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OPTION CONVERGENCE RATE WITH GEOMETRIC RANDOM WALKS APPROXIMATIONS

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Abstract. We describe a broad setting under which, for European options, if the underlying asset form a geometric random walk then, the error with respect to the Black-Scholes model converges to zero at a speed of $1/n$ for continuous payoffs functions, and at a speed of $1/\sqrt{n}$ for discontinuous payoffs functions.

1. Introduction

1.1. Motivation. Throughout this paper we assume a constant risk free rate r. Recall that under the Black-Scholes model, the value S_t of an asset satisfies

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where the drift $r > 0$ is the risk free rate, σ the volatility, and W_t is a Brownian motion. This is equivalent to

$$S_t = S_0 \exp\left(\sigma W_t + \left(r - \frac{1}{2}\sigma^2\right) t\right),$$

for some Brownian motion W_t. We are interested in European options with *maturity* $T > 0$, and we divide the time interval $[0, T]$ into n regular subintervals $[t_{k-1}, t_k]$ defined by the *time steps* $t_k = kT/n$, for $k = 0, 1, ..., n$. Note that the effective yield rate $X[t_{k-1}, t_k]$ of S over the period $[t_{k-1}, t_k]$ is given by

$$X[t_{k-1}, t_k] \stackrel{def}{=} \ln\left(\frac{S_{t_k}}{S_{t_{k-1}}}\right) = \sigma\left(W_{t_k} - W_{t_{k-1}}\right) + \left(r - \frac{1}{2}\sigma^2\right)\frac{T}{n}. \tag{1.1}$$

Well known properties of the Brownian motions guarantee that random variables $X[t_{k-1}, t_k]$, $k = 1, 2, ...$ are independent and identically

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distributed. We set $X \overset{def}{=} X[0, T/n]$ and $A \overset{def}{=} \exp(X)$. Obviously

$$S_t = S_0 \exp\left(\sum_{k=1}^{\lfloor nt/T \rfloor} X[t_{k-1}, t_k]\right),$$

for every $t \in (T/n)\mathbb{N}$.

Consider now geometric random walks $\left\{S^{(n)}\right\}$ of the form

$$S_t^{(n)} = S_0 \exp\left(\sum_{k=1}^{\lfloor nt/T \rfloor} X_n[t_{k-1}, t_k]\right), \tag{1.2}$$

with random variables $X_n[t_{k-1}, t_k]$ independent and identically distributed. Note that

$$X_n[t_{k-1}, t_k] = \ln\left(\frac{S_{t_k}^{(n)}}{S_{t_{k-1}}^{(n)}}\right). \tag{1.3}$$

Set $X_n \overset{def}{=} X_n[0, T/n]$ and $A_n \overset{def}{=} \exp(X_n)$.

The assumption that the variables $X_n[t_{k-1}, t_k]$ are independent and identically distributed is known as the *random walk hypothesis*. It is a *strong* but extensively studied criteria of *market efficiency*. The term 'random walk hypothesis' is due to Nobel laureate in Economics Eugene Fama [12] in his 1965 article '*Random Walks in Stock Market Prices*' and made famous by Malkiel [37] in his book '*A Random Walk Down Wall Street*'. This paper ties up the random walk hypothesis to the Black-Scholes model: we find conditions on X_n such that for a broad class of option payoffs h, the price of security derivatives for which the underlying asset is approximated by $S^{(n)}$ converges to the Black-Scholes price at a speed of $1/n$, providing that the payoff function is continuous, and $1/\sqrt{n}$ otherwise.

The problem of describing and controlling option value errors resulting from evaluating *European options* under *binomial tree scheme* approximations of the Black-Scholes model has attracted the attention of numerous researchers and remains an active and vibrant research topic. An argument to establish a speed of convergence of order n^{-1} for the European call option under a few specific binomial models is proposed in Leisen and Reimer [35]. The authors construct a tree with a convergence of order n^{-2} but this was not actually proven until Joshi [20]. Explicit formulae for the error in *binomial tree schemes* can be found in Walsh [43], Diener and Diener [10, 11], and Leduc [32]. Chang and Palmer [6] also derives explicit error formulae for call and digital options in binomial tree schemes.

Because the convergence of binomial trees displays notoriously oscillatory behaviors, ways to smooth and accelerate the convergence have been sought after. For *European options* under *binomial tree schemes* we have the following results. Chang and Palmer [6] proposed a method to achieve a convergence for European options of the form $\alpha/n+o(1/n)$. A smooth convergence in the sense of achieving an asymptotic expansion in powers of $1/n$ was obtained by Joshi in [19]. Joshi [20] showed how his tree method could reach arbitrarily fast convergence when the number of time steps is odd. His argument was extended in Xiao [45] to even numbers of time steps. Korn and Muller [24] showed how to optimize the constant α in Chang and Palmer [6]. It was shown in Leduc [33] that arbitrarily fast convergence can be reached for a large class of binomial tree schemes. Joshi [21, Chapter 28] contains a thorough coverage of tree methods and its literature.

