

A ROBUST METHOD TO RETRIEVE OPTION IMPLIED RISK NEUTRAL DENSITIES FOR DEFAULTABLE ASSETS

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ABSTRACT. Risk neutral densities recovered from option prices can be used to infer market participants' expectations of future stock returns and are a vital tool for pricing illiquid exotic options. Although there is a broad literature on the subject, most studies do not address the likelihood of default. To fill this gap, in this paper we develop a novel method to retrieve the risk neutral probability density function from call options written on a defaultable asset. The primary advantage of the method is that default probabilities inferred by the model can be analytically expressed and, if available, can be incorporated as an input in a flexible, robust and easily implementable manner.

1. INTRODUCTION

It is well-known that option prices contain information about the future prices of the underlying asset. In their seminal work, Breeden and Litzenberger (1978) illustrated that the discounted risk neutral density (RND) equals the second derivative of a European call option price with respect to its strike price. Therefore, given a series of options and their corresponding strikes, one can estimate the risk neutral density.

A robust method to extract RNDs is of great interest for academicians and practitioners. The connection between risk-neutral distributions and the market views of future prices was studied in Anagnou et al. (2002), Bahra (1997), and de Vincent-Humphreys and Noss (2012). Birru and Figlewski (2012) showed that exceptional detail about investors' expectations were provided by the RND of the S&P500 during the melt down of fall 2008. Du et al. (2012) used RNDs to study the natural gas market. Aït-Sahalia and Lo (2000), Jackwerth (2000), Bliss and Panigirtzoglou (2004), Jackwerth (2004), and others demonstrated how RNDs reveal valuable insights about risk preferences of the market. Dupire (1994) revealed that under minimal conditions and

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given a volatility surface it is possible to retrieve the unique diffusion process (and thus its RND), which is compatible with the volatility surface. Ross (2015) proved that under mild conditions, it is possible to fully recover the real world probability distribution corresponding to the RND. RNDs are needed to price illiquid exotic derivatives in a way that prevents static arbitrage opportunities.

Although one can estimate the risk neutral density from option prices using the Breeden-Litzenberger identity, in practice, a complete set of options prices is never available. Therefore, extracting well-behaved RNDs requires interpolation and extrapolation. This has attracted the attention of many researchers, and a vast literature exists on retrieving RNDs from observed option prices. For example, over 45 different methods are surveyed in Jackwerth (2004). Excellent surveys can also be found in Bahra (1997), Jondeau and Rockinger (2000), Skiadopoulos (2001), Monteiro et al (2008), Figlewski (2009), and Ludwig (2015). A comparison of the performance of non-parametric methods can be found in Lai (2012).

Figlewski (2009) highlighted that interpolation is typically performed in the implied volatility space, which involves fitting a spline or a low-order polynomial to the available data. A popular technique to do this task is to fit implied volatility-based models. Such techniques have excellent empirical performance, but Fengler (2009) and Monteiro et al (2011) pointed out that it is difficult to guarantee the absence of static arbitrage due to the non-linearity of the constraints in the related optimization problem. The resulting RND can take negative values and lead to mispricings. Other popular implied volatility-based models include the SABR model of Hagan et al. (2002), which is omnipresent in interest rate derivative pricing, and the Stochastic Volatility Inspired model (SVI) of Gatheral (2004). It should be noted that some implied volatility-based models can be calibrated in such a way that no static arbitrage violations occur (see for example the implementation of SVI by Gatheral and Jacquier (2013)); however, even these are not applicable to extracting RNDs from options on defaultable assets in a straightforward manner.

Clearly, retrieving RNDs from observed option prices has proved to be a challenging task. The reason for this is that finding an appropriate RND requires, as previously mentioned, the interpolation and extrapolation of call option prices in a way that satisfies several constraints:

- (1) It must well fit the observed prices of the option;
- (2) It must be free of static arbitrage;

- (3) It must smoothly interpolate observed option prices in spite of their noisiness and sparsity, avoiding overfitting,
- (4) It must be flexible enough to allow for extrapolation of option prices in a fashion consistent with the views of the user, avoiding, for instance, imposing unrealistic fat tails on the RNDs;
- (5) It must be easy to calibrate and implement;
- (6) It must be flexible and adaptable to specificities of various markets, such as equities, commodities, currencies, interest rates, etc.

The purpose of this paper is to extend Orosi (2015a) to the case of a defaultable asset. Although the likelihood of default is a critically important feature of numerous asset classes, most techniques developed to extract RNDs fail to adequately address it. Our study is based on the novel parametric implementation of the framework of Orosi (2011) who presented a multi-parameter extension of the models of Figlewski (2002) and Henderson, Hobson, and Kluge (2007). Recent empirical studies Orosi (2011, 2015a) support the use of this approach for its high empirical performance.

Although RNDs with a positive probability of default have been recently considered by Taylor et al. (2014), our work has several differences. First of all, a positive recovery can be incorporated into the model. Moreover, our framework does not make an explicit assumption about the form of the distribution and hence can be easily generalized. For example, models with a large number of parameters or nonparametric extensions are straightforward.

