

Convergence rate of regime-switching trees

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ABSTRACT. Considering a general class of regime-switching geometric random walks and a broad class of piecewise twice differentiable payoff functions, we show that convergence of option prices occurs at a speed of order $\mathcal{O}(n^{-\beta})$, where $\beta = 1/2$ when the payoff is discontinuous and $\beta = 1$ otherwise.

1. Introduction

The acclaimed Black-Scholes model is the *common language* of security derivatives, and option prices are quoted using this model. In spite of this unparalleled triumph, the Black-Scholes model suffers from well known shortcomings. One of them is that the risk-neutral rate r and the volatility σ should not be constant. The regime-switching model provides an enhancement of the Black-Scholes model which alleviates this problem. In this model, the market-related price-determining parameters r and σ of the Black-Scholes model are jointly determined by an externally driven market-related *regime*. While there can be several different forces acting on the price of an option, in this model one force (*regime*) is a dominating factor in setting the price, and the *state* of this regime is modelled to switch back and forth between finitely many modes. For instance, this could be the changes in preferences of the market agents [24] alternating between *bullish* and *bearish* expectations [14, 16] or, as in [2], alternating between *good* and *bad*. It can also be a business cycle [4] recurring from *expansion*, *transition*, and *contraction*. This price-driving force can also be determined by a hidden Markov process such as inside trading [5]. Numerous papers highlight that the regime-switching model is better than the Black-Scholes model in capturing the fat tails exhibited by empirical financial returns [7, 6, 12, 8, 20, 3]. In regime-switching models, asset prices evolve according to models determined by the *state* of some recurrently-switching *regimes* which are driven by unobserved factors resulting in stationary *regime-state* changes following each other independently.

In its simplest form, the regime has two states, 1 and 2, and the risk-free rate and volatility are fully determined by this state. For simplicity, this paper focuses on two-state regime-switching models. In an abstract form, a (two-state) regime-switching model $\Xi := \Xi(\alpha, \xi^1, \xi^2)$ is composed of three independent components: a stochastic model α_t governing the state of the regime, and two independent

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underlying asset stochastic models ξ_t^1 and ξ_t^2 . Starting in a regime-state $\alpha_0 = a$ and at a spot price $\Xi_0 = x$, the value Ξ_t of the underlying asset matches the value of ξ_t^a , that is $\Xi_t = \xi_t^a$, until the first regime-switching occurs at time $\tau_1 = \inf \{t \geq 0 : \alpha_t \neq \alpha_0\}$. From that point on, the asset grows according to $\xi_t^{\alpha_{\tau_1}}$, that is $\Xi_t = (\Xi_{\tau_1}) (\xi_t^{\alpha_{\tau_1}} / \xi_{\tau_1}^{\alpha_{\tau_1}})$, until the regime-state changes again at time τ_2 . The process then continues as $\Xi_t = (\Xi_{\tau_2}) (\xi_t^{\alpha_{\tau_2}} / \xi_{\tau_2}^{\alpha_{\tau_2}})$ until the regime-state changes once more, and this scheme repeats itself forever.

In the "Black-Scholes" regime-switching model the two underlying stochastic processes, ξ^a for $a = 1, 2$, follow the Black-Scholes model with parameters r_a, σ_a , while regime-state changes follow each other after waiting independent exponentially distributed times, the average waiting time being $1/\lambda_a$ when the regime is in state a .

Let T be the maturity of some security derivatives, and let $(T/n)\mathbb{N}$ be a discretization of the time interval. It is natural to be interested in discretizations of the Black-Scholes regime-switching model, namely piecewise constant approximations $\Xi_t^{(n)}$ of Ξ_t . These approximations include binomial and trinomial trees which are essential to price options for which a closed form solution is inexistent or computationally complicated such as in the case for American options. Taking again a high level and abstract view point, we will say that a stochastic process $\Xi^{(n)} := \Xi^{(n)}(\alpha^n, \xi^{(1,n)}, \xi^{(2,n)})$ is a *partially discretized version* of the (Black-Scholes) regime-switching model $\Xi := \Xi(\alpha, \xi^1, \xi^2)$ if the parameters $\alpha^n, \xi^{(1,n)}, \xi^{(2,n)}$ are either discretizations or identical versions of their corresponding parameter in Ξ . A *full discretization* occurs when all tree parameters of $\Xi^{(n)}$ are discretizations of their limiting Ξ -counterparts.

In the trinomial tree method for the Black-Scholes regime-switching model, the two underlying stochastic processes, ξ^a for $a = 1, 2$, are each approximated by a trinomial tree $\xi^{a,n}$ for $a = 1, 2$. Furthermore, given that the regime is in state a at time $t \in (T/n)\mathbb{N}$, the probability that it changes state at time $t + T/n$ is $1 - \exp(-\lambda_a T/n)$. Recall that a self-similar *trinomial* tree $S^{(n)}$ can be seen as a stochastic process which at every positive time t in $(T/n)\mathbb{N}$, has a probability p_n^u of jumping from its current state $S_t^{(n)}$ to the state $S_t^{(n)} u_n$, a probability p_n^d of jumping to the state $S_t^{(n)} d_n$, and a probability $1 - p_n^u - p_n^d$ of jumping to the state $S_t^{(n)} m_n$, for some $u_n, d_n, m_n > 0$.

Trinomial tree methods for regime-switching models have been studied in several papers. Bollen [1] presents a lattice-based method for valuing both European and American-style options and suggests that the regime-switching option values better match implied common volatility smiles in empirical studies. A discretization of the Cox-Ross-Rubinstein type for the regime-switching model is displayed in Guo [5]. Khaliq and Liu [9] compare an implicit schemes with a tree model that generalizes the Cox-Ross-Rubinstein binomial tree model, and with an analytical approximation solution for the two-regime case described in Buffington and Elliott [2]. Liu [13] designs regime-switching recombining tree. Yuen and Yang [23] present a fast and simple tree model to price simple and exotic options in Markov regime-switching models with multiple regime-states. Yoon et al. [21] develop a lattice method for pricing lookback options in a regime-switching market environment. Fuh et al. [4] provide a closed-form formula for the arbitrage-free price of the European call option, and use a tree method, among others, for calculating prices.

Liu [14] introduces a lattice tree method for pricing financial derivatives in a regime-switching mean-reverting model. In Liu and Zhao [16] a lattice approach for option pricing with two underlying assets whose prices are governed by regime-switching models is developed. Yuen et al. [22] incorporate the regime-switching effect in a discrete time binomial model for an asset's prices via the "self-exciting" threshold principle. Costabile et al. [3] present a binomial approach for pricing contingent claims when the parameters governing the underlying asset process follow a regime-switching model. A tree approach to options pricing under a regime-switching jump diffusion model is exhibited in Liu and Nguyen [15].

These natural questions arise: at what speed do option prices converge under typical trinomial tree discretizations? How does this convergence depend on the smoothness of the payoff?

Recently, Ma and Zhu [17, 18] investigated the speed of convergence of European options under Yuen and Yang's trinomial method [23]. The authors considered a European option with maturity T . Letting $a \in \{1, 2\}$ represent the state of the regime, and S be any node of the trinomial tree at time $t_k = Tk/n$, they denote by $\varepsilon_a^k(S) = V(S, t_k, a) - V^k(S, a)$ the difference between the option under the regime-switching model and the same option under the trinomial tree method when the regime-state is a , the spot price is S , and the time is $t_k = Tk/n$. In the main result of their paper, Ma and Zhu state that, for $k = 1, \dots, n-1$ and $a = 1, 2$, $\|\varepsilon_a^k\|_\infty = \mathcal{O}(n^{-1})$, where $\|\varepsilon_a^k\|_\infty := \max_{-n \leq j \leq n} |\varepsilon_a^k(S_j)|$ and $S_j = u^j S_0$. Unfortunately, Ma and Zhu do not specify any conditions for the payoff function. Their main result and its proof are only valid for payoff functions which are *smooth enough* and subject to *boundedness conditions*. This *excludes* call options, put options, binary options, and even payoff functions such as $f(x) = x^2$. To explain this in the simplest manner, we will assume that $\sigma_1 = \sigma_2$ and $r_1 = r_2$ which brings us back to the Black-Scholes model with the parameters r_1, σ_1 . Furthermore, as in Ma and Zhu, we set

$$\sigma := \max(\sigma_1, \sigma_2) + \left(\sqrt{1.5} - 1\right) \frac{\sigma_1 + \sigma_2}{2} = \sqrt{1.5}\sigma_1.$$

Define $\Lambda_1 = \sigma/\sigma_1 > 1$, and let $S_0 = 1$. In this special case, Yuen and Yang's trinomial tree becomes an ordinary trinomial tree approximating the Black-Scholes model with parameters r_1, σ_1 . More specifically, with $\Delta t := T/n$,

$$\begin{aligned} u &= e^{\Lambda_1 \sigma_1 \sqrt{\Delta t}} = e^{\sigma \sqrt{\Delta t}}, m = 1, d = e^{-\Lambda_1 \sigma_1 \sqrt{\Delta t}} = e^{-\sigma \sqrt{\Delta t}}, \\ p_u &= \frac{e^{r_1 h} - e^{-\Lambda_1 \sigma_1 \sqrt{h}} - \left(1 - \frac{1}{(\Lambda_1)^2}\right) \left(1 - e^{-\Lambda_1 \sigma_1 \sqrt{h}}\right)}{e^{\Lambda_1 \sigma_1 \sqrt{h}} - e^{-\Lambda_1 \sigma_1 \sqrt{h}}}, \\ p_d &= \frac{e^{\Lambda_1 \sigma_1 \sqrt{h}} - e^{r_1 h} - \left(1 - \frac{1}{(\Lambda_1)^2}\right) \left(e^{\Lambda_1 \sigma_1 \sqrt{h}} - 1\right)}{e^{\Lambda_1 \sigma_1 \sqrt{h}} - e^{-\Lambda_1 \sigma_1 \sqrt{h}}}, \\ p_m &= 1 - \frac{1}{(\Lambda_1)^2}. \end{aligned}$$

First, consider the smooth payoff $f(x) = x^2$. It is easy to see that

$$\varepsilon_1^{n-1}(x) = x^2 \varepsilon_1^{n-1}(1).$$

As pointed out in section 8 below, the Yuen and Yang model satisfies equation (8.3) below and, with $\gamma = 2$,

$$\varepsilon_1^{n-1}(1) = \mathcal{O}(n^{-2}).$$

Therefore,

$$\max_{-n \leq j \leq n} |\varepsilon_a^{n-1}(S_j)| = \max_{-n \leq j \leq n} |S_j^2| |\varepsilon_a^{n-1}(1)| \geq |S_n^2| \mathcal{O}(n^{-2}) = e^{2\sigma\sqrt{T}\sqrt{n}} \mathcal{O}(n^{-2})$$

and obviously

$$\lim_{n \rightarrow \infty} e^{2\sigma\sqrt{T}\sqrt{n}} \mathcal{O}(n^{-2}) = \infty.$$

Hence $\|\varepsilon_a^{n-1}\|_\infty$ fails to be $\mathcal{O}(n^{-1})$ as it is actually unbounded! Second, consider now the case of a call option with $K = 1 = S_0$, where K is the strike. For $S_0 = d = \exp(-\Lambda_1 \sigma_1 \sqrt{\Delta t}) < 1$ it is clear that the price (discounted expectation) in the one-time-step trinomial tree (maturity Δt) is zero; on the other hand, the price $BS(d, \Delta t)$ of the call option in the Black-Scholes model with a maturity of Δt and a spot price of $S_0 = d$ satisfies

$$BS\left(e^{-\Lambda_1 \sigma_1 \sqrt{\Delta t}}, \Delta t\right) = \frac{\sqrt{\Delta t}}{\sqrt{\pi}} \left(2\sigma_1 e^{-\Lambda_1^2} + \sqrt{\pi} \sigma_1 \Lambda_1 \operatorname{erf}(\Lambda_1)\right) + \mathcal{O}(\Delta t),$$

showing again that

$$\|\varepsilon_1^{n-1}\|_\infty \geq \mathcal{O}\left(n^{-\frac{1}{2}}\right), \quad \|\varepsilon_1^{n-1}\|_\infty \neq \mathcal{O}(n^{-1}).$$

Finally, using the Berry-Esseen theorem, [11] proves that for digital options

$$\sup_{x \geq 0} |\varepsilon_1^0(x)| = \mathcal{O}\left(n^{-\frac{1}{2}}\right),$$

which again does not match the main result in [17]. These three examples show that Ma and Zhu's result [17, 18] fails unless conditions are put on the payoff function.