The speed of convergence for *American options* when the underlying is asset is approximated by *binomial trees* has attracted a lot of interest. In Leisen [34], an argument is proposed to establish a speed of convergence of n^{-1} for put options under some specific models. An analysis of this convergence can be found in Heston and Zhou [14]. In Hu et al. [15], it is shown that uniformly over the nodes of the tree, the error is of order $n^{-1/2}$. As shown in Leduc [30], the price of American options can be decomposed into the sum of a European option and a continuously paying option. Under specific assumptions, a rate of convergence of n^{-1} was obtained in Leduc [31] for continuously paying options.

For *binomial tree approximations* and *game options* a speed of convergence of $n^{-1/4}\left(\ln n\right)^{3/4}$ was established in Kifer [23]. An explicit first order error formula was obtained in Lin and Palmer [36] for *European barrier options*.

Because the results have been so far quite specific in terms of the approximating schemes that were considered with a focus on binomial schemes, Ahn and Song proved in [1] the mere convergence of European and American options for *trinomial trees*. An analysis of the convergence of option prices in *trinomial trees* is done in Xiaoping et al. [46]. Option rate of convergence for *binomial* and *trinomial trees is* also studied in Chung and Shih [8].

Regarding the speed of convergence of European options, Walsh [43] considered a class of payoff functions which is very general for any practical purposes, but his result is limited to the specific random walk obtained when the CRR *binomial tree scheme* is applied to the discounted process. In Diener and Diener [10] a fairly general class of *binomial tree schemes* is considered but the convergence rate of n^{-1} is established only for the *call option*. In Leduc [32], the gap between [43] and [10] is filled and the speed of convergence for European option is established for both a *general class of payoffs* and a general class a *binomial tree schemes*.

As in Lamberton [27], our approximation setting is described through conditions on X_n and on the payoff function h.

Assumption 1. *Random walks* $\left\{S^{(n)}\right\}$ *are of the form (1.2) and satisfy the following conditions:*

$$\mu_n \overset{def}{=} E\left(X_n\right) = \frac{T}{n}\left(\left(r-\frac{1}{2}\sigma^2\right)+\mathcal{O}\left(n^{-\frac{1}{2}}\right)\right), \tag{A1}$$

$$\sigma_n \overset{def}{=} \sqrt{Var\left(X_n\right)} = \sqrt{\frac{T}{n}}\left(\sigma+\mathcal{O}\left(n^{-\frac{1}{2}}\right)\right). \tag{A2}$$

Furthermore, for every real constant γ*,*

$$E\left(\exp\left(\gamma X_n\right)\right) = E\left(\exp\left(\gamma X\right)\right)+\mathcal{O}\left(n^{-2}\right), \tag{A3}$$

$$E\left(\exp\left(\gamma\sqrt{\frac{n}{T}}X_n\right)\right) = \mathcal{O}\left(1\right). \tag{A4}$$

We also describe a general class of payoff functions h for which the rate of convergence will be established. We say that a function h is piecewise $C^{(m)}$, for some integer $m \geq 0$, if there exists countably many intervals $J_\ell := [\beta_\ell, \beta_{\ell+1})$, $\beta_0 < \beta_1 < ...$, forming a partition of $[0,\infty)$ and functions h_ℓ extendible to be $C^{(m)}$ on the closure of J_ℓ, such that

$$h\left(x\right) = \sum_{\ell=0}^{\infty} h_\ell\left(x\right) 1_{[\beta_\ell,\beta_{\ell+1})}\left(x\right).$$

We use I to denote the identity function, that is $I\left(z\right) := z$ for every z. Given an integer k, we set $I^k\left(z\right) := z^k$. We denote by $\mathcal{K}^{(m)}$ the class of piecewise $C^{(m)}$ functions such that h, Ih', ..., $I^m h^{(m)}$ have a limit at infinity and are of bounded variation over $[0,\infty)$. Clearly, for any $h \in \mathcal{K}^{(m)}$, functions h, Ih', ..., $I^m h^{(m)}$ are bounded and we define a norm $\varkappa_m$ on $\mathcal{K}^{(m)}$ by

$$\varkappa_m\left(h\right) = \sum_{k=0}^{m}\left(\mathrm{TV}\left(I^k h^{(k)}\right)+\left\|I^k h^{(k)}\right\|_\infty\right).$$

Finally we denote by $\mathcal{E}_t h\left(x\right)$ and $\mathcal{E}_t^n h\left(x\right)$ the price of a European option with maturity t and payoff function h, when the spot price is x and the underlying asset is respectively S and $S^{(n)}$. The core of this paper is devoted to proving the following theorem:

Theorem 1. *If $\left\{S^{(n)}\right\}$ is a collection of risk neutral geometric random walks satisfying Assumption 1 then, for every $0 < T_1 < T_2 \leq T$, there exists a constant Q which may depend only on T_1, T_2, T, r, σ and $\left\{S^{(n)}\right\}$ such that for every h in $\mathcal{K}^{(2)}$*

$$\sup_{T_1 \leq t \leq T_2} \sup_{x \geq 0} \left|\mathcal{E}_t h\left(x\right) - \mathcal{E}_t^n h\left(x\right)\right| \leq Q \left(\frac{1}{\sqrt{n}} \sum_{x>0} \left|\Delta h\left(x\right)\right| + \frac{1}{n} \varkappa_2\left(h\right) \right).$$

Hinging on Theorem 1, section 2 below extends the rate of convergence of the above result to polynomial payoffs and proves that the requirement of risk neutrality can be dropped.

