 &= e^{-rT} E(V_T) \\ &= \int_k^b \int_{x^+}^b e^{-rT} (S_0 e^{\sigma x} - K) \frac{2(2a - x)}{T\sqrt{2\pi T}} e^{\alpha x - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2a-x)^2} da dx. \end{aligned}$$

If we make the change of variable $z = (2a - x)/\sqrt{T}$ in the inner integral, this becomes

$$\int_k^b \int_{|x|/\sqrt{T}}^{(2b-x)/\sqrt{T}} e^{-rT} (S_0 e^{\sigma x} - K) \frac{z}{\sqrt{2\pi T}} e^{\alpha x - \alpha^2 T/2 - z^2/2} dz dx.$$

Then we swap the order of integration to get

$$\int_{k^+/\sqrt{T}}^{(2b-k)/\sqrt{T}} \int_{h(z)}^{g(z)} e^{-rT} (S_0 e^{\sigma x} - K) \frac{z}{\sqrt{2\pi T}} e^{\alpha x - \alpha^2 T/2 - z^2/2} dx dz,$$

where

$$g(z) = \min\{\sqrt{T} z, 2b - \sqrt{T} z\}, \quad h(z) = \max\{k, -\sqrt{T} z\}.$$

This can be written as

$$\frac{e^{-rT - \alpha^2 T/2}}{\sqrt{T}} \int_{k^+/\sqrt{T}}^{(2b-k)/\sqrt{T}} z \int_{h(z)}^{g(z)} (S_0 e^{(\sigma+\alpha)x} - K e^{\alpha x}) dx \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz = e^{-rT} \int_{k^+/\sqrt{T}}^{(2b-k)/\sqrt{T}} H(z) \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz,$$

where

$$H(z) = \frac{e^{-\alpha^2 T/2}}{\sqrt{T}} z \int_{h(z)}^{g(z)} (S_0 e^{(\sigma+\alpha)x} - K e^{\alpha x}) dx.$$

Since $S_T = S_0 e^{(r - \sigma^2/2)T + \sigma\sqrt{T} Z}$, where Z is a standard normal variable, it follows that the $t = 0$ price of the option is

$$e^{-rT} E(G(S_T)),$$

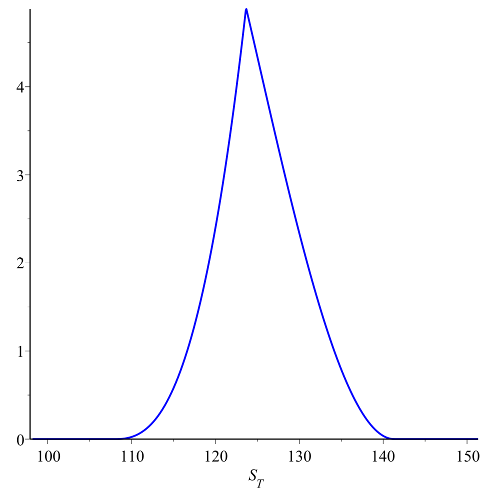

Figure 3: The payoff function under the alternative approach for an up and out call option with strike $K = 105$, maturity $T = 1$ and barrier $B = 120$ when the interest rate is $r = 0.05$, the volatility $\sigma = 0.2$ and the stock price at $t = 0$ is $S_0 = 100$.

where

$$G(S_T) = H\left(\frac{\log(S_T/S_0) - (r-\sigma^2/2)T}{\sigma\sqrt{T}}\right),$$

when $a_1 \le S_T \le a_2$ with $a_1 = \max\{K, S_0\}e^{(r-\sigma^2/2)T}$ and $a_2 = \frac{B^2}{K}e^{(r-\sigma^2/2)T}$, and 0 otherwise.

Integrating, we find that

$$H(z) = \frac{e^{-\alpha^2 T/2}}{\sqrt{T}} z \left[\frac{S_0}{\sigma+\alpha}(e^{(\sigma+\alpha)g(z)} - e^{(\sigma+\alpha)h(z)}) - \frac{K}{\alpha}(e^{\alpha g(z)} - e^{\alpha h(z)})\right],$$

where $\frac{K}{\alpha}(e^{\alpha g(z)} - e^{\alpha h(z)})$ is interpreted as $K(g(z) - h(z))$ when $\alpha = 0$. With

$$z = d(S_T/S_0) = \frac{\log(S_T/S_0) - (r-\sigma^2/2)T}{\sigma\sqrt{T}},$$

we see that

$$g(z) = \frac{1}{\sigma}\min\left\{\log\left(\frac{S_T}{S_0}\right) - (r-\sigma^2/2)T, \log\left(\frac{B^2}{S_T S_0}\right) + (r-\sigma^2/2)T\right\}$$

so that

$$e^{g(z)} = p(S_T)^{1/\sigma}, \quad \text{where } p(S_T) = \min\left\{\frac{S_T e^{-(r-\sigma^2/2)T}}{S_0}, \frac{B^2 e^{(r-\sigma^2/2)T}}{S_T S_0}\right\}.$$

Next we see that

$$h(z) = \frac{1}{\sigma}\max\left\{\log\left(\frac{K}{S_0}\right), -\log\left(\frac{S_T}{S_0}\right) + (r-\sigma^2/2)T\right\}$$

so that

$$e^{h(z)} = q(S_T)^{1/\sigma}, \quad \text{where } q(S_T) = \max\left\{\frac{K}{S_0}, \frac{S_0 e^{(r-\sigma^2/2)T}}{S_T}\right\}.$$

Then for $\max\{K, S_0\}e^{(r-\sigma^2/2)T} \le S_T \le \frac{B^2}{K}e^{(r-\sigma^2/2)T}$, we have

$$\begin{aligned} G(S_T) &= \frac{e^{-\alpha^2 T/2}}{\sqrt{T}} d(S_T/S_0)\left[\frac{2\sigma S_0}{2r+\sigma^2}\left(p(S_T)^{r/\sigma^2+1/2} - q(S_T)^{r/\sigma^2+1/2}\right)\right. \\ &\qquad \left. - \frac{2\sigma K}{2r-\sigma^2}\left(p(S_T)^{r/\sigma^2-1/2} - q(S_T)^{r/\sigma^2-1/2}\right)\right], \end{aligned} \tag{5}$$

where $\frac{2\sigma K}{2r-\sigma^2}\left(p(S_T)^{r/\sigma^2-1/2} - q(S_T)^{r/\sigma^2-1/2}\right)$ is interpreted as $\frac{K}{\sigma}\log(p(S_T)/q(S_T))$ when $\sigma^2 = 2r$; otherwise $G(S_T) = 0$.