2. STATIC ARBITRAGE-FREE CONDITIONS FOR CALL OPTIONS

Recall that the price $C(K, T)$ of a call option with strike K , maturity T , and risk free rate r is given by

$$\begin{aligned} C(K, T) &= e^{-rT} E[\max(S_T - K, 0)] \\ &= e^{-rT} \int_{-\infty}^{\infty} \max(S_T - K, 0) p(S_T, T) dS_T, \end{aligned}$$

where $p(\cdot, \cdot)$ is the risk neutral density of the asset S_T at time T . Differentiating with respect to K , we obtain

$$\frac{\partial C(K, T)}{\partial K} = -e^{-rT} E[\theta(S_T - K)] = -B(K, T),$$

where $\theta(\cdot)$ is the Heaviside function and $B(K, T)$ is a European-style binary option with strike price K and maturity T . It follows that

$$(2.1) \quad -e^{-rT} \leq \frac{\partial C(K, T)}{\partial K} \leq 0$$

since the price of a binary option with strike price K is positive but has less value than a risk free payment of 1 at time T . Hence, the price of a call option is decreasing as a function of K .

Recall that

$$B(K, T) = e^{-rT} P(S_T > K).$$

Hence, differentiating again with respect to K gives

$$\frac{\partial^2 C(K, T)}{\partial K^2} = e^{-rT} p(K, T)$$

since

$$\frac{d}{dK} P(S_T > K) = \frac{d}{dK} (1 - P(S_T \leq K)) = -p(K, T).$$

It follows that

$$(2.2) \quad \frac{\partial^2 C(K, T)}{\partial K^2} \geq 0.$$

Therefore, the price of the call option is convex as a function of strike K .

Finally, the price of a call option with zero strike is given by

$$(2.3) \quad C(0, T) = e^{-rT} \int_0^\infty S_T p(S_T, T) dS_T = S_0 e^{-dT},$$

where d is the continuous dividend yield.

It is known that there are several conditions that ensure no static arbitrage, but under minimal regularity assumptions allowing the differentiations above, conditions (2.1), (2.2) and (2.3) guarantee that at maturity T , the call prices are free of static arbitrage. In this paper, we focus on these conditions.

3. ARBITRAGE-FREE CALL OPTION PRICING MODEL

Our approach closely follows that of Orosi (2011, 2015a) and assumes that the transformed call option price $c = C(K, T) / (S \exp(-dt))$ for a given maturity T with strike price K can be retrieved from the following equation:

$$(3.1) \quad x = 1 - c + f(c) = 1 - c + \frac{F(c)}{c^\alpha} (1 - c)^\beta,$$

where $x = K \exp(-rT) / (S \exp(-dT))$, $\beta \geq 1$, and α is a non-zero positive parameter. This representation with $F(c) = G$, G a constant, $\beta = 1$, and $\alpha = 1$ was originally considered by Henderson, Hobson, and Kluge (2007) who extended a related model of Figlewski (2002). Henderson et al. point out that rather than attempting to determine suitable functions $c = f(x)$, it is easier to look for the inverse function $x = f^{-1}(c)$, where $f^{-1}(c)$ is a decreasing, convex function that satisfies $f^{-1}(c) \geq 1 - c$ (the intrinsic value of call options). A function satisfying these conditions is graphed in the left part of Figure 1.

Given observed option prices $\{C_i\}_{i=1}^n$ at strike values $\{X_i\}_{i=1}^n$, the model's parameters can be determined by minimizing the following objective:

$$\sum_{i=1}^n (c(x_i) - c_i)^2,$$

where $c_i = C_i / (S \exp(-dT))$, $x_i = X_i e^{-rT} / (S \exp(-dT))$, and $c(x_i)$, is determined numerically from

$$x_i = 1 - c(x_i) + \frac{F(c(x_i))(1 - c(x_i))^\beta}{c(x_i)^\alpha}.$$

In this study, we use the following representation for $F(c)$:

$$F(c) = G(Ac + 1)^\gamma.$$

Orosi (2015a) showed that this representation has the following advantages:

- The conditions (2.1), (2.2), and (2.3) are satisfied when the parameters are restricted to one of the following intervals:

$$G > 0, \alpha > 0, \gamma > 1, 0 > A > -1;$$

or

$$G > 0, \alpha > 0, 0 > \gamma, A > 0.$$

It is worthwhile to point out that, in the literature, several interpolants do not satisfy (2.3).

- If $\beta = 1$, the model assumes a non-zero probability of default. In this study, we assume that $\beta = 1$.
- The method is capable of extracting risk neutral densities with complex characteristics, such as bimodal distributions.
- The probability of default can be analytically calculated.
- The model can be easily generalized to incorporate more parameters. For example, considering

$$F(c) = G(Ac + 1)^\gamma g(c)$$

yields arbitrage-free call option prices as long as $g(c)$ is a positive, decreasing, and convex function.