General *geometric random walk* (GRW) approximations, similar to those described here, have been considered in Lamberton's random walk setting: Lamberton [26, 27], Carbone [3], Lamberton and Rogers [29]. Yet, in the case of the American put, only a suboptimal convergence of order $\left(n^{-1}\log n\right)^{4/5}$ could be proved. Furthermore, as pointed out in [10, p. 273], because of the risk neutrality requirement of typical *binomial* and *trinomial trees*, such models do not generally fit into Lamberton's setting. The randomized binomial trees studied in Xiaoping and Jie [47] can also be seen as GRW, but their convergence towards the Black-Scholes model was not analyzed. The purpose of this paper is precisely to develop a broad setting, that includes common binomial and trinomial tree schemes, for which we prove an optimal speed of convergence for European options values of order $1/n$ when the payoff function is continuous and $1/\sqrt{n}$ otherwise. This serves as the foundation under which our results will be extended in a further paper to American options, filling the gap left in [27] and solving a difficult [28, section 4.4.2] and long lasting problem.

1.2. **Notation.** Our results and proofs are best expressed using the notation and conventions of [32] which is gathered in this section.

Parameters r, σ, T are fixed throughout this paper, and expressions in terms of these parameters are considered constants. To simplify the presentation we suppose that $T \leq 1$ although our results remain valid for any $T > 0$. Given n, t_m denotes the m^{th} time step, that is, $t_m = m\frac{T}{n}$. The symbol E_x denotes the expectation when $S_0 = x$. As no ambiguity is possible, we also use the same symbol for the expectation with respect to $S^{(n)}$. We always assume that $S_0^{(n)} = S_0$. Throughout this paper $F_{S_{t_m}}$ and $F_{S_{t_m}^{(n)}}$ denote the cumulative distribution functions of S_{t_m} and $S_{t_m}^{(n)}$. For $t, x \geq 0$ and function h, we denote

$$\mathcal{E}_t h\left(x\right) \stackrel{def}{=} e^{-rt} E_x\left(h\left(S_t\right)\right),$$

and similarly

$$\mathcal{E}_t^n h(x) \stackrel{def}{=} e^{-rt} E_x(h(S_t^{(n)})).$$

Note that $\mathcal{E}$ and $\mathcal{E}^n$ simply denote the discounted expectation. Option prices are given by the discounted expectation of the payoff in the risk neutral world. Hence, when $S_0 = x$, the price of a European option with payoff h and maturity T is $\mathcal{E}_T h(x)$ in model S and $\mathcal{E}_T^n h(x)$ in model $S^{(n)}$. We denote by $\operatorname{Err}_t^n h(x)$ the error between the price of a European option when the underlying S is approximated by $S^{(n)}$. Hence

$$\operatorname{Err}_t^n h(x) \stackrel{def}{=} \mathcal{E}_t h(x) - \mathcal{E}_t^n h(x).$$

We denote by I the identity function: $I(z) \stackrel{def}{=} z$ for every z, and given an integer $k \geq 0$, we have $I^k(z) := z^k$. Among other things, this allows to define for any integer $k \geq 0$, the important quantities

$$\Delta_k^{(n)} \stackrel{def}{=} \operatorname{Err}_{\frac{T}{n}}^n \left((I-1)^k\right)(1)$$

where

$$\operatorname{Err}_{\frac{T}{n}}^n \left((I-1)^k\right)(1) = e^{-r\frac{T}{n}} E_1((S_{\frac{T}{n}} - 1)^k - (S_{\frac{T}{n}}^{(n)} - 1)^k).$$

The terms $n^2 \Delta_k^{(n)}$ are significant as they drive the oscillations of the error $\operatorname{Err}_T^n(h)$ in Theorem 4 below. Lemma 3 in the appendix guarantees that

$$\Delta_k^{(n)} = \mathcal{O}(n^{-\left(2 \vee \frac{k}{2}\right)}).$$

Finally, expressions such as $A = B + C\mathcal{O}(n^{-1})$ mean that there exists a constant Q (which cannot depend on C but may depend only on some constants such as r, σ, T) for which $|A - B| \leq CQn^{-1}$.

1.3. **Outline.** Consider the following conditions on $S^{(n)}$:

P0: $S^{(n)}$ is risk neutral.

P1: For every $0 < t_m \leq T$, $t_m \in \frac{T}{n}\mathbb{N}$,

$$\sup_{x \geq 0} \left| F_{S_{t_m}^{(n)}}(x) - F_{S_{t_m}}(x) \right| = \sqrt{m^{-1}}\mathcal{O}(1).$$

P2: For any fixed integer $M \geq 0$,

$$\mathcal{E}_{\frac{T}{n}}^n \left(|I-1|^M\right)(1) = \mathcal{O}\left(n^{-\frac{M}{2}}\right).$$

P3: For any fixed integer $M \geq 0$,

$$\Delta_M^{(n)} \stackrel{def}{=} \operatorname{Err}_{\frac{T}{n}}^n \left((I-1)^M\right)(1) = \mathcal{O}(n^{-\left(2 \vee \frac{M}{2}\right)}).$$

P4:

$$\mathcal{E}^n_{\frac{T}{n}}\left(\left|\ln\left(I\right)\right|\right)(1)=\mathcal{O}\left(n^{-\frac{1}{2}}\right). \tag{P4a}$$