Figure 3 shows a graph of the function $G(S_T)$ when $S_0 = 100, K = 105, B = 120, r = 0.05, \sigma = 0.2, T = 1$.

2.3 Example

We now give an illustration of the convergence under the binomial method (note we always use the CRR binomial model) when a barrier option is transformed into a path-independent option by an appropriate choice of the payoff function G. We consider an up and out call option with strike $K = 105$, maturity $T = 1$ and barrier $B = 120$ when the stock price at $t = 0$ is $S_0 = 100$, the interest rate is $r = 0.05$ and the volatility $\sigma = 0.2$. The Black-Scholes price at $t = 0$ of the up and out call barrier option is 0.506751. We compute the price under the *traditional CRR* approach for barrier options, as a path-independent option using *Björk's payoff* (1), under the *conditional* approach (4), and under the *alternative* approach (5). In the table below, n is the number of periods and "error" denotes the difference between the binomial price and Black-Scholes price.

n	CRR	$\sqrt{n}$(error)	Björk	$\sqrt{n}$(error)	conditional	n(error)	alternative	n(error)
1000	0.520387	0.431	0.554034	1.50	0.507436	0.685	0.506837	0.0854
2000	0.520162	0.600	0.551815	2.02	0.507739	1.97	0.506602	-0.298
3000	0.508999	0.123	0.361436	-7.96	0.506236	-1.55	0.507296	1.64
4000	0.519242	0.790	0.418624	-5.57	0.506748	-0.00128	0.506476	-1.10
5000	0.525514	1.33	0.571973	4.61	0.506933	0.910	0.506517	-1.17
6000	0.519082	0.955	0.549637	3.32	0.507016	1.59	0.506936	1.11
7000	0.527950	1.77	0.580930	6.21	0.506631	-0.839	0.506599	-1.06
8000	0.518720	1.07	0.455417	-4.59	0.506648	-0.823	0.506604	-1.18
9000	0.520073	1.26	0.553379	4.42	0.506875	1.11	0.506655	-0.868
10000	0.526712	2.00	0.493195	-1.36	0.506856	1.05	0.506906	1.55

We see that the convergence is of order $1/\sqrt{n}$ and $\sqrt{n}$(error) exhibits bounded oscillations for the CRR and Björk approaches. For the CRR method this is consistent with Lin and Palmer (2013) and for the Björk method, in view of Leduc (2016), it is a consequence of the fact that the Björk payoff is not continuous. On the other hand, under the conditional and alternative approaches, the payoff is continuous and only piecewise differentiable so that we observe, as expected, a convergence of order $O(1/n)$, with n(error) exhibiting bounded oscillations. In the next section, for the conditional approach, we calculate the coefficient of $1/n$ in the error.

2.4 Calculation of coefficient of $1/n$ in the error for the up and out call when using the conditional expectation approach

For the conditional expectation approach, we observed that the error was of order $1/n$. This is expected as the payoff function is piecewise smooth continuous. The results in Leduc (2013) enable us to calculate the coefficient c_n of $1/n$ in the asymptotic expansion of the error, that is,

$$C_n - C_{BS} = \frac{c_n}{n} + O\left(\frac{1}{n^{3/2}}\right),$$

where C_n is the price calculated by the $n-$period CRR model using the conditional payoff function and C_{BS} is the Black-Scholes price. We will find that c_n is not a constant, that is, the convergence is not smooth, as expected because the payoff function is not differentiable.

As found in subsection 2.1, the up and out call has the same value as the option with terminal payoff $G(S_T)$, where

$$G(S_T) = \begin{cases} 0 & (S_T \le K) \\ (S_T - K)(1 - (S_T/B)^\beta) & (K < S_T < B) \\ 0 & (S_T \ge B), \end{cases}$$

where $\beta = 2\log(B/S_0)/(\sigma^2 T)$. This is continuous but not differentiable at $S_T = K$ and $S_T = B$. However, if we take

$$\alpha_1 = 1 - (K/B)^\beta, \quad \alpha_2 = \beta(K - B)/B, \tag{6}$$

the payoff

$$J(S_T) = G(S_T) - \alpha_1 \max(S_T - K, 0) + \alpha_2 \max(S_T - B, 0) \tag{7}$$

is differentiable and then we know the coefficient of $1/n$ in the error for this payoff is a constant c. Then the coefficient of $1/n$ in the error for the original payoff G is

$$c_n = c + \alpha_1 a_n(K) - \alpha_2 a_n(B), \tag{8}$$

where $a_n(X)$ is the coefficient of $1/n$ in the error for the call with strike X. In fact, according to Chang and Palmer (2007),

$$a_n(X) = a(X) + b(X)(1 - \Delta_n^2(X)), \tag{9}$$

where

$$\begin{aligned}
d_1(X) &= \frac{\log(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2(X) = d_1(X) - \sigma\sqrt{T},\\
A(X) &= -\sigma^2 T(6 + d_1^2(X) + d_2^2(X)) + 4T(d_1^2(X) - d_2^2(X)) - 12T^2 r^2,\\
a(X) &= \frac{S_0 e^{-d_1^2(X)/2}}{24\sigma\sqrt{2\pi T}} A(X), \quad b(X) = \frac{S_0 e^{-d_1^2(X)/2}}{24\sigma\sqrt{2\pi T}} 12\sigma^2 T,\\
\Delta_n(X) &= 1 - 2\,\mathrm{frac}\left[\frac{\log(S_0/X)}{2\sigma\sqrt{T}}\sqrt{n} - \frac{n}{2}\right],
\end{aligned}$$

and $\mathrm{frac}(x) = x - \mathrm{floor}(x)$. Figure 4 compares the graph of $G(S_T)$ with the graph of $J(S_T)$. The latter is continuously differentiable because at every jump of of $G'(S_T)$, we have subtracted $\Delta G'(S_T)$ call options, where $\Delta G'(S_T)$ is the jump.

Now according to Leduc (2013), if C_n is the $n-$period CRR binomial approximation to the option with payoff $J(S_T)$ and C_{BS} is the corresponding Black-Scholes price,

$$C_n - C_{BS} = \frac{c}{n} + O\left(\frac{1}{n^{3/2}}\right),$$

where

$$c = -\left(\frac{\Delta_2 S_0^2}{2} V_0''(S_0) + \frac{\Delta_3 S_0^3}{6} V_0'''(S_0) + \frac{\Delta_4 S_0^4}{24} V_0^{(4)}(S_0)\right), \tag{10}$$

with

$$V_0(x) = e^{-rT} E(J(S_T)|S_0 = x) \quad \text{with} \quad \beta = 2\log(B/S_0)/(\sigma^2 T) \quad \text{fixed},$$

and

$$\Delta_2 = T^2(r^2 + r\sigma^2 + 5\sigma^4/12), \quad \Delta_3 = 2\sigma^2 T^2(r + \sigma^2), \quad \Delta_4 = 2\sigma^4 T^2. \tag{11}$$

Omitting the details of the proof, we calculate $V_0(x)$.