The reason for these problems comes from the use in Ma and Zhu [17, eq. (18) and (19)] of Taylor's theorem to compute $V(S_j, t_k, i) - V(S_j, t_{k+1}, i)$ and $V(S_{j+1}, t_{k+1}, i) - V(S_j, t_{k+1}, i)$, *requiring that the reminders be of order $\mathcal{O}(\Delta t^2)$ uniformly in t_k and S_j* . Unfortunately, for this to be true, the payoff function should be sufficiently smooth and subject to boundedness conditions. Indeed, still considering the case of the call option (a non-differentiable payoff) when the two regimes are identical (Black-Scholes with parameters r_1, σ_1), and letting $BS(S, t)$ denote the value of the option in the Black-Scholes model at time t and spot price S , it is easy to calculate that

$$\begin{aligned} BS(K, 2\Delta t) - BS(K, \Delta t) + \left(\frac{\partial}{\partial t} BS(K, \Delta t)\right) \Delta t \\ = \sqrt{\Delta t} \frac{K \sigma_1}{\sqrt{\pi}} (2\sqrt{2} - 1) + \mathcal{O}(\Delta t), \end{aligned}$$

which contradicts [17, equation (18)] when $t_k = (n-2)\Delta t$.

In summary, both the result and Ma and Zhu's proof [17, 18] can be valid *only* for options with *smooth* payoff functions subject to boundedness conditions, excluding the most interesting cases which only have a *piecewise smooth* payoff function such as *put options, call options, and digital options*.

This problem is addressed in this article and constitutes its main result. We show that for a broad family of *piecewise smooth* payoff functions (including call, put, and digital options) and for a large class of discretizations of the two-state regime-switching model, convergence occurs at a rate of $\mathcal{O}(n^{-1})$ for continuous payoff functions, and at a rate of $\mathcal{O}(n^{-1/2})$ when the payoff is discontinuous. These discretizations include, but are not limited to, trinomial trees and other lattice methods. In particular, Yuen and Yang's trinomial model [23] falls under our setting.

2. Settings

This section describes the building blocks on which this paper relies.

2.1. Payoff function class. We say that a function h is piecewise $C^{(m)}$, for some integer $m \geq 0$, if there exists countably many intervals $J_\ell := [\beta_\ell, \beta_{\ell+1})$, $\beta_0 < \beta_1 < \dots$, forming a partition of $[0, \infty)$ and functions h_ℓ extendible to be $C^{(m)}$ on the closure of J_ℓ , such that

$$h(x) = \sum_{\ell=0}^{\infty} h_\ell(x) 1_{[\beta_\ell, \beta_{\ell+1})}(x).$$

We use I to denote the identity function, that is $I(z) := z$ for every z . Given an integer k , we set $I^k(z) := z^k$. We denote by $\mathcal{K}^{(m)}$ the class of piecewise $C^{(m)}$ functions such that $h, Ih', \dots, I^m h^{(m)}$ have a limit at infinity and are of bounded variation over $[0, \infty)$. Clearly, for any $h \in \mathcal{K}^{(m)}$, functions $h, Ih', \dots, I^m h^{(m)}$ are bounded and we define a norm \varkappa_m on $\mathcal{K}^{(m)}$ as

$$\varkappa_m(h) = \sum_{k=0}^m \left(\text{TV} \left(I^k h^{(k)} \right) + \left\| I^k h^{(k)} \right\|_{\infty} \right)$$

where $\text{TV}(g)$ is the total variation of g over the interval $[0, \infty)$.

2.2. Black-Scholes discretization. Let $t_m := Tm/n$, for $m = 0, 1, \dots$. In this article, the Black-Scholes model is discretized using geometric random walks. These are piecewise constant stochastic processes $\{\xi^{(n)}\}$ of the form

$$(2.1) \quad \xi_t^{(n)} = \xi_0 \exp \left(\sum_{k=1}^{\lfloor nt/T \rfloor} X_n [t_{k-1}, t_k] \right),$$

with random variables $X_n [t_{k-1}, t_k]$ independent and identically distributed as

$$X_n \stackrel{\text{def}}{=} X_n [0, T/n].$$

We consider only discrete approximations $\{\xi^{(n)}\}$ of the the Black-Scholes model with parameters r and σ for which the following assumptions on X_n hold:

$$(A1) \quad \mu_n \stackrel{\text{def}}{=} E(X_n) = \frac{T}{n} \left(\left(r - \frac{1}{2} \sigma^2 \right) + \mathcal{O} \left(n^{-\frac{1}{2}} \right) \right),$$

$$(A2) \quad \sigma_n \stackrel{\text{def}}{=} \sqrt{\text{Var}(X_n)} = \sqrt{\frac{T}{n}} \left(\sigma + \mathcal{O} \left(n^{-\frac{1}{2}} \right) \right).$$

Furthermore, for every real constant γ ,

$$(A3) \quad E(\exp(\gamma X_n)) = \exp\left(\frac{1}{2} \frac{T}{n} \gamma (2r - \sigma^2 + \sigma^2 \gamma)\right) + \mathcal{O}(n^{-2}),$$

$$(A4) \quad E\left(\exp\left(\gamma \sqrt{\frac{n}{T}} X_n\right)\right) = \mathcal{O}(1).$$

For binomial trees, it was shown in [11] that all put options, all call options, and all options with polynomial payoffs converge at a rate of n^{-1} to the Black-Scholes price with risk-free rate r and volatility σ if and only if assumptions A1-A4 hold.

REMARK 1. *In a risk-neutral setting, the price of a European option coincides with the discounted expectation of the payoff at maturity. Assumption A3 amounts to what is called quasi risk neutrality in [11], where the discounted expectation of the asset price over an interval of time of size T/n is equal to the spot price plus an error of order $1/n^2$. For mere simplicity of the presentation, we will assume in this paper that ξ^1 and ξ^2 are risk-neutral.*

2.3. Regime-state discretizations. Recall that α_t , the regime-state stochastic process, is piecewise constant which values in $\{1, 2\}$, and it changes value after waiting independent exponentially distributed times, the average waiting time being $1/\lambda_a$ when the regime is in state a . In this article, we will approximate α_t by a piecewise constant process α_t^n which, at every *time step* t_m , changes from state $a \in \{1, 2\}$ to state $a' \neq a$ with probability p_a^n , and remains in state a with probability $1 - p_a^n$. One possible instance of α_t^n is the process which changes state at time t_m , $m \geq 1$, if and only if α_t experiences at least one jump in the interval $(t_{m-1}, t_m]$. In this case $p_a^n = 1 - \exp(-\lambda_a T/n)$. We will call this specific instance of α_t^n the *default discretized regime-state space*. It differs from the *snapshot discretized regime-state space* $\widehat{\alpha}_t^n$ defined as

$$\widehat{\alpha}_t^n = \sum_{m=0}^{\infty} \alpha_{t_m} 1_{[t_m, t_{m+1})}(t).$$

Note that, on any trajectory $t \mapsto \alpha_t$ where a maximum of one jumps occurs in any interval $[t_{m-1}, t_m)$, $m = 1, \dots, n$, we have

$$\sup_{0 \leq t \leq T} |\widehat{\alpha}_t^n - \alpha_t^n| = 0.$$

Throughout this paper we use $\lambda \stackrel{def}{=} \lambda_1 \vee \lambda_2$. Furthermore, for every $0 \leq s \leq t$, $N[s, t]$ denote the number of jumps of α_t in the interval $[s, t]$, while $N_\lambda[s, t]$ denote the number of jumps of a Poisson process with intensity λ in the same interval. Recall that for any $0 \leq s \leq T$, $\Delta \alpha_s = \alpha_s - \alpha_{s-}$.

We will denote by \mathcal{A} the event that, over the interval of time $[0, T]$, the trajectory $t \mapsto \alpha_t$ has a maximum of one jump in any interval $(t_{m-1}, t_m]$. That is

$$(2.2) \quad \mathcal{A} = \bigcap_{m=1}^n \mathcal{A}_m$$

where

$$\mathcal{A}_m = \left\{ \omega : \sum_{t_{m-1} < s \leq t_m} |\Delta \alpha_s| \leq 1 \right\}.$$

Consider now the probability of the complement \mathcal{A}_m^c of \mathcal{A}_m , that is the probability that over the interval $(t_{m-1}, t_m]$ the trajectory $t \mapsto \alpha_t$ has at least two jumps. Note

that $P(\mathcal{A}_m^c)$ is smaller than the probability that $t \mapsto N_\lambda[0, t]$ has at least two jumps over the same interval. Hence,

$$P(\mathcal{A}_m^c) \leq 1 - e^{-\frac{\lambda T}{n}} \left(1 + \left(\frac{\lambda T}{n} \right) \right) = \mathcal{O}(n^{-2}).$$

It follows that

$$(2.3) \quad P(\mathcal{A}) = 1 - \mathcal{O}(n^{-1}), \quad P(\mathcal{A}^c) = \mathcal{O}(n^{-1}).$$

This allows us to define the *pseudo discretized regime-state space*, $\bar{\alpha}_t^n$, as

$$\bar{\alpha}_t^n = 1_{\mathcal{A}} \alpha_t^n + 1_{\mathcal{A}^c} \alpha_t.$$

While this stochastic process is neither a discretization nor a Markov process, it plays an important role in this paper because it is precisely equal to the *default discretized regime-state space* α_t^n except on some ‘negligible’ event of probability $\mathcal{O}(n^{-1})$. It allows us to strictly focus our efforts only on those ‘non-negligible’ trajectories in \mathcal{A} .