Furthermore, for any fixed real number γ,

$$\mathcal{E}^n_{\frac{T}{n}}\left(I^{\gamma}\right)(1)=\mathcal{E}_{\frac{T}{n}}\left(I^{\gamma}\right)(1)+\mathcal{O}\left(n^{-2}\right), \tag{P4b}$$

and

$$\mathcal{E}^n_{\frac{T}{n}}\left(\left|I^{\gamma}-1\right|\right)(1)=\mathcal{O}\left(n^{-\frac{1}{2}}\right), \tag{P4c}$$

$$\max_{j=0,\ldots,n}\left|\mathcal{E}^n_{\frac{jT}{n}}\left(I^{\gamma}\right)(x)-\mathcal{E}_{\frac{jT}{n}}\left(I^{\gamma}\right)(x)\right|=x^{\gamma}\mathcal{O}\left(n^{-1}\right), \tag{P4d}$$

$$\max_{j=0,\ldots,n}\left|\mathcal{E}^n_{\frac{jT}{n}}\left(I^{\gamma}\right)(x)\right|=x^{\gamma}\mathcal{O}\left(1\right). \tag{P4e}$$

P5: For any fixed integer γ and any fixed integer $M\geq 0$,

$$\mathcal{E}^n_{\frac{T}{n}}\left(\left|\int_1^I u^{\gamma}\left(I-u\right)^M du\right|\right)(1)=\mathcal{O}(n^{-\frac{M+1}{2}}).$$

In section 4 we prove that under Assumption 1, conditions **P1-P5** hold. In section 3 we assume that **P0-P5** hold to prove Theorem 1. Section 2 discusses Assumption 1 and extends Theorem 1 to a class of polynomially bounded payoff functions while dropping the condition **P0** of risk neutrality for $S^{(n)}$.

2. Discussion and main result

In this section we examine more closely Assumption 1. First we explain why **A3** is equivalent to having all options with a polynomial payoff converging at a rate of n^{-1}. Then we show that the risk neutral requirement of Theorem 1 can be dropped. Putting this together, we obtain the main result of this paper. Next we survey the literature of binomial and trinomial trees to see which ones fall under Assumption 1 and if those who don't have a slower convergence rate. We prove that, for binomial trees, Assumption 1 is equivalent to having a convergence of order n^{-1} for all put options, all call options, and all options with a polynomial payoff. Finally we show how that the class of geometric random walks considered in Lamberton [27] fall under Assumption 1.

2.1. **Main result.** Let $\mathcal{P}$ denote the set of all polynomials with real coefficients and real exponents. Note that **P4d** is equivalent to saying that a convergence of order n^{-1} occurs for every polynomial in $\mathcal{P}$. Furthermore, **P4d** is equivalent to **A3** according to Lemma 2 in section 4.2 below. Thus condition **A3** is *equivalent* to having a convergence of order n^{-1} for every polynomial in $\mathcal{P}$. It follows easily from **P4d** that, given $0 < T_1 \leq T_2 \leq T$ and a real γ,

$$\sup_{T_1 \leq t \leq T_2} \left| \mathrm{Err}_t^n \left(I^\gamma \right) (x) \right| = x^\gamma \mathcal{O}\left(n^{-1} \right). \tag{2.1}$$

Note that, since the payoff of a put option belong to $\mathcal{K}^{(2)}$, the put-call parity implies that a convergence at a speed of n^{-1} also occurs for call options. Yet it must be underlined that because of (2.1), only a pointwise convergence in x can be obtained, not a uniform one.

We now show how the risk neutrality requirement of Theorem 1 can be dropped. First note that, with $\gamma = 1$, condition **A3** already requires $S^{(n)}$ to be risk neutral *up to a negligible term of order* n^{-2}. Indeed from **A3,**

$$E\left(\exp\left(X_n\right)\right) = e^{r\frac{T}{n}} + \delta_n,$$

where $\delta_n = \mathcal{O}\left(n^{-2}\right)$. Define

$$\alpha_n := \ln\left(\frac{e^{r\frac{T}{n}}}{e^{r\frac{T}{n}} + \delta_n}\right),$$
$$Y_n := X_n + \alpha_n,$$

and note that

$$\alpha_n = \ln\left(1 - \frac{\delta_n}{e^{r\frac{T}{n}} + \delta_n}\right) = \mathcal{O}\left(n^{-2}\right).$$

Clearly

$$E\left(\exp\left(Y_n\right)\right) = e^{\alpha_n} E\left(\exp\left(X_n\right)\right) = e^{r\frac{T}{n}}.$$

Thus Y_n determines a risk neutral family of random walks $\widetilde{S}^{(n)}$. This family obviously satisfies Assumption 1**,** given that $S^{(n)}$ does.