With $\beta = 2\log(B/S_0)/(\sigma^2 T)$ fixed,

$V_0(x) = e^{-rT} E(J(S_T)|S_0 = x)$

$= (1 - \alpha_1)C(x, K, r, \sigma, T) - (1 - \alpha_2)C(x, B, r, \sigma, T) + (K - B)D(x, B, r, \sigma, T)$

$-\frac{e^{\beta(r+(\beta+1)\sigma^2/2)T} x^\beta}{B^\beta}[C(x, K, r + \beta\sigma^2, \sigma, T) - C(x, B, r + \beta\sigma^2, \sigma, T) + (K - B)D(x, B, r + \beta\sigma^2, \sigma, T)],$

where $C(x, K, r, \sigma, T)$ (resp. $D(x, K, r, \sigma, T)$) is the time $t = 0$ Black-Scholes price of the call (resp. digital call) option with strike K, maturity T when the interest rate is r, the volatility σ and the $t = 0$ stock price is x.

After getting the second, third and fourth derivatives of this function with respect to x and setting $x = S_0$, we are able to calculate c. Then we calculate $a_n(K)$ and $a_n(B)$ from (9) and finally c_n from (8).

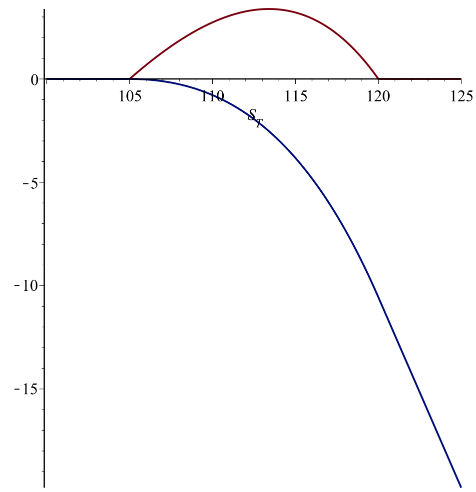

Figure 4: The graph of the smoothed function $J(S_T)$ (the lower graph) compared to the graph of the original conditional payoff function $G(S_T)$ for an up and out call option with strike $K = 105$, maturity $T = 1$ and barrier $B = 120$ when the stock price $S_0 = 100$ at $t = 0$, interest rate $r = 0.05$ and volatility $\sigma = 0.2$. Discontinuities in the derivatives of $G(S_T)$ were removed by subtracting appropriate call option payoffs.

Example: We consider the $t = 0$ price of an up and out call option with strike $K = 105$, maturity $T = 1$ and barrier $B = 120$ with stock price $S_0 = 100$ at $t = 0$, interest rate $r = 0.05$ and volatility $\sigma = 0.2$. The conditional payoff $G(S_T)$ is shown in Figure 1. We find from (6) that

$$\alpha_1 = 0.703966474488, \quad \alpha_2 = -1.13950972996.$$

The smoothed out payoff $J(S_T)$ as given in (7) is shown in Figure 4. We calculate $c = -0.0145928896929$ using (10) and, using (9) and (8),

$$\begin{aligned} c_n &= c + \alpha_1 a_n(K) - \alpha_2 a_n(B) \\ &= 2.18617073592 - 2.79267187864\,\Delta_n^2\,(K) - 3.88274748116\,\Delta_n^2\,(B)\,. \end{aligned}$$

It is the quantities $\Delta_n^2(K)$ and $\Delta_n^2(B)$ which cause the oscillation. The Black-Scholes price is $C_{BS} = 0.506751$. In the table below, n is the number of periods in the CRR binomial model, C_n is the price obtained using the CRR model with payoff $G(S_T)$, $\Delta_n(K)$ and $\Delta_n(B)$ are as in the equation following (9), c_n is the coefficient of $1/n$ in $C_n - C_{BS}$. $C_n^* = C_n - c_n/n$ is what we call the *corrected conditional approach*. We calculate

$$n^{3/2}(\text{error}) = n^{3/2}(C_n - C_{BS} - c_n/n)$$

in order to verify the correctness of c_n. As expected, it remains bounded. Also we note that C_n^* converges much faster than C_n to the Black-Scholes price.

n	C_n	C_n^*	$\Delta_n\,(K)$	$\Delta_n\,(B)$	c_n	$n^{3/2}$(error)
1000	0.507436	0.506790	0.714402	-0.172431	0.645430	1.243162
2000	0.507739	0.506761	-0.090188	-0.231661	1.955082	0.880395
3000	0.506236	0.506750	0.361737	0.930815	-1.543335	-0.117772
4000	0.506748	0.506746	0.428805	0.655139	0.006172	-1.197684
5000	0.506933	0.506757	0.249928	-0.539595	0.881216	2.070304
6000	0.507016	0.506754	-0.103651	-0.387165	1.574158	1.353329
7000	0.506631	0.506753	-0.589610	-0.729421	-0.850507	0.998497
8000	0.506648	0.506749	0.819625	0.536679	-0.808229	-1.355984
9000	0.506875	0.506753	0.143207	-0.517292	1.089909	1.920429
10000	0.506856	0.506750	-0.604918	0.160778	1.063893	-1.254161

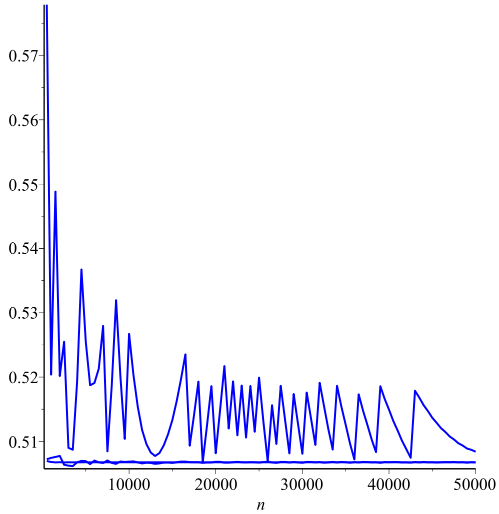

Figure 5: The prices obtained using the *conditional approach* and the *corrected conditional approach* (the two lower graphs) are almost indistinguishable from their limit, the Black-Scholes value. The convergence of the price obtained by the *traditional CRR approach* (the upper graph) is far slower.