2.4. Occupation time random variables. Throughout this paper, the maturity T is fixed. For $a \in \{1, 2\}$, the occupation time random variable L_a is, by definition, equal to the total time spent in state a by the regime-state process α_t over the interval $[0, T]$. Hence,

$$L_a = \int_0^T \delta_a(\alpha_t) dt.$$

Note that $0 \leq L_a \leq T$, $L_1 + L_2 = T$. We define in the same manner L_a^n , \widehat{L}_a^n and \bar{L}_a^n to be the occupation time random variables of α_t^n , $\widehat{\alpha}_t^n$, and $\bar{\alpha}_t^n$. Recall \mathcal{A} from section 2.3, the event that over the interval of time $[0, T]$, the trajectory $t \mapsto \alpha_t$ has a maximum of one jump in any interval $(t_{m-1}, t_m]$. Obviously,

$$0 = \sup_{0 \leq t \leq T} 1_{\mathcal{A}} \left| L_a^n - \widehat{L}_a^n \right| = \sup_{0 \leq t \leq T} 1_{\mathcal{A}} \left| L_a^n - \bar{L}_a^n \right| = \sup_{0 \leq t \leq T} 1_{\mathcal{A}^c} \left| \bar{L}_a^n - L_a \right|.$$

It follows that, for any bounded measurable function f ,

$$\begin{aligned} E(f(L_1^n, L_2^n)) &= E\left(f\left(\widehat{L}_1^n, \widehat{L}_2^n\right)\right) + \|f\|_\infty \mathcal{O}(n^{-1}) \\ &= E\left(f\left(\bar{L}_1^n, \bar{L}_2^n\right)\right) + \|f\|_\infty \mathcal{O}(n^{-1}). \end{aligned}$$

3. Option price closed-form formula

Throughout this paper, the price of a European option refers to the discounted expectation of the payoff at maturity. The purpose of this section is to introduce the notation related to expectations and discounted expectations of regime-switching models. Furthermore, we give a closed-form formula for the regime-switching Black-Scholes model. Here again $\Xi := \Xi(\alpha, \xi^1, \xi^2)$ is a regime-switching Black-Scholes model, and $\Xi^{(n)} := \Xi^{(n)}(\alpha^n, \xi^{1,n}, \xi^{2,n})$ is a *partial* or *full* discretization discretization of Ξ .

3.1. Black-Scholes model. We denote respectively by E_x^a and $E_x^{a,n}$, the conditional *expectation* given that $\xi_0^a = x$ and $\xi_0^{a,n} = x$, where a is the regime-state, $a = 1, 2$. Additionally, for any payoff function ψ , we denote the *discounted* expectations of the payoff under a Black-Scholes model, ξ^a , and its discretization, $\xi^{a,n}$, by

$$\begin{aligned}\mathcal{E}_t^a \psi(x) &\stackrel{def}{=} e^{-r_a t} E_x^a(\psi(\xi_t^a)), \\ \mathcal{E}_t^{a,n} \psi(x) &\stackrel{def}{=} e^{-r_a t} E_x^{a,n}(\psi(\xi_t^{a,n})).\end{aligned}$$

3.2. Regime-switching Black-Scholes model. In the Black-Scholes model the price of an asset ξ_t^a satisfies

$$d\xi_t^a = r_a \xi_t^a dt + \sigma_a \xi_t^a dW_t^a$$

where W_t^a is a Brownian motion. Assuming that $\xi_0^a = 1$, this solves as

$$\xi_t^a = \exp\left(\sigma_a W_t^a + \left(r_a - \frac{1}{2}\sigma_a^2\right)t\right).$$

On the other hand, in the regime-switching Black-Scholes model the price of an asset Ξ_t satisfies

$$d\Xi_t = (r_{\alpha_t}) \Xi_t dt + (\sigma_{\alpha_t}) \Xi_t dW_t$$

for some independent Brownian motion W_t . This solves as

$$\begin{aligned}\Xi_t &= \Xi_0 \exp\left(\int_0^t \left(r_{\alpha_s} - \frac{\sigma_{\alpha_s}^2}{2}\right) ds - \int_0^t (\sigma_{\alpha_s}) dW_s\right) \\ &= \Xi_0 \Pi_{a=1}^2 \exp\left(\sigma_a W_{L_a}^a + \left(r_a - \frac{1}{2}\sigma_a^2\right)L_a\right) \\ &= \Xi_0 \xi_{L_1}^1 \xi_{L_2}^2\end{aligned}$$

where $L_a := L_a([0, t])$ is the occupation time in state a over the interval $[0, t]$, that is,

$$L_a = \int_0^t \delta_a(\alpha_s) ds.$$

Now for any payoff function ψ , let us denote by $\mathcal{E}_t \psi(a, x)$ the *discounted* expectation of the payoff $\psi(\Xi_t)$ at time t given that $(\alpha_0, \Xi_0) = (a, x)$. Furthermore, we denote by $E_a^{L_1}$ the expectation with respect to L_1 given that $\alpha_0 = a$. To avoid unnecessary complications we sometimes simply use E to denote the expectation given the initial value of the stochastic processes involved. Then,

$$(3.1) \quad \mathcal{E}_t \psi(a, x) = E\left(e^{-r_1 L_1} e^{-r_2 L_2} \psi\left(x \xi_{L_1}^1 \xi_{L_2}^2\right)\right).$$

Moreover, because of the independence of L_1 , ξ^1 and ξ^2 ,

$$\begin{aligned}\mathcal{E}_t \psi(a, x) &= E_a^{L_1}\left(E_1^1\left[e^{-r_1 L_1} E_1^2\left\{e^{-r_2 L_2} \psi\left(x \xi_{L_1}^1 \xi_{L_2}^2\right)\right\}\right]\right) \\ &= E_a^{L_1}\left(E_1^1\left[e^{-r_1 L_1} \mathcal{E}_{L_2}^2 h\left(x \xi_{L_1}^1\right)\right]\right) \\ &= E_a^{L_1}\left(\mathcal{E}_{L_1}^1\left(\mathcal{E}_{L_2}^2 h\right)(x)\right).\end{aligned}$$

In a similar manner,

$$\mathcal{E}_t \psi(a, x) = E_a^{L_1}\left(\mathcal{E}_{L_2}^2\left(\mathcal{E}_{L_1}^1 h\right)(x)\right).$$

Note that random variables L_1 and L_2 are related by $L_1 + L_2 = T$. Furthermore, L_1 and L_2 have a density. We denote by $f_a^1(t)$ and $f_a^2(t)$ the density functions of

L_1 and L_2 given that $\alpha_0 = a$. A closed form formula for $f_a^1(t)$ and $f_a^2(t)$ can be found in [19, 5, 4, 3]. Then,

$$\mathcal{E}_t\psi(a, x) = \int_0^T \mathcal{E}_t^1(\mathcal{E}_{T-t}^2 h)(x) f_a^1(t) dt = \int_0^T \mathcal{E}_{T-t}^1(\mathcal{E}_t^2 h)(x) f_a^2(t) dt.$$

It is easy to see that $\mathcal{E}_t^1(\mathcal{E}_{T-t}^2 h)(x) = \mathcal{E}_t^*(h)(x)$, where $\mathcal{E}_t^*(h)(x)$ is the price of an option in the Black-Scholes model with maturity T , spot price x , payoff function h , risk-neutral rate $r_*(t)$, and volatility $\sigma_*(t)$ where

$$\begin{aligned} r_*(t) &= \frac{t}{T}r_1 + \frac{T-t}{T}r_2, \\ \sigma_*^2(t) &= \frac{t}{T}\sigma_1^2 + \frac{T-t}{T}\sigma_2^2. \end{aligned}$$

This is a simple reformulation of a formula in [19, 3]. As in [24], it produces a closed form formula requiring only one integral,

$$(3.2) \quad \mathcal{E}_t\psi(a, x) = \int_0^T \mathcal{E}_t^*(h)(x) f_a^1(t) dt.$$

In particular, for a European put we get

$$\begin{aligned} \mathcal{E}_t^*(h)(x) &= \Phi(-d_2^*(t))Ke^{-r_*(t)T} - \Phi(-d_1^*(t))x, \\ d_1^*(t) &= \frac{1}{\sigma_*(t)\sqrt{T}} \left(\ln\left(\frac{x}{K}\right) + \left(r_*(t) + \frac{\sigma_*^2(t)}{2}\right)T \right), \\ d_2^*(t) &= d_1^*(t) - \sigma_*(t)\sqrt{T}. \end{aligned}$$

For the digital put option we have

$$\mathcal{E}_t^*(h)(x) = e^{-r_*(t)T}\Phi(-d_2^*(t)).$$

REMARK 2. Let \mathbb{L} be any measurable subset of $[0, T]$ and \mathbb{L}^c its complement. Clearly,

$$\begin{aligned} \mathcal{E}_t\psi(a, x) &= \int_0^t 1_{\mathbb{L}}(s) \mathcal{E}_s^1(\mathcal{E}_{t-s}^2 h)(x) f_a^1(s) ds \\ &\quad + \int_0^t 1_{\mathbb{L}^c}(s) \mathcal{E}_s^1(\mathcal{E}_{t-s}^2 h)(x) f_a^1(s) ds \end{aligned}$$

or, written differently,

$$\begin{aligned} \mathcal{E}_t\psi(a, x) &= \int_0^t 1_{\mathbb{L}}(s) \mathcal{E}_s^1(\mathcal{E}_{t-s}^2 h)(x) f_a^1(s) ds \\ &\quad + \int_0^t 1_{\mathbb{L}^c}(s) \mathcal{E}_{t-s}^2(\mathcal{E}_s^1 h)(x) f_a^1(s) ds. \end{aligned}$$

Analogue expressions are valid for the expectation and conditional expectation with respect to some partial or full discretization $\Xi^{(n)}$. In particular,

$$\begin{aligned} \mathcal{E}_t^n\psi(a, x) &= E \left(e^{-r_1 L_1^n} e^{-r_2 L_2^n} \psi \left(x \xi_{L_1^n}^{1,n} \xi_{L_2^n}^{2,n} \right) \right) \\ &= E_a^{L_1^n} \left(1_{\mathbb{L}}(L_1^n) \mathcal{E}_{L_1^n}^{1,n} \left(\mathcal{E}_{L_2^n}^{2,n} h \right) (x) \right) \\ &\quad + E_a^{L_1^n} \left(1_{\mathbb{L}^c}(L_1^n) \mathcal{E}_{L_2^n}^{2,n} \left(\mathcal{E}_{L_1^n}^{1,n} h \right) (x) \right). \end{aligned}$$

where $E_a^{L_1^n}$ is the expectation with respect to the random variable L_1^n , given that $\alpha_0 = a$.