Fix constants T_1 and T_2, $0 < T_1 \leq T_2 \leq T$. For any $t \in [T_1, T_2]$, let $\mathfrak{m} := \mathfrak{m}(t) = \lfloor nt/T \rfloor$ be the largest integer such that $t_{\mathfrak{m}} = \mathfrak{m}T/n \leq t$. For any $h \in \mathcal{K}^{(2)}$ we have

$$e^{-rt} E_x\left(h\left(\widetilde{S}_t^{(n)}\right)\right) = e^{-rt} E_x\left(h\left(e^{\theta_n} S_t^{(n)}\right)\right),$$

where $\theta_n := \theta_n(t)$ is defined by $\theta_n = \mathfrak{m}\alpha_n$. Note that,

$$0 < \left|\theta_n\right| \leq n \left|\alpha_n\right| = \mathcal{O}\left(n^{-1}\right).$$

Define now for every real $z \geq 0$,

$$h^{\pm}_{\theta_n}(z) := h\left(e^{\pm\theta_n} z\right).$$

Note that

$$e^{-rt} E_x\left(h\left(\widetilde{S}^{(n)}_t\right)\right) = e^{-rt} E_x\left(h^{+}_{\theta_n}\left(S^{(n)}_t\right)\right),$$
$$e^{-rt} E_x\left(h^{-}_{\theta_n}\left(\widetilde{S}^{(n)}_t\right)\right) = e^{-rt} E_x\left(h\left(S^{(n)}_t\right)\right).$$

Moreover, if

$$\widetilde{\mathrm{Err}}^{n}_t(h)(x) \overset{def}{=} e^{-rt} E_x\left(h\left(\widetilde{S}^{(n)}_t\right)\right) - e^{-rt} E_x\left(h\left(S_t\right)\right),$$

then

$$\widetilde{\mathrm{Err}}^{n}_t\left(h^{-}_{\theta_n}\right)(x) = \mathrm{Err}^{n}_t(h)(x) + e^{-rt}\left(E_x\left(h\left(S_t\right)\right) - E_x\left(h\left(e^{-\theta_n} S_t\right)\right)\right).$$

Now since

$$e^{-\theta_n} = 1 + \mathcal{O}\left(n^{-1}\right),$$
$$E_x\left(h\left(e^{-\theta_n} S_t\right)\right) = E_{xe^{-\theta_n}}\left(h\left(S_t\right)\right),$$

one verifies that, if h is a digital option, then (uniformly on the strike)

$$\sup_{T_1 \leq t \leq T_2} \sup_{x \geq 0} \left|E_x\left(h\left(S_t\right)\right) - E_x\left(h\left(e^{-\theta_n} S_t\right)\right)\right| = \mathcal{O}\left(n^{-1}\right).$$

On the other hand if $h \in \mathcal{K}^{(2)}$ is continuous then, from

$$\left|h(z) - h^{-}_{\theta_n}(z)\right| \leq \|Ih'\|_{\infty} \mathcal{O}\left(n^{-1}\right),$$

we get

$$\sup_{T_1 \leq t \leq T_2} \sup_{x \geq 0} \left|E_x\left(h\left(S_t\right)\right) - E_x\left(h\left(e^{-\theta_n} S_t\right)\right)\right| \leq \|Ih'\|_{\infty} \mathcal{O}\left(n^{-1}\right).$$

Therefore, decomposing h into a sum of digital options and a continuous member of $\mathcal{K}^{(2)}$, it is easy to see that

$$\sup_{T_1 \leq t \leq T_2} \sup_{x \geq 0} \left|E_x\left(h\left(S_t\right)\right) - E_x\left(h\left(e^{-\theta_n} S_t\right)\right)\right| = \varkappa_2(h) \mathcal{O}\left(n^{-1}\right).$$

Hence

$$\widetilde{\mathrm{Err}}^{n}_t\left(h^{-}_{\theta_n}\right)(x) = \mathrm{Err}^{n}_t(h)(x) + \varkappa_2(h) \mathcal{O}\left(n^{-1}\right).$$

Noting that

$$\varkappa_2\left(h^{-}_{\theta_n}\right) = \mathcal{O}(1) \varkappa_2(h),$$
$$\sum_{x>0} \left|\Delta h^{-}_{\theta_n}(x)\right| = \sum_{x>0} \left|\Delta h(x)\right|,$$

we obtain from Theorem 1 that

$$\sup_{T_1 \leq t \leq T_2} \sup_{x \geq 0} \left|\widetilde{\mathrm{Err}}^{n}_t\left(h^{-}_{\theta_n}\right)(x)\right| = \mathcal{O}\left(n^{-\frac{1}{2}}\right) \sum_{x>0} \left|\Delta h(x)\right| + \varkappa_2(h) \mathcal{O}\left(n^{-1}\right).$$

Thus

$$\sup_{T_1\leq t\leq T_2}\sup_{x\geq 0}\left|\mathrm{Err}_t^n(h)(x)\right| = \mathcal{O}\left(n^{-\frac{1}{2}}\right)\sum_{x>0}\left|\Delta h(x)\right| + \varkappa_2(h)\,\mathcal{O}\left(n^{-1}\right).$$

Putting this together with (2.1) we obtain the main result of this paper:

Theorem 2 (Uniform speed of convergence for European options)**.** *If* $\left\{S^{(n)}\right\}$ *is a collection of geometric random walks satisfying Assumption 1 then, for every* $0 < T_1 < T_2 \leq T$*, there exists a constant* Q *which may depend only on* T_1*,* T_2*,* T*,* r*,* σ *and* $\left\{S^{(n)}\right\}$ *such that for every* h *in* $\mathcal{K}^{(2)}$

$$\sup_{T_1\leq t\leq T_2}\sup_{x\geq 0}\left|\mathrm{Err}_t^n(h)(x)\right| \leq Q\left(\frac{1}{\sqrt{n}}\sum_{x>0}\left|\Delta h(x)\right| + \frac{1}{n}\varkappa_2(h)\right);$$

furthermore for every $x > 0$ *and every real* γ*,*

$$\sup_{T_1\leq t\leq T_2}\left|\mathrm{Err}_t^n(I^\gamma)(x)\right| = x^\gamma\mathcal{O}\left(n^{-1}\right).$$

2.2. **A survey of binomial and trinomial trees.** Let $\Delta t := T/n$. Recall that a self-similar *trinomial* tree $S^{(n)}$ is a stochastic process which at every positive time t in $(\Delta t)\,\mathbb{N}$, has a probability p_n^u of jumping from its current state $S_t^{(n)}$ to the state $S_t^{(n)}u_n$, a probability p_n^d of jumping to the state $S_t^{(n)}d_n$, and a probability $1-p_n^u-p_n^d$ of jumping to the state $S_t^{(n)}m_n$, for some $u_n, d_n, m_n > 0$. A self-similar *binomial* tree is a special case with $p_n^u+p_n^d = 1$. This corresponds, in our setting, to a GRW $S^{(n)}$ where the random variable X_n takes the value $\ln(u_n)$ with probability p_n^u, the value $\ln(m_n)$ with probability $1-p_n^u-p_n^d$, and the value $\ln(d_n)$ with probability p_n^d.

Note that, given $\ln(u_n) = \mathcal{O}(\sqrt{\Delta t})$ and $\ln(d_n) = \mathcal{O}(\sqrt{\Delta t})$, condition **A4** is trivially satisfied. Furthermore, given expansions of u_n, d_n, m_n, p_n^u, and p_n^d in powers of $\sqrt{\Delta t}$, it is always elementary to check if a tree satisfy the other conditions of Assumption 1. Essentially following Joshi [21, Chapter 28] which provides an extensive review of the literature, we surveyed over 30 binomial and trinomial trees from 26 different publications to see which ones fall under Assumption 1.

The binomial trees surveyed are from Cox, Ross and Rubinstein [9], Jarrow and Rudd [17], Trigeorgis [41], Tian [39], Chriss [7], Leisen and Reimer [35], Lamberton [25], Wilmott [44], Tian [40], Jarrow and Turnbull [18], Van den Berg [42], Jabbour et al. [16], Walsh [43], Diener and Diener [10], Chang and Palmer [6], Chance [5], Joshi [19, 20], Korn and Muller [24], Leduc [32, 33]. The trinomial trees surveyed are from

Boyle [2], Kamrad and Ritchken [22], Tian [39], Ahn and Song [1], and Chan et al. [4].

We found that among all the trees surveyed, those reaching a speed of convergence of order n^{-1} all satisfy Assumption 1. Two trees did not reach a speed of convergence of order n^{-1} and therefore did not fall under Assumption 1. These two binomial trees are: (a) the special case in Diener and Diener [10] when $\mu \neq \nu$; (b) the trees introduced by Chance [5] when $\pi \neq 1/2$.

For Diener and Diener [10] when $\mu \neq \nu$,

$$\begin{aligned} u_n &= 1 + \sigma\sqrt{\Delta t} + \mu\Delta t + \mathcal{O}\left(\Delta t^{\frac{3}{2}}\right), \\ d_n &= 1 - \sigma\sqrt{\Delta t} + \nu\Delta t + \mathcal{O}\left(\Delta t^{\frac{3}{2}}\right), \\ p_n^u &= \frac{e^{r\Delta t} - d_n}{u_n - d_n}. \end{aligned}$$

Diener and Diener proved that, in this case, the speed of convergence for call options is of order $n^{-1/2}$. The corresponding random variable X_n does do not satisfy condition **A3** because

$$E\left(e^{\gamma X_n}\right) = E\left(e^{\gamma X}\right) + \frac{1}{2}\sigma\gamma\left(\gamma - 1\right)\left(\mu - \nu\right)\Delta t^{\frac{3}{2}} + \mathcal{O}\left(\Delta t^2\right).$$

In the case of Chance [5] when $\pi \neq 1/2$,

$$\begin{aligned} u_n &= \frac{\exp\left(r\Delta t + \sigma\frac{\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}\right)}{\pi\exp\left(\sigma\frac{\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}\right) + (1-\pi)}, \\ d_n &= \frac{\exp\left(r\Delta t\right)}{\pi\exp\left(\sigma\frac{\sqrt{\Delta t}}{\sqrt{\pi(1-\pi)}}\right) + (1-\pi)}, \\ p_n^u &= \pi = \frac{e^{r\Delta t} - d_n}{u_n - d_n}. \end{aligned}$$

Random variable X_n does do not satisfy condition **A3** because

$$E\left(e^{\gamma X_n}\right) = E\left(e^{\gamma X}\right) + \Delta t^{\frac{3}{2}}\left(-\frac{\sigma^3\gamma\left(2\pi - 1\right)\left(\gamma^2 - 1\right)}{6\sqrt{\pi}\sqrt{1-\pi}}\right) + \mathcal{O}\left(\Delta t^2\right).$$

It follows from Lemma 2 in section 4.2 that the tree converges at a speed of only $n^{-1/2}$ for polynomial payoffs.