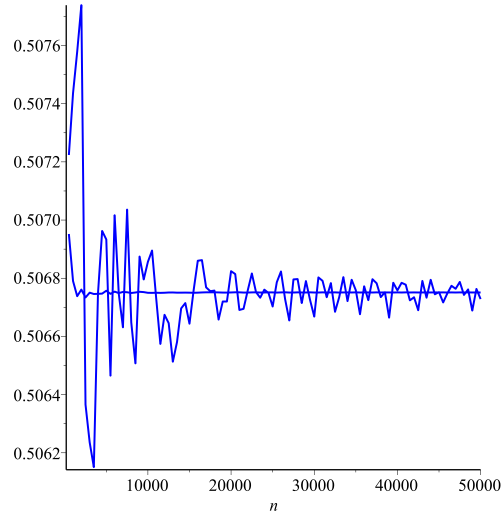

Figure 6: Subtracting the error c_n/n from the values C_n of the *conditional approach* results in a striking improvement in the convergence. Here the values obtained by the *corrected conditional approach* appears almost flat when compared to those obtained by the *conditional approach*.

Figure 5 displays the value of the option calculated using the classical CRR method, our *conditional approach* and the *corrected conditional approach*. The option values were calculated for all multiples of $n = 500$ up to a maximum of $50,000$. The conditional approach and the corrected conditional approach converge much faster than the usual CRR method. Because their error is asymptotically infinitely smaller than the error of the CRR approach, they appear almost indistinguishable from the limiting value C_{BS}. However, when compared to one another in Figure 6, it is clear that the corrected conditional approach converges far faster than the uncorrected one.

Remark: Leduc (2013) gives another way to calculate c. In fact,

$$c = -e^{-rT}[c_1 I_1 + c_2 I_2 + c_3(I_3 - I_1)],$$

where

$$c_1 = \frac{1}{2}\Delta_2 - \frac{1}{3}\Delta_3 + \frac{1}{4}\Delta_4, \quad c_2 = \frac{1}{24}\frac{4\Delta_3 - 5\Delta_4}{\sigma\sqrt{T}}, \quad c_3 = \frac{1}{24}\frac{\Delta_4}{\sigma^2 T}$$

with Δ_2, Δ_3, Δ_4 as in (11) and for $i = 0, 1, 2, 3$

$$I_i = E(S_T^2 Z^i G''(S_T)) \quad \text{with } Z = \frac{\log(S_T/S_0) - (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

It turns out that the method we used above is more convenient here.

3 Lookback options

We consider a lookback call with fixed strike K. We want to price it at time 0 but we suppose we started taking the maximum at some previous time $t_0 \leq 0$. So the payoff is

$$V_T = \max\left\{\max\left\{L, \max_{0\leq t\leq T} S_t\right\} - K, 0\right\},$$

where $L = \max_{t_0 \leq t \leq 0} S_t \geq S_0$. A put with floating strike has payoff

$$\max\left\{\max\left\{L, \max_{0\leq t\leq T} S_t\right\} - S_T, 0\right\} = \max\left\{L, \max_{0\leq t\leq T} S_t\right\} - S_T.$$

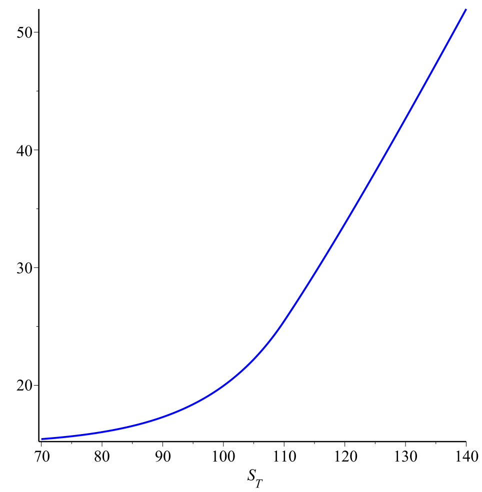

Figure 7: This is the graph of the conditional payoff function $G(S_T)$ for a lookback call option with strike $K = 95$, maturity $T = 1$ and with $L = 110$ as the maximum stock price up to time $t = 0$. The stock price at $t = 0$ is $S_0 = 100$, the interest rate $r = 0.05$ and the volatility $\sigma = 0.2$. This function is continuously differentiable.

So we include the case of a put with floating strike by taking $K = 0$ in the call with fixed strike case (and subtracting S_T). A lookback put with fixed strike K has payoff

$$V_T = \max\left\{K - \min\left\{L, \min_{0\le t\le T} S_t\right\}, 0\right\},$$

where $L = \min_{t_0\le t\le 0} S_t \le S_0$. A call with floating strike has payoff

$$\max\left\{S_T - \min\left\{L, \min_{0\le t\le T} S_t\right\}, 0\right\} = S_T - \min\left\{L, \min_{0\le t\le T} S_t\right\}.$$

So we include the case of a call with floating strike by taking $K = S_0$ in the put with fixed strike case and adding $S_T - S_0$.

Here we only consider lookback calls with fixed strike, that is,

$$V_T = \max\left\{\max\left\{L, \max_{0\le t\le T} S_t\right\} - K, 0\right\} = \max\left\{\max_{0\le t\le T} S_t - B, 0\right\} + B - K,$$

where

$$B = \max\{K, L\}.$$

3.1 Conditional Expectation Approach

First we find the payoff using the conditional expectation approach. So, for every $y > 0$, we calculate

$$G(y) = E(V_T|S_T = y) = E(V_T|S_0 e^{\sigma \widehat{W}_T} = y) = E(V_T|\widehat{W}_T = \log(y/S_0)/\sigma),$$

where, as in Shreve (2004), we take $\alpha = (r - \sigma^2/2)/\sigma$, $\widehat{W}_t = \alpha t + W_t$ and $\widehat{M}_T = \max_{0\le t\le T} \widehat{W}_t$ so that $S_t = S_0 e^{\sigma \widehat{W}_t}$. So for real x, we calculate

$$\begin{aligned} g(x) &= E(V_T|\widehat{W}_T = x) \\ &= E\left(\max\left\{\max_{0\le t\le T} S_t - B, 0\right\} \middle| \widehat{W}_T = x\right) + B - K \\ &= E\left(\max\left\{S_0 e^{\sigma \widehat{M}_T} - B, 0\right\} \middle| \widehat{W}_T = x\right) + B - K. \end{aligned}$$

Recall from (2) the conditional density of $\widehat{M}_T$

$$f_{\widehat{M}_T}(a|\widehat{W}_T = x) = \frac{2(2a-x)}{T}e^{-2(a^2-ax)/T} = h'(a),$$

where $h(a) = -e^{-2(a^2-ax)/T}$ if $x^+ \leq a$ and 0 otherwise. Then

$$g(x) = \int_{x^+}^{\infty} p(a)h'(a)da + B - K \quad \text{where } p(a) = \max\{S_0 e^{\sigma a} - B, 0\}.$$

Note that $p'(a) = \sigma S_0 e^{\sigma a}$ when $a \geq b = \log(B/S_0)/\sigma$ and 0 otherwise.