4. Outline of the paper

Given a regime-switching Black-Scholes model $\Xi := \Xi(\alpha, \xi^1, \xi^2)$, this paper considers the following families of partial and full discretizations: $\widehat{\Xi}^{(n)} := \Xi(\alpha^n, \xi^1, \xi^2)$ and $\Xi^{(n)} := \Xi(\alpha^n, \xi^{1,n}, \xi^{2,n})$, where $\xi^{1,n}$ and $\xi^{2,n}$ are discrete approximations of Black-Scholes models ξ^1 and ξ^2 satisfying assumptions A1-A4, and where α^n is the *default discretized regime-state space* of section 2.3. In section 5, we show that for an option with payoff h in $\mathcal{K}^{(2)}$, the pricing error resulting from using model $\widehat{\Xi}^{(n)}$ instead of the regime-switching Ξ is of order $\mathcal{O}(n^{-\beta})$, where β depends on whether or not h is continuous. More specifically, we show that, given an initial state of the regime of a and a spot price of x , if $\widehat{\mathcal{E}}_t^n h(a, x)$ and $\mathcal{E}_t^n h(a, x)$ are respectively the discounted expectation of the payoff at maturity t under model $\widehat{\Xi}^{(n)}$ and Ξ , then there exists a constant Q , which does not depend on h or x , such that

$$\left| \mathcal{E}_t h(a, x) - \widehat{\mathcal{E}}_t^n h(a, x) \right| \leq \chi_2(h) Q n^{-\beta},$$

where $\beta = 1/2$ if h is discontinuous and $\beta = 1$ otherwise. In section 6, we show that for the same option, the pricing error resulting from using model $\Xi^{(n)}$ instead of model $\widehat{\Xi}^{(n)}$ is of order $\mathcal{O}(n^{-1})$ when h is continuous but of order $\mathcal{O}(n^{-1/2})$ otherwise. More specifically we show that there exists a constant Q , which does not depend on h or x , such that

$$\left| \mathcal{E}_t^n h(a, x) - \widehat{\mathcal{E}}_t^n h(a, x) \right| \leq \chi_2(h) Q n^{-\beta},$$

where $\beta = 1/2$ if h is discontinuous and $\beta = 1$ otherwise. This proves that

$$|\mathcal{E}_t^n h(a, x) - \mathcal{E}_t h(a, x)| \leq \chi_2(h) Q n^{-\beta},$$

where $\beta = 1/2$ if h is discontinuous and $\beta = 1$ otherwise. In section 7, focussing on payoffs of the form $h(x) = x^\gamma$, for any real γ , we show that there exists a constant Q , which does not depend on h or x , such that for every $x > 0$,

$$|\mathcal{E}_t^n h(a, x) - \mathcal{E}_t h(a, x)| \leq Q x^\gamma n^{-1}.$$

In section 8, we explain how the trinomial method of Yuen and Yang [23] corresponds to the a full discretizations of the form $\Xi^{(n)} := \Xi(\alpha^n, \xi^{1,n}, \xi^{2,n})$ where $\xi^{1,n}$ and $\xi^{2,n}$ are discrete approximations of the Black-Scholes models ξ^1 and ξ^2 satisfying assumptions A1-A4, and where α^n is the *default discretized regime-state space*. Section 9 provides numerical results illustrating our findings. Auxiliary results are found in Section 10.

5. Partial discretization error

Here we investigate the error resulting from replacing the parameter α in $\Xi(\alpha, \xi^1, \xi^2)$ by α^n . This can be seen alternatively as replacing L_a by L_a^n in the price formula (3.1). More specifically, we show the following:

PROPOSITION 1 (Regime-state discretization error). *Assume that properties A1-A4 hold and let h belong to $\mathcal{K}^{(2)}$. Then,*

$$E\left(e^{-r_1 L_1^n} e^{-r_2 L_2^n} h\left(x \xi_{L_1^n}^1 \xi_{L_2^n}^2\right)\right) = E\left(e^{-r_1 L_1} e^{-r_2 L_2} h\left(x \xi_{L_1}^1 \xi_{L_2}^2\right)\right) + \chi_2(h) \mathcal{O}(n^{-1}),$$

where the $\mathcal{O}(n^{-1})$ term is uniform in h and x .

PROOF. Recall \bar{L}_a^n and \mathcal{A} from section 2. Because h is bounded, because \bar{L}_a^n coincides with L_a^n on \mathcal{A} and because $P(\mathcal{A}^c) = \mathcal{O}(n^{-1})$, it follows that

$$\begin{aligned} E\left(e^{-r_1 L_1^n} e^{-r_2 L_2^n} h\left(x \xi_{L_1^n}^1 \xi_{L_2^n}^2\right)\right) &= E\left(e^{-r_1 \bar{L}_1^n} e^{-r_2 \bar{L}_2^n} h\left(x \xi_{L_1^n}^1 \xi_{L_2^n}^2\right)\right) \\ &\quad + \chi_2(h) \mathcal{O}(n^{-1}). \end{aligned}$$

Note that, from Lemma 1,

$$(5.1) \quad E\left(\left|\bar{L}_a^n - L_a\right|\right) = \mathcal{O}(n^{-1}),$$

$$(5.2) \quad P\left(\left|\bar{L}_a^n - L_a\right| > \frac{T}{4}\right) = \mathcal{O}(n^{-1}).$$

Define $\mathbb{L}_1 \stackrel{def}{=} \{L_1 \geq \frac{T}{2}\}$ and its complement $\mathbb{L}_2 \stackrel{def}{=} \{L_2 > \frac{T}{2}\}$. Recall that

$$\begin{aligned} E\left(e^{-r_1 \bar{L}_1^n} e^{-r_2 \bar{L}_2^n} h\left(x \xi_{L_1^n}^1 \xi_{L_2^n}^2\right)\right) &= E\left(e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}_1} \mathcal{E}_{L_1^n}^1 h\left(x \xi_{L_2^n}^2\right)\right) \\ &\quad + E\left(e^{-r_1 \bar{L}_1^n} 1_{\mathbb{L}_2} \mathcal{E}_{L_2^n}^2 h\left(x \xi_{L_1^n}^1\right)\right), \\ &= E\left(1_{\mathbb{L}_1} \mathcal{E}_{L_2^n}^2 \mathcal{E}_{L_1^n}^1 h(x)\right) \\ &\quad + E\left(1_{\mathbb{L}_2} \mathcal{E}_{L_1^n}^1 \mathcal{E}_{L_2^n}^2 h(x)\right). \end{aligned}$$

The above equations remain true with \bar{L}_1^n and \bar{L}_2^n replaced with L_1 and L_2 . Hence we only need to show that

$$(5.3) \quad \begin{aligned} E\left(e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}_1} \mathcal{E}_{L_1^n}^1 h\left(x \xi_{L_2^n}^2\right)\right) &= E\left(e^{-r_2 L_2} 1_{\mathbb{L}_1} \mathcal{E}_{L_1}^1 h\left(x \xi_{L_2}^2\right)\right) \\ &\quad + \chi_2(h) \mathcal{O}(n^{-1}), \\ E\left(e^{-r_1 \bar{L}_1^n} 1_{\mathbb{L}_2} \mathcal{E}_{L_2^n}^2 h\left(x \xi_{L_1^n}^1\right)\right) &= E\left(e^{-r_1 L_1} 1_{\mathbb{L}_2} \mathcal{E}_{L_2}^2 h\left(x \xi_{L_1}^1\right)\right) \\ &\quad + \chi_2(h) \mathcal{O}(n^{-1}). \end{aligned}$$

The two cases are symmetric so we need to prove only the first equation. From Taylor's theorem,

$$1_{\mathbb{L}_1} \mathcal{E}_{L_1^n}^1 h\left(x \xi_{L_2^n}^2\right) = 1_{\mathbb{L}_1} \mathcal{E}_{L_1}^1 h\left(x \xi_{L_2}^2\right) + 1_{\mathbb{L}_1} \int_{L_1}^{\bar{L}_1^n} \frac{\partial}{\partial t} \mathcal{E}_t^1 h\left(x \xi_{L_2^n}^2\right) dt.$$

Note that according to Lemma 2,

$$\sup_{t \geq \frac{T}{4}} \sup_{z \geq 0} \left\| \frac{\partial}{\partial t} \mathcal{E}_t^1 h(z) \right\|_{\infty} = O(1) \chi_2(h).$$

Hence,

$$1_{\mathbb{L}_1} \left| \int_{L_1}^{\bar{L}_1^n} \frac{\partial}{\partial t} \mathcal{E}_t^1 h\left(x \xi_{L_2^n}^2\right) dt \right| \leq O(1) \chi_2(h) \left| L_1 - \bar{L}_1^n \right|$$

Thus from (5.1),

$$\begin{aligned} E\left(1_{\mathbb{L}_1} e^{-r_2 \bar{L}_2^n} \mathcal{E}_{L_1^n}^1 h\left(x \xi_{L_2^n}^2\right)\right) &= E\left(1_{\mathbb{L}_1} e^{-r_2 \bar{L}_2^n} \mathcal{E}_{L_1}^1 h\left(x \xi_{L_2^n}^2\right)\right) + \chi_2(h) \mathcal{O}(n^{-1}) \\ &= E\left(1_{\mathbb{L}_1} \mathcal{E}_{L_2^n}^2 \mathcal{E}_{L_1}^1 h(x)\right) + \chi_2(h) \mathcal{O}(n^{-1}) \end{aligned}$$

Hence, in order to prove (5.3), we need only to show that

$$E \left(1_{\mathbb{L}_1} \mathcal{E}_{\bar{L}_2}^2 \mathcal{E}_{L_1}^1 h(x) \right) = E \left(1_{\mathbb{L}_1} \mathcal{E}_{L_2}^2 \mathcal{E}_{L_1}^1 h(x) \right) + \chi_2(h) \mathcal{O}(n^{-1}).$$

Now let

$$\begin{aligned} \mathbb{L}'_1 &:= \left\{ L_1 \geq \frac{T}{2} \right\} \cap \left\{ L_2 \geq \bar{L}_2 \right\}, \\ \mathbb{L}''_1 &:= \left\{ L_1 \geq \frac{T}{2} \right\} \cap \left\{ L_2 < \bar{L}_2 \right\}. \end{aligned}$$

Because \mathbb{L}_1 is the disjoint union of \mathbb{L}'_1 and \mathbb{L}''_1 , in order to establish (5.3), we only need to show that

$$(5.4) \quad \begin{aligned} E \left(1_{\mathbb{L}'_1} \mathcal{E}_{\bar{L}_2}^2 \left(\mathcal{E}_{L_1}^1 h \right) (x) \right) &= E \left(1_{\mathbb{L}'_1} \mathcal{E}_{L_2}^2 \left(\mathcal{E}_{L_1}^1 h \right) (x) \right) + \chi_2(h) \mathcal{O}(n^{-1}), \\ E \left(1_{\mathbb{L}''_1} \mathcal{E}_{\bar{L}_2}^2 \left(\mathcal{E}_{L_1}^1 h \right) (x) \right) &= E \left(1_{\mathbb{L}''_1} \mathcal{E}_{L_2}^2 \left(\mathcal{E}_{L_1}^1 h \right) (x) \right) + \chi_2(h) \mathcal{O}(n^{-1}), \end{aligned}$$

The two cases can be treated in a completely symmetrical manner: using Taylor's expansion around $x\xi_{\bar{L}_2}^2$ in the first case, and around $x\xi_{L_2}^2$ in the second case. We will leave the second case as an exercise for the reader. Note that in order to establish (5.4), that is,

$$E \left(e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}'_1} \mathcal{E}_{L_1}^1 h \left(x\xi_{\bar{L}_2}^2 \right) \right) = E \left(e^{-r_2 L_2} 1_{\mathbb{L}'_1} \mathcal{E}_{L_1}^1 h \left(x\xi_{L_2}^2 \right) \right) + \chi_2(h) \mathcal{O}(n^{-1}),$$

we can replace $e^{-r_2 L_2}$ by $e^{-r_2 \bar{L}_2^n}$ on the right hand side, that is, we only need to prove that

$$(5.5) \quad \begin{aligned} E \left(e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}'_1} \mathcal{E}_{L_1}^1 h \left(x\xi_{\bar{L}_2}^2 \right) \right) &= E \left(e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}'_1} \mathcal{E}_{L_1}^1 h \left(x\xi_{L_2}^2 \right) \right) \\ &\quad + \chi_2(h) \mathcal{O}(n^{-1}). \end{aligned}$$