4. CALCULATING THE PROBABILITY OF DEFAULT

With $\beta = 1$, (3.1) becomes

$$(4.1) \quad x = 1 - c + \frac{F(c)}{c^\alpha} (1 - c).$$

Implicitly differentiating this equation with respect to x , and making use of

$$\begin{aligned} B(K, T) &= -\frac{\partial C(K, T)}{\partial K} \\ &= -\frac{\partial(C)}{\partial x} \frac{\partial(x)}{\partial(K)} \\ &= -\frac{\partial(cSe^{-dT})}{\partial x} \frac{\partial\left(\frac{Ke^{-rT}}{Se^{-dT}}\right)}{\partial(K)} \\ &= -\frac{\partial c}{\partial x} e^{-rT}, \end{aligned}$$

we obtain

$$1 = -\left(\frac{F(c)}{c^\alpha} + 1\right) \frac{\partial c}{\partial x} + (1 - c) \frac{d\left(\frac{F(c)}{c^\alpha} + 1\right)}{dc} \frac{\partial c}{\partial x}.$$

Substituting $c = 1$ and using

$$-\frac{\partial c}{\partial x} = e^{rT} B(0, T) = P(S_T > 0)$$

gives

$$1 = (F(1) + 1) P(S_T > 0).$$

It follows that the probability of the asset not defaulting prior to T can be expressed as

$$(4.2) \quad P(S_T > 0) = \frac{1}{F(1) + 1} = 1 - PD,$$

where PD is the probability of default.

5. AN APPLICATION TO A DEFAULTABLE STOCK

To illustrate the applicability of our method, we calibrate our model to options on the General Motors stock with $T = 1.63$ on May 30, 2008, when the market saw a high probability of default before the option's expiry. After calculating option premiums using market observed volatilities with the Black-Scholes formula, our model was calibrated to the resulting European-style option values. The best fit parameters were obtained as follows: $G = 0.4742$, $\alpha = 0.4577$, $A = 0.0121$, and $\gamma = -100$. Using (4.2), it can be inferred that $PD = 12.5\%$. The resulting cumulative distribution and probability density are shown in Figures 2 and 3, respectively.

6. INCORPORATING THE PROBABILITY OF DEFAULT

Often, the probability of default PD can be reliably estimated; see Finkelstein (2001) for instance. When this is the case, $P(S_T > 0) = 1 - PD$ is given, and equation (4.2) can be rearranged to yield the following restriction on parameter G :

$$G = \frac{PD}{(1 - PD)(A + 1)^\gamma}.$$

Hence, if the probability of the asset defaulting by time T is known *a priori*, it can be incorporated into our model as a parameter. This approach is illustrated by calibrating the model to previously considered General Motors stock options with various default probability values. The resulting cumulative distributions and probability densities are shown in Figures 4 and 5, while the resulting call option values are depicted in Table 2. Although the different values of default probabilities lead to almost identical European call option prices, it can be observed from Figure 4 that the resulting binary prices vary significantly. Hence, when probabilities of defaults are available, exotic European options can be priced with higher accuracy.

7. NON-ZERO RECOVERY

In this section, we consider the case when the value of the stock in the case of default is given by R , a non-zero value. Moreover, Chang and Orosi (2016) show that

$$(7.1) \quad C(K, T) = \begin{cases} Se^{-dT} - Ke^{-rT} & \text{if } K \leq R \end{cases} .$$

Then, the transformed call prices must satisfy

$$c = 1 - x$$

when $K \leq R$, or equivalently,

$$x = 1 - c,$$

and transformed call, c^* , at $K = R$ is given by

$$c^* = 1 - R \exp(-rT) / (S \exp(-dT)).$$

Then, letting $c^* = 1 - R \exp(-rT) / (S \exp(-dT))$, the function

$$(7.2) \quad x = \begin{cases} 1 - c & \text{if } K \leq R \\ 1 - c + G \frac{(c^* - c)}{c^\alpha} & \text{if } K > R \end{cases},$$

is convex, monotone, and continuously differentiable as long as $G > 0$, and $\alpha > 0$. Moreover, it can be shown that PD is given by

$$\begin{aligned} PD &= 1 - P(S_T > R) = \\ &= 1 - \frac{1}{1 + \frac{G}{(c^*)^\alpha}}. \end{aligned}$$

Therefore, using our model with (7.2) in lieu of (4.1), one can retrieve an RND in the presence of positive recovery. Orosi (2015b) points out that the options on General Motors stock on April 15, 2009 must have assumed a non-zero recovery because the stock price was $S = 1.89$ and the value of the option at $K = 1$ is given by (7.1) (see Table 3). To illustrate the applicability of our method, we calibrated our model with $R = 1$. The best fit parameters were obtained as follows: $G = 0.4671$ and $\alpha = 0.8979$ that imply that $PD = 47.8\%$.

8. CONCLUSION

In this paper, we employ a new, flexible, robust, and easy to implement method to calculate the risk neutral densities extracted from call options written on a defaultable asset. An interesting feature of our method is that default probabilities can be expressed analytically or, when available, incorporated as an input. To demonstrate the applicability of the method, we fit the model to vanilla options written on General Motors stock. Our results show that the probability default has a significant impact on the resulting binary options and little impact on the vanilla option prices. We additionally showed how our model can be extended to incorporate a positive recovery that can also be inferred by the model or given as an input.

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