Then, integrating by parts,

$$g(x) = p(a)h(a)\Big|_{x^+}^{\infty} - \int_{\max\{b,x^+\}}^{\infty} \sigma S_0 e^{\sigma a} h(a)da + B - K = I_1 + I_2 + B - K,$$

where

$$I_1 = -p(x^+)h(x^+) = \begin{cases} 0 & (x \leq b) \\ S_0 e^{\sigma x} - B & (x > b) \end{cases}$$

and

$$I_2 = \int_{\max\{b,x^+\}}^{\infty} \sigma S_0 e^{\sigma a} e^{-2(a^2-ax)/T} da = \frac{\sigma S_0 \sqrt{2\pi T} e^{d_1^2/2}}{2} N\left(d_1 - \frac{2\max\{x,b\}}{\sqrt{T}}\right),$$

where $d_1 = \frac{x+\sigma T/2}{\sqrt{T}}$ and N is the standard normal cumulative distribution function. Then

$$g(x) = \begin{cases} B - K + \dfrac{\sigma S_0 \sqrt{2\pi T} e^{d_1^2/2}}{2} N(d_2) & (x \leq b) \\ \\ S_0 e^{\sigma x} - K + \dfrac{\sigma S_0 \sqrt{2\pi T} e^{d_1^2/2}}{2} N(d_3) & (x > b), \end{cases}$$

where

$$d_1 = \frac{x + \sigma T/2}{\sqrt{T}}, \quad d_2 = d_1 - \frac{2b}{\sqrt{T}}, \quad d_3 = d_1 - \frac{2x}{\sqrt{T}} = -d_1 + \sigma\sqrt{T}.$$

Thus we have shown that

$$G(S_T) = g(\log(S_T/S_0)/\sigma) = \begin{cases} B - K + Fe^{d_1^2/2} N(d_2) & (S_T \leq B) \\ \\ S_T - K + Fe^{d_1^2/2} N(d_3) & (S_T > B) \end{cases},$$

where

$$F = \frac{1}{2}\sigma S_0 \sqrt{2\pi T}, \quad d_1 = \frac{\log(S_T/S_0) + \sigma^2 T/2}{\sigma\sqrt{T}}, \quad d_2 = d_1 + \frac{2\log(S_0/B)}{\sigma\sqrt{T}}, \quad d_3 = -d_1 + \sigma\sqrt{T}.$$

Smoothness: We show that G is continuously differentiable. The continuity at B follows from the fact that when $S_T = B$,

$$d_3 = -d_1 + \sigma\sqrt{T} = -\frac{\log(B/S_0)}{\sigma\sqrt{T}} + \sigma\sqrt{T}/2 = d_1 + \frac{2\log(S_0/B)}{\sigma\sqrt{T}} = d_2.$$

Next the derivative of $G(S_T)$ from the left at $S_T = B$ is

$$\frac{Fe^{d_1^2/2}}{\sigma\sqrt{T}B}[d_1 N(d_2) + N'(d_2)]$$

and the derivative of $G(S_T)$ from the right at $S_T = B$ is

$$1 + \frac{Fe^{d_1^2/2}}{\sigma\sqrt{T}B}[d_1 N(d_3) - N'(d_3)] = 1 + \frac{Fe^{d_1^2/2}}{\sigma\sqrt{T}B}[d_1 N(d_2) - N'(d_2)].$$

The difference between these is

$$1 - \frac{2Fe^{d_1^2/2}}{\sigma\sqrt{T}B}N'(d_2) = 1 - \frac{S_0}{B}e^{(d_1^2-d_2^2)/2} = 0$$

since when $S_T = B$,

$$d_1^2 - d_2^2 = d_1^2 - \left(d_1 + \frac{2\log(S_0/B)}{\sigma\sqrt{T}}\right)^2 = 2\log(B/S_0).$$

Hence G is continuously differentiable as claimed.

Computational issues: Note that $d_2 = d_1 - \beta$, where $\beta = 2\log(B/S_0)/(\sigma\sqrt{T})$. When $S_T \to 0$, both d_1 and $d_2 \to -\infty$. Then calculating $e^{d_1^2/2}N(d_2)$ involves multiplying a large number by a small number. Instead we write

$$e^{d_1^2/2}N(d_2) = \frac{e^{\beta d_2+\beta^2/2}}{\sqrt{2\pi}}e^{(-d_2)^2/2}\int_{-d_2}^{\infty} e^{-t^2/2}dt = \frac{e^{\beta d_2+\beta^2/2}}{\sqrt{2\pi}}g(-d_2),$$

where

$$g(x) = e^{x^2/2}\int_x^{\infty} e^{-t^2/2}dt = \int_0^{\infty} e^{-xt-t^2/2}dt.$$

There are no problems calculating this last integral when x is large.

Now $d_3 = -d_1 + \sigma\sqrt{T}$. When $S_T \to \infty$, $d_1 \to \infty$ but $d_3 \to -\infty$. Then we write

$$e^{d_1^2/2}N(d_3) = \frac{e^{d_1^2/2}}{\sqrt{2\pi}}\int_{d_1-\sigma\sqrt{T}}^{\infty} e^{-t^2/2}dt = \frac{e^{-\sigma\sqrt{T}d_3+\sigma^2T/2}}{\sqrt{2\pi}}g(-d_3).$$

A graph of the function G for $S_0 = 100$, $K = 95$, $L = 110$, $r = 0.05$, $\sigma = 0.2$, $T = 1$ is shown in Figure 7.

3.2 Alternative approach for lookbacks

Here is an alternative approach suggested by Carbone (2004). As above, we take $\alpha = (r - \sigma^2/2)/\sigma$, $\widehat{W}_t = \alpha t + W_t$ and $\widehat{M}_T = \max_{0\le t\le T}\widehat{W}_t$ so that $S_t = S_0e^{\sigma\widehat{W}_t}$ and $\max_{0\le t\le T}S_t = S_0e^{\sigma\widehat{M}_T}$.