Indeed, it easily follows from (5.1) that

$$E \left(e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}'_1} \mathcal{E}_{L_1}^1 h \left(x\xi_{\bar{L}_2}^2 \right) \right) = E \left(e^{-r_2 L_2} 1_{\mathbb{L}'_1} \mathcal{E}_{L_1}^1 h \left(x\xi_{L_2}^2 \right) \right) + \|h\|_\infty \mathcal{O}(n^{-1}).$$

Using Taylor's theorem, we write

$$\begin{aligned} e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}'_1} \mathcal{E}_{L_1}^1 h \left(x\xi_{\bar{L}_2}^2 \right) &= e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}'_1} \mathcal{E}_{L_1}^1 h \left(x\xi_{\bar{L}_2^n}^2 \right) \\ &\quad + e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}'_1} \frac{\partial}{\partial x} \mathcal{E}_{L_1}^1 h \left(x\xi_{\bar{L}_2^n}^2 \right) \left(x\xi_{L_2}^2 - x\xi_{\bar{L}_2^n}^2 \right) \\ &\quad + e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}'_1} \int_{x\xi_{\bar{L}_2^n}^2}^{x\xi_{L_2}^2} \left(\frac{\partial^2}{\partial x^2} \mathcal{E}_{L_1}^1 h(z) \right) \left(z - x\xi_{\bar{L}_2^n}^2 \right) dz. \end{aligned}$$

To obtain (5.5) we will show that the expectation of the last two terms has the form $\chi_2(h) \mathcal{O}(n^{-1})$. Let us denote

$$\begin{aligned} \Delta L_2 &:= L_2 - \bar{L}_2^n, \\ \bar{\xi}_{\Delta L_2}^2 &:= \frac{\xi_{L_2}^2}{\xi_{\bar{L}_2^n}^2}. \end{aligned}$$

Note that, as $\Delta L_2 \stackrel{def}{=} L_2 - \bar{L}_2^n > 0$ on \mathbb{L}'_1 . Basic properties of the geometric Brownian motion guarantee that $\bar{\xi}_{\Delta L_2}^2$ is independent of $\xi_{\bar{L}_2^n}^2$ and identically distributed

as $\xi_{\Delta L_2}^2$. Obviously,

$$\begin{aligned} x\xi_{L_2}^2 &= \left(x\xi_{L_2^n}^2\right)\bar{\xi}_{\Delta L_2}^2, \\ x\xi_{L_2}^2 - x\xi_{L_2^n}^2 &= x\xi_{L_2^n}^2 \left(\bar{\xi}_{\Delta L_2}^2 - 1\right). \end{aligned}$$

The independence of $\xi_{L_2^n}^2$ and $\bar{\xi}_{\Delta L_2}^2$ gives that

$$\begin{aligned} &\left| E \left(e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}'_1} \left(\frac{\partial}{\partial x} \mathcal{E}_{L_1}^1 h \left(x\xi_{L_2^n}^2 \right) x\xi_{L_2^n}^2 \right) \left(\bar{\xi}_{\Delta L_2}^2 - 1 \right) \right) \right| \\ &= \left| E \left(e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}'_1} \left(\frac{\partial}{\partial x} \mathcal{E}_{L_1}^1 h \left(x\xi_{L_2^n}^2 \right) x\xi_{L_2^n}^2 \right) \right) \right| \left| E \left(\bar{\xi}_{\Delta L_2}^2 - 1 \right) \right| \end{aligned}$$

Invoking Lemma 2 we achieve

$$\begin{aligned} \left| E \left(e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}'_1} \frac{\partial}{\partial x} \mathcal{E}_{L_1}^1 h \left(x\xi_{L_2^n}^2 \right) x\xi_{L_2^n}^2 \left(\bar{\xi}_{\Delta L_2}^2 - 1 \right) \right) \right| &\leq \chi_2(h) |E(\xi_{\Delta L_2}^2 - 1)| \\ &= \chi_2(h) |E(e^{r\Delta L_2} - 1)| \\ &= \chi_2(h) \mathcal{O}(1) |E(r\Delta L_2)|, \end{aligned}$$

and from (5.1) we conclude that

$$\left| E \left(e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}'_1} \frac{\partial}{\partial x} \mathcal{E}_{L_1}^1 h \left(x\xi_{L_2^n}^2 \right) x\xi_{L_2^n}^2 \left(\bar{\xi}_{\Delta L_2}^2 - 1 \right) \right) \right| \leq \chi_2(h) \mathcal{O}(n^{-1}).$$

Now we define \mathcal{L} as

$$\mathcal{L} := E \left(e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}'_1} \int_{x\xi_{L_2^n}^2}^{(x\xi_{L_2^n}^2)\bar{\xi}_{\Delta L_2}^2} \left(\frac{\partial^2}{\partial x^2} \mathcal{E}_{L_1}^1 h(z) \right) (z - x\xi_{L_2^n}^2) dz \right).$$

To complete this proof, we need to show that $\mathcal{L} = \chi_2(h) \mathcal{O}(n^{-1})$. With the change of variables $z = x\xi_{L_2^n}^2 y$ we get

$$\begin{aligned} \mathcal{L} &= E \left(e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}'_1} \int_1^{\bar{\xi}_{\Delta L_2}^2} \frac{\partial^2}{\partial x^2} \mathcal{E}_{L_1}^1 \left(x\xi_{L_2^n}^2 y \right) \left(x\xi_{L_2^n}^2 y - x\xi_{L_2^n}^2 \right) \left[x\xi_{L_2^n}^2 \right] dy \right) \\ &= E \left(e^{-r_2 \bar{L}_2^n} 1_{\mathbb{L}'_1} \int_1^{\bar{\xi}_{\Delta L_2}^2} \frac{\partial^2}{\partial x^2} \mathcal{E}_{L_1}^1 \left(x\xi_{L_2^n}^2 y \right) \left(x\xi_{L_2^n}^2 y \right)^2 \left(\frac{y-1}{y^2} \right) dy \right). \end{aligned}$$

On the other hand, according to Lemma 2, there exists a constant Q depending only on the parameters σ_1, r_1, T such that

$$\sup_{\frac{T}{2} \leq L_1 \leq T} \sup_{z \geq 0} \left| z^2 \frac{\partial^2}{\partial x^2} \mathcal{E}_{L_1}^1 h(z) \right| \leq Q\chi_2(h).$$

This gives

$$\begin{aligned} 1_{\mathbb{L}'_1} \left| \int_1^{\bar{\xi}_{\Delta L_2}^2} \frac{\partial^2}{\partial x^2} \mathcal{E}_{L_1}^1 \left(x\xi_{L_2^n}^2 y \right) \left(x\xi_{L_2^n}^2 y \right)^2 \left(\frac{y-1}{y^2} \right) dy \right| \\ \leq Q\chi_2(h) 1_{\mathbb{L}'_1} \int_1^{\bar{\xi}_{\Delta L_2}^2} \left(\frac{y-1}{y^2} \right) dy. \end{aligned}$$

Hence,

$$\begin{aligned}
|\mathcal{L}| &\leq Q\chi_2(h) E \left(1_{L_1'} \left(\ln \xi_{\Delta L_2}^2 - 1 + (\xi_{\Delta L_2}^2)^{-1} \right) \right) \\
&= Q\chi_2(h) E \left(1_{L_1'} \left(\Delta L_2 \left(r - \frac{1}{2}\sigma^2 \right) - 1 + \exp(-\Delta L_2 (r - \sigma_2^2)) \right) \right) \\
&\leq \chi_2(h) \mathcal{O}(1) E |\Delta L_2| \\
&= \chi_2(h) \mathcal{O}(n^{-1}).
\end{aligned}$$

□

6. Full discretization error

The following theorem is the main result of this paper.

THEOREM 1 (Regime-switching discretization error). *Assume that properties A1-A4 hold and that h belongs to $\mathcal{K}^{(2)}$. Let $\beta = 1/2$ if h is discontinuous and $\beta = 1$ otherwise. Then,*

$$\mathcal{E}_t^n \psi(a, x) = \mathcal{E}_t \psi(a, x) + \chi_2(h) \mathcal{O}(n^{-\beta}),$$

where the $\mathcal{O}(n^{-\beta})$ term is uniform in h and x .

PROOF. Here we want to show that

$$\begin{aligned}
E \left(e^{-r_1 L_1^n} e^{-r_2 L_2^n} h \left(x \xi_{L_1^n}^{1,n} \xi_{L_2^n}^{2,n} \right) \right) &= E \left(e^{-r_1 L_1} e^{-r_2 L_2} h \left(x \xi_{L_1}^1 \xi_{L_2}^2 \right) \right) \\
&\quad + \chi_2(h) \mathcal{O}(n^{-\beta}).
\end{aligned}$$

First we write

$$\begin{aligned}
E \left(e^{-r_1 L_1^n} e^{-r_2 L_2^n} h \left(x \xi_{L_1^n}^{1,n} \xi_{L_2^n}^{2,n} \right) \right) &= E \left(1_{L_1^n \geq \frac{T}{2}} \mathcal{E}_{L_2^n}^{2,n} \left(\mathcal{E}_{L_1^n}^{1,n} h \right) (x) \right) \\
&\quad + E \left(1_{L_2^n > \frac{T}{2}} \mathcal{E}_{L_1^n}^{1,n} \left(\mathcal{E}_{L_2^n}^{2,n} h \right) (x) \right).
\end{aligned}$$

Because ξ^2 and ξ^1 satisfy A1-A4, we obtain from Theorem 3 in the appendix that

$$\begin{aligned}
\sup_{\frac{T}{2} \leq t \leq T} \sup_{z \geq 0} \left| \mathcal{E}_t^{1,n} h(z) - \mathcal{E}_t^1 h(z) \right| &= \chi_2(h) \mathcal{O}(n^{-\beta}), \\
\sup_{\frac{T}{2} \leq t \leq T} \sup_{z \geq 0} \left| \mathcal{E}_t^{2,n} h(z) - \mathcal{E}_t^2 h(z) \right| &= \chi_2(h) \mathcal{O}(n^{-\beta}).
\end{aligned}$$

As a result,

$$\begin{aligned}
E \left(1_{L_1^n \geq \frac{T}{2}} \mathcal{E}_{L_2^n}^{2,n} \left(\mathcal{E}_{L_1^n}^{1,n} h \right) (x) \right) &= E \left(1_{L_1^n \geq \frac{T}{2}} \mathcal{E}_{L_2^n}^{2,n} \left(\mathcal{E}_{L_1^n}^1 h \right) (x) \right) \\
&\quad + \chi_2(h) \mathcal{O}(n^{-\beta}), \\
E \left(1_{L_2^n > \frac{T}{2}} \mathcal{E}_{L_1^n}^{1,n} \left(\mathcal{E}_{L_2^n}^{2,n} h \right) (x) \right) &= E \left(1_{L_2^n > \frac{T}{2}} \mathcal{E}_{L_1^n}^{1,n} \left(\mathcal{E}_{L_2^n}^2 h \right) (x) \right) \\
&\quad + \chi_2(h) \mathcal{O}(n^{-\beta}).
\end{aligned}$$