2.3. **A characterization of binomial trees with convergence rate of order n^{-1}.** We explains here why the result of the previous section are not surprising. Let μ_1, μ_2 be real numbers $\sigma_1, \sigma_2 > 0$, and consider self-similar binomial trees of the form

$$u_n = 1 + \sigma_1\sqrt{\Delta t} + \mu_1 \Delta t + \mathcal{O}\left(\Delta t^{\frac{3}{2}}\right),$$
$$d_n = 1 - \sigma_2\sqrt{\Delta t} + \mu_2 \Delta t + \mathcal{O}\left(\Delta t^{\frac{3}{2}}\right).$$

The existence of expansions of u_n and d_n in powers of $\sqrt{\Delta t}$ up to a term of order 3/2 appears to be quite typical in the literature. For such binomial trees we have:

Theorem 3. *All put options, all call options and all options with polynomial payoffs converge at a rate of n^{-1} to the Black-Scholes price with risk free rate r and volatility σ if and only if the binomial tree falls under Assumption 1. Moreover, in this case $\sigma = \sigma_1 = \sigma_2$ and $\mu_1 = \mu_2$.*

Proof. We know from Theorem 2, that Assumption 1 is a sufficient condition for all put options, all call options and all options with polynomial payoffs to converge at a rate of n^{-1} to the Black-Scholes price with risk free rate r and volatility σ. We need only to prove the converse.

First assume risk neutrality so that $p_n^u = (e^{r\Delta t} - d_n)/(u_n - d_n)$. Taylor's expansion theorem gives that the corresponding random variable X_n satisfies

$$E(X_n) = (r - \frac{1}{2}\sigma^2)\Delta t + \left(\sigma^2 - \sigma_1\sigma_2\right)\Delta t + \mathcal{O}\left(\Delta t^{\frac{3}{2}}\right),$$
$$\sqrt{Var(X_n)} = \sigma\sqrt{\Delta t} - \sqrt{\Delta t}\,(\sigma - \sqrt{\sigma_1}\sqrt{\sigma_2}) + \mathcal{O}(\Delta t),$$
$$E(\exp(\gamma X_n)) = E(\exp(\gamma X)) - \frac{\Delta t}{2}\gamma(\gamma - 1)\left(\sigma^2 - \sigma_1\sigma_2\right),$$
$$- \frac{\Delta t^{\frac{3}{2}}}{6}\gamma(\gamma - 1) A + \mathcal{O}\left(\Delta t^2\right)$$

where

$$A := 3r\sigma_2 - 3r\sigma_1 + 3\sigma_1\mu_2 - 3\sigma_2\mu_1 - 2\sigma_1\sigma_2^2 + 2\sigma_1^2\sigma_2 + \gamma\sigma_1\sigma_2^2 - \gamma\sigma_1^2\sigma_2.$$

As γ is a varies, it is easy to see that convergence at a speed of order n^{-1} for all payoffs of the form I^γ (in other words **A3**) can occur only if $\sigma = \sigma_1 = \sigma_2$ and $\mu_1 = \mu_2$. In that case Assumption 1 holds.

Suppose now that the binomial tree is not risk neutral. As in section 2.1, set

$$\begin{aligned}\delta_n &:= E\left(\exp\left(X_n\right)\right) - \exp\left(r\Delta t\right),\\ \alpha_n &:= \ln\left(\frac{\exp\left(r\Delta t\right)}{\exp\left(r\Delta t\right)+\delta_n}\right),\\ Y_n &:= X_n + \alpha_n.\end{aligned}$$

Then Y_n determines a risk neutral binomial tree so that

$$E^n\left(Y_n\right) = \exp\left(n\alpha_n\right) E^n\left(\exp\left(X_n\right)\right) = \exp\left(rT\right).$$

Now $E^n\left(\exp\left(X_n\right)\right) = E\left(S_T^{(n)}\right)$ and since, by assumption, options with a polynomial payoff converge at a rate of n^{-1},

$$E^n\left(\exp\left(X_n\right)\right) = E\left(S_T\right) + \mathcal{O}\left(n^{-1}\right) = \exp\left(rT\right)\left(1+\mathcal{O}\left(n^{-1}\right)\right).$$

Hence

$$\exp\left(n\alpha_n\right)\exp\left(rT\right)\left(1+\mathcal{O}\left(n^{-1}\right)\right) = \exp\left(rT\right).$$

This gives

$$\begin{aligned}-n\alpha_n &= \ln\left(1+\mathcal{O}\left(n^{-1}\right)\right) = \mathcal{O}\left(n^{-1}\right),\\ \alpha_n &= \mathcal{O}\left(n^{-2}\right) = \mathcal{O}\left(\Delta t^2\right).\end{aligned}$$

Now Y_n determines a risk neutral binomial tree. Let $\widetilde{u}_n$ and $\widetilde{d}_n$ be its corresponding up and down factors. Note that

$$\begin{aligned}\widetilde{u}_n &= \exp\left(\alpha_n\right) u_n = u_n + \mathcal{O}\left(\Delta t^2\right),\\ \widetilde{d}_n &= \exp\left(\alpha_n\right) d_n = d_n + \mathcal{O}\left(\Delta t^2\right).\end{aligned}$$