Recall from (3) the joint density, $f_{\widehat{M}_T,\widehat{W}_T}(a,x)$, of $(\widehat{M}_T,\widehat{W}_T)$. So the $t=0$ price of the option with payoff

$$V_T = \max\left\{\max\left\{L, \max_{0\le t\le T}S_t\right\} - K, 0\right\} = \max\left\{S_0e^{\sigma\widehat{M}_T} - B, 0\right\} + B - K,$$

where $B = \max\{K, L\}$, is

$$e^{-rT}\int_0^{\infty}\int_{-\infty}^{a}(\max\{S_0e^{\sigma a} - B, 0\} + B - K)\frac{2(2a-x)}{T\sqrt{2\pi T}}e^{\alpha x-\alpha^2T/2-(2a-x)^2/(2T)}dxda.$$

Setting $z = (2a-x)/\sqrt{T}$, this becomes

$$e^{-rT}\int_0^{\infty}\int_{-z\sqrt{T}}^{z\sqrt{T}}\left(\max\left\{S_0e^{\sigma(x+z\sqrt{T})/2}\} - B, 0\right\} + B - K\right)\frac{ze^{\alpha x-\alpha^2T/2-z^2/2}}{\sqrt{2\pi T}}dxdz,$$

which equals

$$e^{-rT}\int_{-\infty}^{\infty}g(z)\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz$$

where

$$g(z) = \frac{e^{-\alpha^2T/2}}{\sqrt{T}}z\int_{-z\sqrt{T}}^{z\sqrt{T}}e^{\alpha x}\left(\max\left\{S_0e^{\sigma(x+z\sqrt{T})/2}\} - B, 0\right\} + B - K\right)dx \quad \text{if} \quad z \ge 0,$$

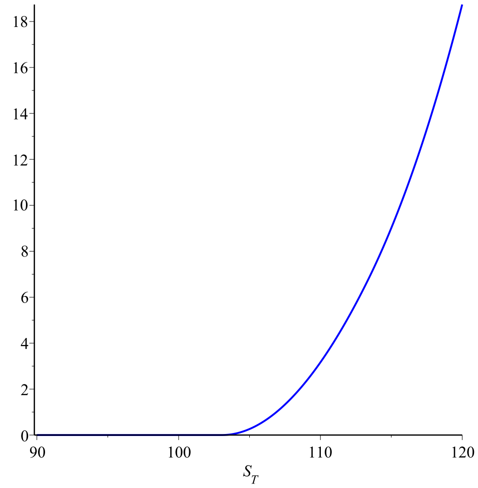

Figure 8: This is the graph of the alternative payoff function $G(S_T)$ for a lookback call option with strike $K = 95$, maturity $T = 1$ and with $L = 110$ as the maximum stock price up to time $t = 0$. The stock price at $t = 0$ is $S_0 = 100$, the interest rate $r = 0.05$ and the volatility $\sigma = 0.2$. This function is continuously differentiable.

and 0 otherwise. Since $S_T = S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z}$, where Z is a standard normal variable, then the $t = 0$ price of the option is

$$e^{-rT}E(G(S_T)),$$

where

$$G(S_T) = g\left(\frac{\log\left(S_T/S_0\right) - \left(r - \sigma^2/2\right)T}{\sigma\sqrt{T}}\right).$$

If $0 \le z\sqrt{T} \le b = \log(B/S_0)/\sigma$, the integral in $g(z)$ equals

$$(B-K)\int_{-z\sqrt{T}}^{z\sqrt{T}} e^{\alpha x}dx = \frac{B-K}{\alpha}[e^{\alpha z\sqrt{T}} - e^{-\alpha z\sqrt{T}}]$$

which is interpreted as $2(B-K)z\sqrt{T}$ when $\alpha = 0$. Then if $z\sqrt{T} > b$, the integral equals

$$= \int_{2b-z\sqrt{T}}^{z\sqrt{T}} e^{\alpha x}(S_0 e^{\sigma(x+z\sqrt{T})/2} - K)dx + \int_{-z\sqrt{T}}^{2b-z\sqrt{T}} (B-K)e^{\alpha x}dx$$

$$= \frac{S_0 e^{\sigma z\sqrt{T}/2}}{\alpha+\sigma/2}[e^{(\alpha+\sigma/2)z\sqrt{T}} - e^{(\alpha+\sigma/2)(2b-z\sqrt{T})}] + \frac{B}{\alpha}(e^{\alpha(2b-z\sqrt{T})} - e^{-\alpha z\sqrt{T}}) - \frac{K}{\alpha}(e^{\alpha z\sqrt{T}} - e^{-\alpha z\sqrt{T}})$$

with the obvious interpretation when $\alpha = 0$. So when $\alpha \ne 0$, that is $2r \ne \sigma^2$, the time $t = 0$ price of the option is

$$e^{-rT}E(G(S_T)),$$

where, taking $u_T = S_T/(S_0 e^{(r-\sigma^2/2)T})$,

$$G(S_T) = \frac{e^{-\alpha^2 T/2}}{\sqrt{T}} d(S_T/S_0)\begin{cases} 0 & (u_T < 1) \\ c_0\left[u_T^{r/\sigma^2-1/2} - u_T^{-r/\sigma^2+1/2}\right] & (1 \le u_T \le B/S_0), \\ c_1 u_T^{r/\sigma^2+1/2} + c_2 u_T^{r/\sigma^2-1/2} + c_3 u_T^{-r/\sigma^2+1/2} & (u_T > B/S_0), \end{cases}$$

where

$$d(z) = \frac{\log(z) - (r - \sigma^2/2)T}{\sigma\sqrt{T}}, \quad c_0 = \frac{2\sigma(B-K)}{2r - \sigma^2}$$

and

$$c_1 = \frac{\sigma S_0}{r}, \quad c_2 = -\frac{2\sigma K}{2r - \sigma^2}, \quad c_3 = \frac{\sigma^3 B}{r(2r - \sigma^2)} \left(\frac{B}{S_0}\right)^{2r/\sigma^2 - 1} - c_0.$$

When $\alpha = 0$, that is $2r = \sigma^2$, we get

$$G(S_T) = \frac{2}{\sigma\sqrt{T}} d(S_T/S_0) \begin{cases} 0 & (S_T < S_0), \\ (B-K)\log(S_T/S_0) & (S_0 \le S_T \le B), \\ S_T - K\log(S_T/S_0) + B(\log(B/S_0) - 1) & (S_T > B). \end{cases}$$

One verifies that G is continuously differentiable. A graph of G for $S_0 = 100$, $K = 95$, $L = 110$, $r = 0.05$, $\sigma = 0.2$, $T = 1$ is shown in Figure 8.