Note that for $a = 1, 2$ and for every $t > 0$, function $x \mapsto \mathcal{E}_t^a h(x)$ belongs to $C^{(\infty)} \cap \mathcal{K}^2$ and, furthermore, thanks to Lemma 2, there exists a constant Q such that

$$\sup_{t \geq \frac{T}{2}} \chi_2(\mathcal{E}_t^a h) \leq Q\chi_2(h).$$

Therefore, from Theorem 3 in the appendix,

$$\begin{aligned} \sup_{0 \leq t_m < \frac{T}{2}} \sup_{z \geq 0} \left| \mathcal{E}_{t_m}^{1,n} \mathcal{E}_{T-t_m}^2 h(z) - \mathcal{E}_{t_m}^1 \mathcal{E}_{T-t_m}^2 h(z) \right| &= \chi_2(h) \mathcal{O}(n^{-1}), \\ \sup_{0 \leq t_m < \frac{T}{2}} \sup_{z \geq 0} \left| \mathcal{E}_{t_m}^{2,n} \mathcal{E}_{T-t_m}^1 h(z) - \mathcal{E}_{t_m}^2 \mathcal{E}_{T-t_m}^1 h(z) \right| &= \chi_2(h) \mathcal{O}(n^{-1}). \end{aligned}$$

As a result,

$$\begin{aligned} E \left(1_{L_1^n \geq \frac{T}{2}} \mathcal{E}_{L_2^n}^{2,n} \left(\mathcal{E}_{L_1^n}^{1,n} h \right) (x) \right) &= E \left(1_{L_1^n \geq \frac{T}{2}} \mathcal{E}_{L_2^n}^2 \left(\mathcal{E}_{L_1^n}^1 h(z) \right) (x) \right) \\ &\quad + \chi_2(h) \mathcal{O}(n^{-\beta}), \\ E \left(1_{L_2^n > \frac{T}{2}} \mathcal{E}_{L_1^n}^{1,n} \left(\mathcal{E}_{L_2^n}^{2,n} h \right) (x) \right) &= E \left(1_{L_2^n > \frac{T}{2}} \mathcal{E}_{L_1^n}^1 \left(\mathcal{E}_{L_2^n}^2 h(z) \right) (x) \right) \\ &\quad + \chi_2(h) \mathcal{O}(n^{-\beta}). \end{aligned}$$

Hence,

$$\begin{aligned} E \left(e^{-r_1 L_1^n} e^{-r_2 L_2^n} h \left(x \xi_{L_1^n}^{1,n} \xi_{L_2^n}^{2,n} \right) \right) &= E \left(e^{-r_1 L_1^n} e^{-r_2 L_2^n} h \left(x \xi_{L_1^n}^1 \xi_{L_2^n}^2 \right) \right) \\ &\quad + \chi_2(h) \mathcal{O}(n^{-\beta}). \end{aligned}$$

But according to Proposition 1,

$$\begin{aligned} E \left(e^{-r_1 L_1^n} e^{-r_2 L_2^n} h \left(x \xi_{L_1^n}^1 \xi_{L_2^n}^2 \right) \right) &= E \left(e^{-r_1 L_1} e^{-r_2 L_2} h \left(x \xi_{L_1}^1 \xi_{L_2}^2 \right) \right) \\ &\quad + \chi_2(h) \mathcal{O}(n^{-1}). \end{aligned}$$

We have just proved that

$$\begin{aligned} E \left(e^{-r_1 L_1^n} e^{-r_2 L_2^n} h \left(x \xi_{L_1^n}^{1,n} \xi_{L_2^n}^{2,n} \right) \right) &= E \left(e^{-r_1 L_1} e^{-r_2 L_2} h \left(x \xi_{L_1}^1 \xi_{L_2}^2 \right) \right) \\ &\quad + \chi_2(h) \mathcal{O}(n^{-\beta}), \end{aligned}$$

as we wanted to. \square

7. Polynomial payoffs

THEOREM 2. *Assume that properties A1-A4 hold. For every $\gamma \in \mathbb{R}$, every $0 < T_1 < T_2 \leq T$, and every $x > 0$,*

$$\sup_{T_1 \leq t \leq T_2} \left| \mathcal{E}_t^n(I^\gamma)(a, x) - \mathcal{E}_t(I^\gamma)(a, x) \right| = x^\gamma \mathcal{O}(n^{-1}),$$

where the $\mathcal{O}(n^{-1})$ term is uniform in x .

PROOF. Fix $0 < T_1 < T_2 \leq T$, choose any arbitrary $t \in [T_1, T_2]$, and let L_1, L_2, L_1^n, L_2^n be the usual occupation time random variables over the interval $[0, t]$. We have

$$\begin{aligned} \mathcal{E}_t^n(I^\gamma)(a, x) &= E \left(e^{-r_1 L_1^n} e^{-r_2 L_2^n} \left(x \xi_{L_1^n}^{1,n} \xi_{L_2^n}^{2,n} \right)^\gamma \right) \\ &= x^\gamma E \left(e^{-r_1 L_1^n} \left(\xi_{L_1^n}^{1,n} \right)^\gamma e^{-r_2 L_2^n} \left(\xi_{L_2^n}^{2,n} \right)^\gamma \right) \\ &= x^\gamma E \left(\mathcal{E}_{L_1^n}^{1,n}(I^\gamma)(1) \mathcal{E}_{L_2^n}^{2,n}(I^\gamma)(1) \right) \end{aligned}$$

and, invoking Theorem 3, we can continue with

$$= x^\gamma E \left(\mathcal{E}_{L_1^n}^1(I^\gamma)(1) \mathcal{E}_{L_2^n}^2(I^\gamma)(1) \right) + x^\gamma \mathcal{O}(n^{-1})$$

where the $\mathcal{O}(n^{-1})$ term is uniform in x . Note that for any constant $\mathcal{L} \geq 0$

$$\mathcal{E}_{\mathcal{L}}^a(I^\gamma)(1) = \exp(\beta_a \mathcal{L}),$$

where, for $a = 1, 2$,

$$\beta_a = \frac{1}{2}(\gamma - 1)(\gamma\sigma_a^2 + 2r_a).$$

Recall from (2.3) that $L_a^n = \bar{L}_a^n$ for $a = 1, 2$ except on a set \mathcal{A} with $P(\mathcal{A}^c) = \mathcal{O}(n^{-1})$. Hence,

$$E\left(\mathcal{E}_{L_1^n}^1(I^\gamma)(1)\mathcal{E}_{L_2^n}^2(I^\gamma)(1)\right) = E\left(\mathcal{E}_{\bar{L}_1^n}^1(I^\gamma)(1)\mathcal{E}_{\bar{L}_2^n}^2(I^\gamma)(1)\right) + \mathcal{O}(n^{-1}).$$

Finally,

$$\begin{aligned} E\left(\mathcal{E}_{L_1^n}^1(I^\gamma)(1)\mathcal{E}_{L_2^n}^2(I^\gamma)(1)\right) &= E\left(e^{\beta_1 \bar{L}_1^n} e^{\beta_2 \bar{L}_2^n}\right) \\ &= E\left(e^{\beta_1 L_1} e^{\beta_2 L_2}\right) + \mathcal{O}(1) E\left(\left|\bar{L}_1^n - L_1\right|\right) \\ &= E\left(\mathcal{E}_{L_1}^1(I^\gamma)(1)\mathcal{E}_{L_2}^2(I^\gamma)(1)\right) \\ &\quad + \mathcal{O}(1) E\left(\left|\bar{L}_1^n - L_1\right|\right). \end{aligned}$$

The result follows from Lemma 1. \square

8. Yuen and Yang trinomial model

Recall that the Yuen and Yang [23] trinomial model for regime-switching options is defined by

$$\begin{aligned} \sigma &> \max(\sigma_1, \sigma_2), \Lambda_a = \frac{\sigma}{\sigma_a}, \\ u &= e^{\sigma\sqrt{\Delta t}}, m = 1, d = e^{-\sigma\sqrt{\Delta t}}, \\ p_u^a &= \frac{e^{r_a h} - e^{-\sigma\sqrt{h}} - \left(1 - \frac{1}{(\Lambda_a)^2}\right) \left(1 - e^{-\sigma\sqrt{h}}\right)}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}, \\ p_d^a &= \frac{e^{\sigma\sqrt{h}} - e^{r_a h} - \left(1 - \frac{1}{(\Lambda_a)^2}\right) \left(e^{\sigma\sqrt{h}} - 1\right)}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}, \\ p_m^a &= 1 - \frac{1}{(\Lambda_a)^2}, \end{aligned}$$

where $\Delta t = T/n$. Assume that at time $t \in (\Delta t)\mathbb{N}$, the regime-state is $\alpha_t^n = a$ and the asset price is $\Xi_t^{(n)} = x$. Let $a' = 2$ if $a = 1$ and $a' = 1$ otherwise. In the Yuen and Yang [23] model, six outcomes are possible at time $t + \Delta t$: (a) with a probability of $(p_u^a) \exp(-\lambda_a \Delta t)$, the regime remains in state a and the asset price jumps up to xa ; (b) with a probability of $(p_m^a) \exp(-\lambda_a \Delta t)$, the regime remains in state a and the asset price also remains at level x ; (c) with a probability of $(p_d^a) \exp(-\lambda_a \Delta t)$, the regime remains in state a and the asset price jumps down to xd ; (d) with a probability of $(p_u^a) (1 - \exp(-\lambda_a \Delta t))$, the regime switches to state a' and the asset price jumps up to xa ; (e) with a probability of $(p_m^a) (1 - \exp(-\lambda_a \Delta t))$, the regime switches to state a' and the asset price remains at level x ; (f) with a probability of $(p_d^a) (1 - \exp(-\lambda_a \Delta t))$, the regime switches to state a' and the asset price jumps down to xd .

We show here that the trinomial trees $\xi^{a,n}$, $a = 1, 2$, defined by the Yuen and Yang model fall under assumptions A1-A4. To be specific, for $a = 1, 2$, let X_n^a be the random variable which takes the value $\ln(u) = \sigma\sqrt{\Delta t}$ with probability p_u^a , the value $\ln(m) = 0$ with probability p_m^a , and the value $\ln(d) = -\sigma\sqrt{\Delta t}$ with probability p_d^a . Tedious but otherwise simple calculations that can easily be carried out by a computer algebra system give

$$(8.1) \quad E(X_n^a) = \left(r_a - \frac{1}{2}\sigma_a^2\right)\Delta t + \mathcal{O}(\Delta t^2),$$

$$(8.2) \quad \sqrt{\text{Var}(X_n^a)} = \sigma\sqrt{\Delta t} + \mathcal{O}(\Delta t^{\frac{3}{2}}),$$

and for any real constant γ ,

$$(8.3) \quad E(\exp(\gamma X_n^a)) = \exp\left(\frac{1}{2}(\Delta t)\gamma(2r_a - \sigma_a^2 + \sigma_a^2\gamma)\right) + \mathcal{O}(\Delta t^2),$$

$$(8.4) \quad E\left(\exp\left(\gamma\sqrt{\frac{1}{\Delta t}}X_n^a\right)\right) \leq \exp(|\gamma|\sigma_a\Lambda_a) = \mathcal{O}(1).$$

Equations (8.1)-(8.4) precisely say that, for $a = 1, 2$, the trinomial tree $\xi^{a,n}$ approximating ξ^a satisfies conditions A1-A4.

9. Numerical results

To illustrate the convergence behavior of security derivatives with piecewise smooth payoff functions in lattice methods for the two-state regime-switching model, we study two different kinds of options. We chose a European put option to represent the class of continuous payoff functions, and a digital put option, to represent the class of discontinuous payoff functions. The prices of these options are calculated using the Yuen and Yang trinomial model. We consider the case where the strike price is $K = 100$, and the time to maturity is $T = 1$. We choose the interest rate and the volatility to be $r_1 = 0.04, \sigma_1 = 0.25$ for regime 1, and $r_2 = 0.06, \sigma_2 = 0.35$ for regime 2. We set the jump intensity to be $\lambda_1 = \lambda_2 = 2$ for both regimes. In order to cover the three main cases, *In The Money* (ITM), *At The Money* (ATM), and *Out of The Money* (OTM), we select the value of the initial stock price S_0 to be 90, 100, and 110. Numerical errors

$$\text{Err}_T^n h(a, x) = \mathcal{E}_T h(a, x) - \mathcal{E}_T^n h(a, x)$$

are calculated by subtracting the numerical approximations from the benchmark value obtained from our closed-form solution (3.2) of Section 3.2. Furthermore, we examine the value of $n^\beta \times \text{Err}_T^n h(a, x)$ to verify a convergence speed of order $\mathcal{O}(n^{-\beta})$. As we discussed in the previous section, European put options have a convergence of order $\mathcal{O}(n^{-1})$ while the convergence occurs at a speed of $\mathcal{O}(n^{-1/2})$ for digital options. Hence the values of $n \times \text{Err}_T^n h(a, x)$ and $n^{1/2} \times \text{Err}_T^n h(a, x)$ should be bounded for these two options.

Tables 1 to 3 collect specific values for the European put option in the ITM, ATM, and OTM cases. In each situation we illustrate the error starting with either Regime 1 or Regime 2. We can see that when the spot price is *in the money* or *out of the money*, the Yuen and Yang trinomial model price oscillates around the true price. On the other hand, when the spot price is *at the money*, the convergence is monotone and smooth.

European put						
N	Regime 1			Regime 2		
	Price	Error	N \times Error	Price	Error	N \times Error
20	13.6292	-0.0383	-0.7666	14.1202	0.0528	1.0566
100	13.6135	-0.0226	-2.2632	14.1764	-0.0034	-0.3363
200	13.6019	-0.0111	-2.2172	14.1745	-0.0015	-0.2942
500	13.5949	-0.0041	-2.0448	14.1733	-0.0003	-0.1354
1000	13.5906	0.0003	0.2501	14.1710	0.0020	1.9809
2500	13.5917	-0.0008	-2.1103	14.1731	-0.0001	-0.1964
5000	13.5911	-0.0002	-0.9555	14.1728	0.0002	0.8687

TABLE 1. European put option with $S_0 = 90$

European put						
N	Regime 1			Regime 2		
	Price	Error	N \times Error	Price	Error	N \times Error
20	9.0866	0.0129	0.2581	9.6875	0.1102	2.2045
100	9.0967	0.0028	0.2844	9.7757	0.0220	2.2017
200	9.0981	0.0014	0.2875	9.7867	0.0110	2.2014
500	9.0990	0.0006	0.2894	9.7933	0.0044	2.2011
1000	9.0993	0.0003	0.2900	9.7955	0.0022	2.2011
2500	9.0994	0.0001	0.2904	9.7968	0.0009	2.2010
5000	9.0995	0.0001	0.2904	9.7972	0.0004	2.2009

TABLE 2. European put option with $S_0 = 100$

European put						
N	Regime 1			Regime 2		
	Price	Error	N \times Error	Price	Error	N \times Error
20	5.9254	-0.0004	-0.0087	6.5337	0.0966	1.9330
100	5.9444	-0.0195	-1.9466	6.6292	0.0011	0.1083
200	5.9333	-0.0083	-1.6644	6.6285	0.0018	0.3694
500	5.9265	-0.0016	-0.7837	6.6279	0.0024	1.1865
1000	5.9266	-0.0016	-1.6215	6.6299	0.0004	0.4090
2500	5.9259	-0.0010	-2.3977	6.6304	-0.0001	-0.3104
5000	5.9252	-0.0002	-1.1108	6.6301	0.0002	0.8830

TABLE 3. European put option with $S_0 = 110$

Figures 1 to 6 show in detail the relationship between $n \times \text{Err}_T^n h(a, x)$ and the number of time steps n , which varies from $n = 20$ to $n = 5,000$ with an increment of 1. The oscillations of $n \times \text{Err}_T^n h(a, x)$ in the ITM and OTM cases are unmistakable in Figures 1, 2, 5, and 6. Nonetheless, these oscillations are bounded, illustrating that the convergence is of order $\mathcal{O}(n^{-1})$. By contrast, while both curves in Figures 3 and 4 are also bounded, they display a smooth and monotone convergence. This numerically supports a convergence of order $\mathcal{O}(n^{-1})$.

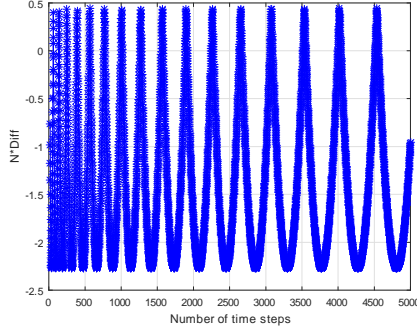


Figure 1. $S_0 = 90$ and $\alpha_0 = 1$

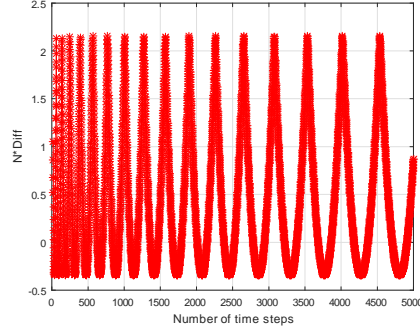


Figure 2. $S_0 = 90$ and $\alpha_0 = 2$

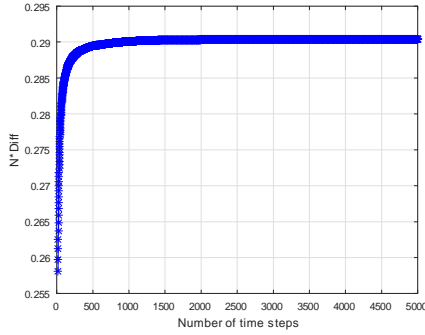


Figure 3. $S_0 = 100$ and $\alpha_0 = 1$

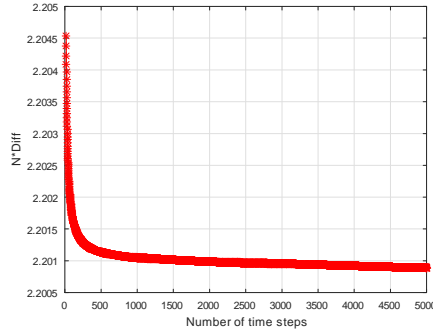


Figure 4. $S_0 = 100$ and $\alpha_0 = 2$

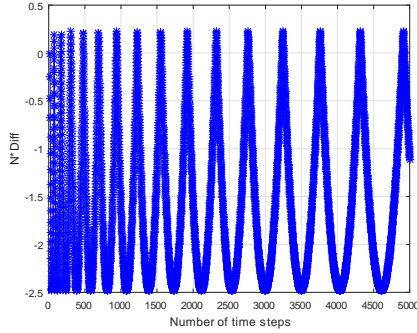


Figure 5. $S_0 = 110$ and $\alpha_0 = 1$

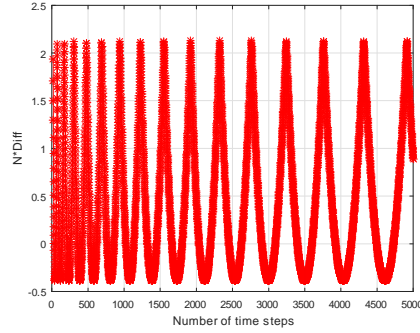


Figure 6. $S_0 = 110$ and $\alpha_0 = 2$

The numerical results for the digital put are similar to those of the European put option apart from the fact that the convergence speed is of order $\mathcal{O}(n^{-1/2})$. Tables 4 to 6 show the results for the digital put options when computed using the same set of parameters as for the put option, the same strike and the same maturity. From Tables 4 to 6 we can again draw the conclusion that in the ATM case the convergence is smooth and monotone while in the other two cases oscillations occur. The relationship between $\sqrt{n} \times \text{Err}_T^2 h(a, x)$ and n is displayed in Figures 7 to 12. Clearly all curves in the plots are bounded, illustrating that the speed of convergence of digital put options is of order $\mathcal{O}(n^{-1/2})$.

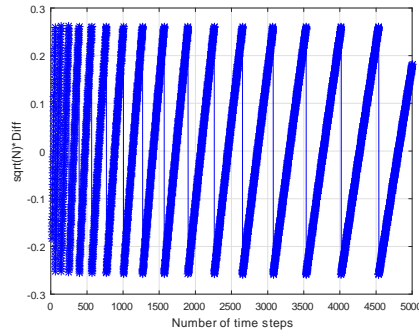
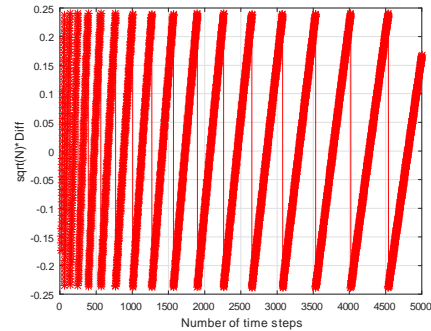
Digital put						
N	Regime 1			Regime 2		
	Price	Error	$\sqrt{N} \times \text{Error}$	Price	Error	$\sqrt{N} \times \text{Error}$
20	0.6477	-0.0410	-0.1835	0.6355	-0.0389	-0.1738
100	0.6052	0.0014	0.0140	0.5954	0.0012	0.0119
200	0.6040	0.0026	0.0372	0.5942	0.0024	0.0336
500	0.6033	0.0034	0.0756	0.5935	0.0031	0.0693
1000	0.5987	0.0080	0.2517	0.5893	0.0073	0.2318
2500	0.6054	0.0013	0.0629	0.5954	0.0012	0.0578
5000	0.6041	0.0026	0.1812	0.5942	0.0024	0.1669