3.3 Example

Now we compute the $t = 0$ price of a lookback call with fixed strike $K = 95$ and maturity $T = 1$, when the maximum stock price up to time $t = 0$ is $L = 110$. We suppose the current stock price $S_0 = 100$, the interest rate $r = 0.05$ and the volatility $\sigma = 0.2$. The Black-Scholes price is $C_{BS} = 25.475463$. We compute the CRR binomial price C_n in three ways. First by using the method in Cheuk and Vorst (1997), then using the conditional payoff and thirdly using the alternative payoff. In the table below n denotes the number of periods and "error" means $C_n - C_{BS}$.

n	Cheuk-Vorst	$\sqrt{n}$(error)	conditional	n(error)	alternative	n(error)
1000	25.652643	5.60	25.475654	0.192	25.463169	-12.29
2000	25.564515	3.98	25.475561	0.197	25.469319	-12.29
3000	25.573158	5.35	25.475524	0.184	25.471368	-12.28
4000	25.555958	5.09	25.475509	0.184	25.472377	-12.34
5000	25.589251	8.05	25.475502	0.195	25.473005	-12.29
6000	25.560545	6.59	25.475494	0.187	25.473409	-12.33
7000	25.557603	6.87	25.475490	0.192	25.473707	-12.29
8000	25.570866	8.53	25.475486	0.184	25.473922	-12.33
9000	25.524024	4.61	25.475484	0.191	25.474095	-12.31
10000	25.558672	8.32	25.475482	0.193	25.474232	-12.31

As expected, the error for the Cheuk-Vorst method is of order $1/\sqrt{n}$. In the other cases, the error is of order $1/n$ and the convergence is smooth in the other two cases, as expected since the payoff is differentiable.

3.4 Coefficient of $1/n$ for lookbacks using the conditional expectation approach

For the conditional expectation approach, we observed that the error was of order $1/n$ and the convergence was smooth. This is expected as the payoff function is differentiable. The results in Leduc (2013) enable us to calculate the coefficient c of $1/n$ in the asymptotic expansion of the error, that is,

$$C_n - C_{BS} = \frac{c}{n} + O\left(\frac{1}{n^{3/2}}\right),$$

where C_n is the price calculated by the $n-$period CRR model using the conditional payoff function and C_{BS} is the Black-Scholes price.

The payoff under the conditional expectation approach is

$$G(S_T) = \begin{cases} B - K + Fe^{d_1^2/2}N(d_2) & (S_T \le B) \\ \\ S_T - K + Fe^{d_1^2/2}N(d_3) & (S_T > B), \end{cases}$$

where

$$B = \max\{K, L\}, \quad F = \frac{1}{2}\sigma S_0\sqrt{2\pi T},$$

and

$$d_1 = \frac{\log(S_T/S_0) + \sigma^2 T/2}{\sigma\sqrt{T}}, \quad d_2 = d_1 + \frac{2\log(S_0/B)}{\sigma\sqrt{T}}, \quad d_3 = -d_1 + \sigma\sqrt{T}.$$

Now we calculate

$$V_0(x) = e^{-rT}E(G(S_T)|S_0 = x).$$

This means $S_T = xe^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z}$, where Z is $N(0,1)$, but the S_0's in the d_i remain as they are. It is straightforward to prove that

$$V_0(x) = e^{-rT}(B-K) + xN(d_4) - Be^{-rT}N(d_5) - \frac{\sigma\sqrt{T}x}{2d_6}\left((S_0e^{-rT})^{-\beta+1}x^{\beta-1}N(d_7) - N(d_4)\right),$$

where

$$\beta = \frac{2\log(B/S_0)}{\sigma^2 T}, \quad d_4 = \frac{\log(x/B) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_5 = d_4 - \sigma\sqrt{T}$$

and

$$d_6 = \frac{\log(x/S_0) + rT}{\sigma\sqrt{T}}, \quad d_7 = -d_4 + (-\beta+1)\sigma\sqrt{T}.$$

Then we calculate the coefficient as

$$-\left(\frac{\Delta_2 S_0^2}{2}V_0''(S_0) + \frac{\Delta_3 S_0^3}{6}V_0'''(S_0) + \frac{\Delta_4 S_0^4}{24}V_0^{(4)}(S_0)\right),$$

where

$$\Delta_2 = T^2(r^2 + r\sigma^2 + 5\sigma^4/12), \quad \Delta_3 = 2\sigma^2T^2(r+\sigma^2), \quad \Delta_4 = 2\sigma^4T^2.$$

Remark: As remarked earlier, Leduc (2013) gives another way to calculate the coefficient. It turns out in this case also that the method we just used above is more convenient.

Example: For $S_0 = 100$, $K = 95$, $L = 110$, $r = 0.05$, $\sigma = 0.2$, $T = 1$, we calculate the value C_n of the call option using our conditional expectation approach. We calculate the coefficient c to be 0.188528. Recall that the Black-Scholes price is $C_{BS} = 25.475463$. The value of C_n can be improved by subtracting the error term c/n. Let us call $C_n^* = C_n - c/n$, the *corrected* option value. With 'error' meaning $C_n - C_{BS}$, and 'error*' meaning the *corrected* error, $C_n^* - C_{BS} = C_n - C_{BS} - c/n$, the term n(error) converges slowly to c, which is reflected in the fact that $n^{3/2}$(error*) remains bounded.

n	conditional	n(error)	corrected	$n^{3/2}$(error*)
1000	25.475654	0.192	25.475465	-0.37
2000	25.475561	0.197	25.475467	0.099
3000	25.475524	0.184	25.475461	-0.48
4000	25.475509	0.184	25.475462	0.36
5000	25.475502	0.195	25.475464	0.32
6000	25.475494	0.187	25.475463	-0.22
7000	25.475490	0.192	25.475463	-0.36
8000	25.475486	0.184	25.475462	-0.29
9000	25.475484	0.191	25.475463	0.072
10000	25.475482	0.193	25.475463	0.45

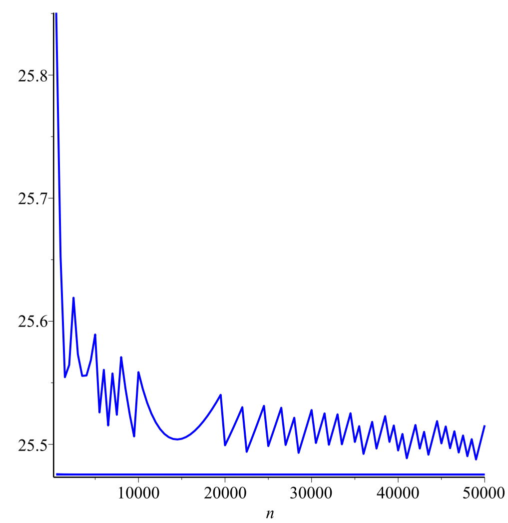

Figure 9: The *conditional approach* and the *corrected conditional approach* both significantly improve the convergence of the *Cheuk-Vorst approach*. The values obtained using the *Cheuk-Vorst approach* are shown in the upper graph. The graphs of the values obtained by the other two approaches coincide with the Black-Scholes value which is along the horizontal axis.