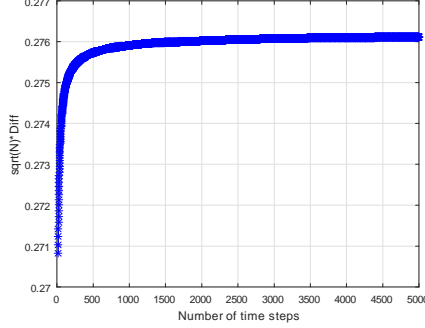
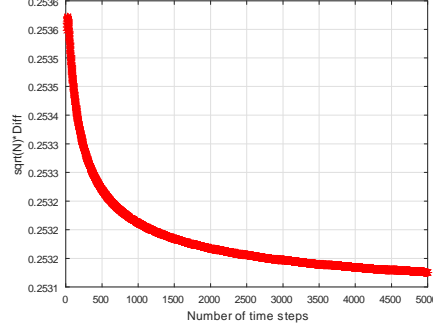
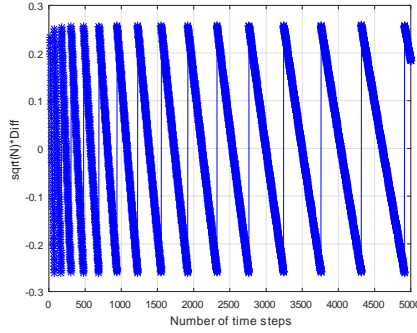
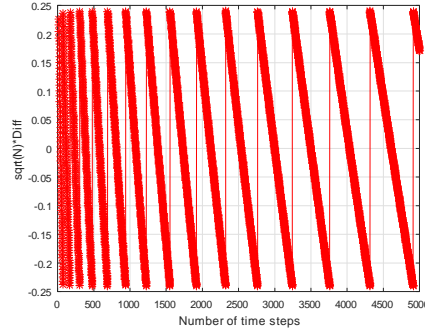
TABLE 4. Digital put option with $S_0 = 90$

Digital put						
N	Regime 1			Regime 2		
	Price	Error	$\sqrt{N} \times \text{Error}$	Price	Error	$\sqrt{N} \times \text{Error}$
20	0.4093	0.0606	0.2708	0.4144	0.0567	0.2735
100	0.4423	0.0275	0.2747	0.4457	0.0253	0.2535
200	0.4503	0.0195	0.2753	0.4532	0.0179	0.2534
500	0.4575	0.0123	0.2757	0.4597	0.0113	0.2533
1000	0.4611	0.0087	0.2759	0.4631	0.0080	0.2532
2500	0.4643	0.0055	0.2760	0.4660	0.0051	0.2532
5000	0.4659	0.0039	0.2761	0.4675	0.0036	0.2531

TABLE 5. Digital put option with $S_0 = 100$

Digital put						
N	Regime 1			Regime 2		
	Price	Error	$\sqrt{N} \times \text{Error}$	Price	Error	$\sqrt{N} \times \text{Error}$
20	0.2943	0.0522	0.2333	0.3069	0.0507	0.2265
100	0.3355	0.0109	0.1091	0.3471	0.0105	0.1046
200	0.3367	0.0098	0.1379	0.3483	0.0092	0.1304
500	0.3374	0.0091	0.2024	0.3490	0.0085	0.1893
1000	0.3419	0.0045	0.1438	0.3533	0.0043	0.1345
2500	0.3456	0.0009	0.0428	0.3567	0.0008	0.0403
5000	0.3438	0.0026	0.1833	0.3551	0.0024	0.1705

TABLE 6. Digital put option with $S_0 = 110$ Figure 7. $S_0 = 90$ and $\alpha_0 = 1$ Figure 8. $S_0 = 90$ and $\alpha_0 = 2$


 Figure 9. $S_0 = 100$ and $\alpha_0 = 1$

 Figure 10. $S_0 = 100$ and $\alpha_0 = 1$

 Figure 11. $S_0 = 110$ and $\alpha_0 = 1$

 Figure 12. $S_0 = 110$ and $\alpha_0 = 1$

10. Auxiliary results

10.1. Rate of convergence for Black-Scholes discretizations. The following result is from Leduc [11]. Recall that $\text{Err}_t^n(h)(x) := \mathcal{E}_t h(x) - \mathcal{E}_t^n h(x)$.

THEOREM 3 (Black-Scholes convergence rate for European options). *Assume that properties A1-A4 hold. Then, for every $0 < T_1 < T_2 \leq T$, and every h in $\mathcal{K}^{(2)}$,*

$$\sup_{T_1 \leq t \leq T_2} \sup_{x \geq 0} |\text{Err}_t^n(h)(x)| \leq \varkappa_2(h) \mathcal{O}(n^{-\beta}),$$

where the $\mathcal{O}(n^{-\beta})$ term is uniform in h , and where $\beta = 1/2$ if h is discontinuous and $\beta = 1$ otherwise. Furthermore, for every $x > 0$ and every real γ ,

$$\sup_{T_1 \leq t \leq T_2} |\text{Err}_t^n(I^\gamma)(x)| = x^\gamma \mathcal{O}(n^{-1}),$$

where the $\mathcal{O}(n^{-1})$ term is uniform in x .

10.2. Occupation time discretization error.

LEMMA 1. *Assume that \mathcal{L}_a^n is either L_a^n , \widehat{L}_a^n or \overline{L}_a^n , that is \mathcal{L}_a^n is either the default, snapshot or pseudo regime-state discretization defined in section 2.3. Then,*

$$\begin{aligned} E(|\mathcal{L}_a^n - L_a|) &= \mathcal{O}(n^{-1}), \\ P\left(|\mathcal{L}_a^n - L_a| > \frac{T}{4}\right) &= \mathcal{O}(n^{-1}). \end{aligned}$$

PROOF. Recall event \mathcal{A} from (2.2) where the state process α_t can have at most one jump in any subinterval $(t_{m-1}, t_m]$. Then, for any $\omega \in \mathcal{A}$, $\widehat{L}_a^n(\omega)$ differs from $L_a(\omega)$ only by aggregate errors of size less than or equal to T/n near each jump. In other words,

$$1_{\mathcal{A}} \left| \widehat{L}_a^n - L_a \right| \leq (T/n) N[0, T],$$

for $a = 1, 2$, where $N[0, T]$ is the number of jumps of α_t in the interval $[0, T]$. Furthermore, as α_t^n changes state at time t_m if and only if α_t changed state during the interval $(t_{m-1}, t_m]$, it follows that

$$1_{\mathcal{A}} \left| L_a^n - \widehat{L}_a^n \right| = 0.$$

Thus,

$$1_{\mathcal{A}} |L_a^n - L_a| \leq \left(\frac{T}{n} \right) N[0, T].$$

Then,

$$E(1_{\mathcal{A}} |L_a^n - L_a|) \leq \left(\frac{T}{n} \right) E(N[0, T]) \leq \left(\frac{T}{n} \right) E(N_{\lambda}[0, T]) = \mathcal{O}(n^{-1}),$$

where $N_{\lambda}[0, T]$ is a Poisson process with parameter $\lambda = \lambda_1 \vee \lambda_2$. Therefore,

$$P\left(1_{\mathcal{A}} |L_a^n - L_a| > \frac{T}{4}\right) \leq \frac{4E(1_{\mathcal{A}} |L_a^n - L_a|)}{T} = \mathcal{O}(n^{-1}).$$

But, according to (2.3), $P(\mathcal{A}^c) = \mathcal{O}(n^{-1})$, and because $|L_a^n - L_a| \leq T$, it follows that

$$(10.1) \quad E(|L_a^n - L_a|) = \mathcal{O}(n^{-1}),$$

$$(10.2) \quad P\left(|L_a^n - L_a| > \frac{T}{4}\right) = \mathcal{O}(n^{-1}).$$

If now \mathcal{L}_a^n is either \widehat{L}_a^n or \overline{L}_a^n , then because these random variables are bounded by T , and because they are identical on \mathcal{A} , which is a set of probability $1 - \mathcal{O}(n^{-1})$, it follows that (10.1) and (10.2) also hold when L_a^n is replaced by \mathcal{L}_a^n . \square

10.3. About the norm χ_2 on $\mathcal{K}^{(2)}$. In the task of controlling the error of option values under approximations ξ^n of geometric Brownian motions (GBM) ξ , the norm χ_2 is quite practical because it disentangles the \mathcal{O} terms from the payoff function h . This is particularly useful when considering options for which the payoff function is itself an option value of the form $\mathcal{E}_t h(x)$.

LEMMA 2. *Let ξ be a GBM with drift r and volatility σ . For every $0 \leq T_0 \leq T$ and every integer $\ell \geq 0$, there exists a constant Q such that for every $h \in C^{(0)} \cap \mathcal{K}^{(2)}$,*

$$\sup_{T_0 \leq t \leq T} \sup_{x \geq 0} \left(\left| \frac{\partial}{\partial t} \mathcal{E}_t h(x) \right| + \sum_{k=0}^{\ell} \left| x^k \frac{\partial^k}{\partial x^k} \mathcal{E}_t h(x) \right| \right) \leq Q \chi_2(h),$$

$$\sup_{T_0 \leq t \leq T} \chi_2(\mathcal{E}_t h) \leq Q \chi_2(h).$$

PROOF. Let φ be the density function of a standard normal random variable, and let

$$\mathfrak{E}_t^{(\ell)} h(x) = e^{-rt} \int_{-\infty}^{\infty} h\left(xe^{\sqrt{t}\sigma z + (r - \frac{1}{2}\sigma^2)t}\right) \varphi^{(j)}(z) dz.$$

According to [10],

$$(10.3) \quad x^k \frac{\partial^k}{\partial x^k} \mathcal{E}_t h(x) = \sum_{\ell=1}^k \alpha_\ell \sqrt{t}^{-\ell} \mathfrak{E}_t^{(\ell)} h(x).$$

Note that, for any given integer $k \geq 0$,

$$\sup_{t, x \geq 0} \left| \mathfrak{E}_t^{(k)} h(x) \right| \leq \|h\|_\infty \int_{-\infty}^{\infty} \left| \varphi^{(k)}(z) \right| dz \leq O(1) \chi_2(h).$$

It follows, in particular, that for any integer $k \geq 0$,

$$(10.4) \quad \sup_{T_0 \leq t \leq T} \sup_{x \geq 0} \left| x^k \frac{\partial^k}{\partial x^k} \mathcal{E}_t h(x) \right| \leq O(1) \chi_2(h).$$

The Black-Scholes equation

$$\frac{\partial}{\partial t} \mathcal{E}_t h(x) = r \mathcal{E}_t h(x) - r x \frac{\partial}{\partial x} \mathcal{E}_t h(x) - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2}{\partial x^2} \mathcal{E}_t h(x)$$

guarantees that

$$(10.5) \quad \sup_{T_0 \leq t \leq T} \sup_{x \geq 0} \left| \frac{\partial}{\partial t} \mathcal{E}_t h(x) \right| \leq O(1) \chi_2(h).$$

Now note that for every $t \geq 0$,

$$\text{TV} \left(\mathfrak{E}_t^{(\ell)} h \right) \leq \text{TV}(h) \int_{-\infty}^{\infty} \left| \varphi^{(j)}(z) \right| dz.$$

Hence, it follows from (10.3) that for every integer $k \geq 0$,

$$(10.6) \quad \sup_{T_0 \leq t \leq T} \text{TV} \left(x^k \frac{\partial^k}{\partial x^k} \mathcal{E}_t h(x) \right) = O(1) \text{TV}(h) \leq O(1) \chi_2(h).$$

Putting together (10.4), (10.5) and (10.6) completes the proof. \square

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