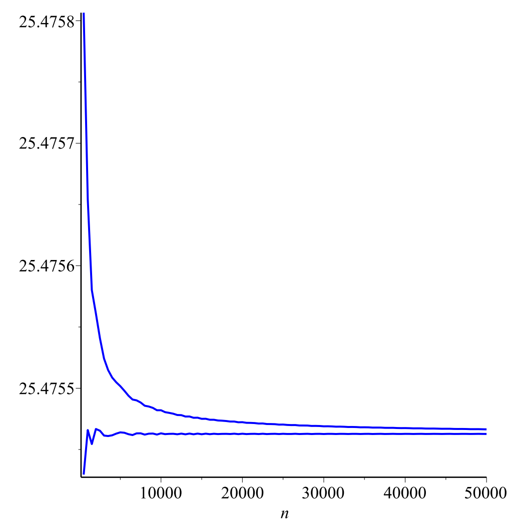

Figure 10: The graphs of the values obtained from the *corrected conditional approach* (the lower graph) and those obtained from the *conditional approach* are shown here. The *corrected conditional approach* exhibits an unmistakable improvement over the *conditional approach*.

The *corrected* option value C_n^* converges at a speed of $n^{-3/2}$, while the option value C_n converges at a speed of n^{-1}. By comparison, the value of the option under the Cheuk-Vorst approach converges at a speed of $n^{-1/2}$.

Figure 9 displays these three methods of calculating the option value: option values were calculated for multiples of $n = 500$ up to a maximum of $50,000$. The figure illustrates that the *conditional approach* and the *corrected conditional approach* both strikingly outperform the *Cheuk-Vorst approach*. Zooming in, Figure 10 compares the values obtained using the *conditional approach* to those obtained using the *corrected conditional approach*. These two approaches are indistinguishable in Figure 9, because they both improve the speed of convergence from $n^{-1/2}$ to n^{-1}. However, we see from Figure 10 that the *corrected conditional approach* significantly improves the behavior of the convergence of the *conditional approach*. This is because it accelerates the speed of convergence from n^{-1} to $n^{-3/2}$.

Declaration of interest

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References

Appolloni, E., M. Gaudenzi, and A. Zanette (2014). The binomial interpolated lattice method for step double barrier options. *International Journal of Theoretical and Applied Finance* **17**(06), 1450035. DOI: https://doi.org/10.1142/S0219024914500356

Babbs, S. (1992). Binomial valuation of lookback options. *Working Paper*.

Babbs, S. (2000). Binomial valuation of lookback options. *Journal of Economic Dynamics and Control* **24**(11-12), 1499–1525. https://doi.org/10.1016/S0165-1889(99)00085-8

Björk, T. (2009). *Arbitrage theory in continuous time.* Oxford University Press.

Bock, A. and R. Korn (2016). Improving convergence of binomial schemes and the Edgeworth expansion. *Risks* **4**(2), 15. DOI: https://doi.org/10.3390/risks4020015

Boyle, P. P. and S. H. Lau (1994). Bumping up against the barrier with the binomial method. *The Journal of Derivatives* **1**(4), 6–14. DOI: https://doi.org/10.3905/jod.1994.407891

Broadie, M., P. Glasserman, and S.-G. Kou (1999). Connecting discrete and continuous path-dependent options. *Finance and Stochastics* **3**(1), 55–82. DOI: https://doi.org/10.1007/s007800050052

Carbone, R. (2004). Binomial approximation of Brownian motion and its maximum. *Statistics and Probability Letters* **69**(3), 271–285. DOI: https://doi.org/10.1016/j.spl.2004.06.020

Chang, L. and K. Palmer (2007). Smooth convergence in the binomial model. *Finance and Stochastics* **11**(1), 91–105. DOI: https://doi.org/10.1007/s00780-006-0020-6

Cheuk, T. H. and T. C. Vorst (1997). Currency lookback options and observation frequency: a binomial approach. *Journal of International Money and Finance* **16**(2), 173–187. Retrieved from http://hdl.handle.net/1765/10155

Dai, M. (2000). A modified binomial tree method for currency lookback options. *Acta Mathematica Sinica* **16**(3), 445–454. DOI: https://doi.org/10.1007/s101140000068

Dai, T.-S. and Y.-D. Lyuu (2010). The bino-trinomial tree: a simple model for efficient and accurate option pricing. *Journal of Derivatives* **17**(4), 7. DOI: https://doi.org/10.3905/jod.2010.17.4.007

Derman, E., I. Kani, D. Ergener, and I. Bardhan (1995). Enhanced numerical methods for options with barriers. *Financial Analysts Journal* **51**(6), 65–74. DOI: https://doi.org/10.2469/faj.v51.n6.1951

Gaudenzi, M. and M. A. Lepellere (2006). Pricing and hedging American barrier options by a modified binomial method. *International Journal of Theoretical and Applied Finance* **9**(04), 533–553. DOI: https://doi.org/10.1142/S0219024906003664

Grosse-Erdmann, K. and F. Heuwelyckx (2016). The pricing of lookback options and binomial approximation. *Decisions in Economics and Finance* **39**(1), 33–67. DOI: https://doi.org/10.1007/s10203-016-0171-7

Heuwelyckx, F. (2014). Convergence of European lookback options with floating strike in the binomial model. *International Journal of Theoretical and Applied Finance* **17**(04), 1450025. DOI: https://doi.org/10.1142/S0219024914500253

Kennedy, D. (2016). *Stochastic Financial Models.* Chapman and Hall/CRC.

Leduc, G. (2013). A European option general first-order error formula. *The ANZIAM Journal* **54**(4), 248–272. DOI: https://doi.org/10.1017/S1446181113000254

Leduc, G. (2016). Option convergence rate with geometric random walks approximations. *Stochastic Analysis and Applications* **34**(5), 767–791. DOI: https://doi.org/10.1080/07362994.2016.1171721

Lin, J. and K. Palmer (2013). Convergence of barrier option prices in the binomial model. *Mathematical Finance* **23**(2), 318–338. DOI: https://doi.org/10.1111/j.1467-9965.2011.00501.x

Lyuu, Y.-D. (1998). Very fast algorithms for barrier option pricing and the ballot problem. *Journal of Derivatives* **5**(3), 68–79. DOI: https://doi.org/10.3905/jod.1998.407999

Rogers, L. and E. Stapleton (1997). Fast accurate binomial pricing. *Finance and Stochastics* **2**(1), 3–17. DOI: https://doi.org/10.1007/s007800050029

Shreve, S. (2004). *Stochastic Calculus for Finance: Continuous-Time Models.* Springer-Verlag, New York.