It follows that $\sigma_1 = \sigma_2 = \sigma$ and $\mu_1 = \mu_2$, as wanted. In this case Assumption 1 holds for Y_n. But it is simple to see that therefore this must also be the case for X_n. □

2.4. **Lamberton's geometric random walks.** Having in mind the representation of a geometric Brownian motions S as

$$S_t = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right)$$

for some Brownian motion B, it is natural to consider approximations $P^{(n)} := E\left(e^{-rT} f\left(\mu T + B_T^{(n)}\right)\right)$ of $P := E\left(e^{-rT} f\left(\mu T + B_T\right)\right)$ where

$$B_t^{(n)} = \sqrt{\frac{T}{n}} \sum_{k=1}^{\lfloor nt/T \rfloor} Y_k$$

for some i.i.d. random variables Y_k distributed as Y satisfying some natural conditions. This is Lamberton's setting [25, 26, 29, 27, 3] for approximating the underlying asset in the Black-Scholes model. Indeed Lamberton [27] points out that with $f(x) := h(S_0 e^{\sigma x})$ and $\mu := (r - \sigma^2/2)/\sigma$ one gets

$$f\left(\mu T + B_T^{(n)}\right) = h\left(S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma B_T^{(n)}}\right) = h\left(S_T^{(n)}\right)$$

where

$$S_t^{(n)} = S_0 \exp\left(\sum_{k=1}^{\lfloor nt/T \rfloor} X_n [t_{k-1}, t_k]\right),$$

and where the random variables

$$X_n [t_{k-1}, t_k] := \left(r - \frac{\sigma^2}{2}\right)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}} Y_k$$

are i.i.d. This boils down to studying the special class of those random variables X_n, in the setting of this paper, for which X_n has the form

$$X_n := X_n[0, \frac{T}{n}] = \left(r - \frac{\sigma^2}{2}\right)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}} Y, \tag{2.2}$$

for some random variable Y *which doesn't depend on* n.

The same setting allows to study American options by replacing T with the optimal stopping time. Under the condition that

(C) $\quad Y$ is bounded, $E(Y) = 0$, $E\left(Y^2\right) = 1$ and $E\left(Y^3\right) = 0$,

Lamberton [27] established that the price of the American put converges at a speed of at least $\left(n^{-1}\log n\right)^{4/5}$.

It is worth pointing out that because Y does not vary with n, Lamberton's setting is inherently incompatible with risk neutrality. Indeed, if for instance [25] $P(X = 1) = 1/2$ and $P(X = -1) = 1/2$ then, for every $n \geq 1$ sufficiently large,

$$e^{-r\frac{T}{n}} E\left(e^{\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}} X}\right) = 1 - \left(\frac{T}{n}\right)^2 \left(\frac{\sigma^4}{12}\right) + \mathcal{O}\left(\left(\frac{T}{n}\right)^3\right) < 1.$$

Hence the geometric random walk is not risk neutral. Conversely, for the classical Cox, Ross and Rubinstein [9] binomial tree $S^{(n)}$ we can write

$$S_{\frac{T}{n}}^{(n)} = e^{\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}} Y}$$

where the random variable Y is given by

$$Y = \begin{cases} 1 - \frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{\frac{T}{n}}}{\sigma} & \text{with probability } p_n := \frac{e^{r\frac{T}{n}} - e^{-\sigma\sqrt{\frac{T}{n}}}}{e^{\sigma\sqrt{\frac{T}{n}}} - e^{-\sigma\sqrt{\frac{T}{n}}}} \\ -1 - \frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{\frac{T}{n}}}{\sigma} & \text{with probability } 1 - p_n. \end{cases}$$

Unlike in Lamberton's setting, the distribution of Y depends on n and moreover $E(Y) \neq 0$.

While the geometric random walks coming from the setting of Lamberton can hardly be risk neutral, that doesn't prevent them from being *quasi risk neutral* in the sense that they satisfy condition **A3**. In fact, under the condition **L1-L2** below which forms a milder version of **C**,

$$\text{(L1)} \qquad E(Y) = 0, E\left(Y^2\right) = 1, E\left(Y^3\right) = 0,$$

$$\text{(L2)} \qquad M_Y(\gamma) := E(\exp(\gamma Y)) < \infty \text{ for every real } \gamma,$$

these geometric random walks fall under Assumption 1. To verify this, set X_n as in (2.2) and choose Y satisfying conditions **L1-L2.** Note first that

$$\mu_n \overset{def}{=} E(X_n) = \frac{T}{n}(r - \frac{1}{2}\sigma^2),$$

$$\sigma_n \overset{def}{=} \sqrt{Var(X_n)} = \sqrt{\frac{T}{n}}\sigma,$$

and for every real γ,

$$E\left(\exp\left(\gamma\sqrt{\frac{n}{T}}X_n\right)\right) = \exp\left(\left(r - \frac{\sigma^2}{2}\right)\sqrt{\frac{T}{n}}\right) M_Y(\gamma\sigma) = \mathcal{O}(1).$$

In other words conditions **A1**, **A2** and **A4** are satisfied.

Now fix γ and recall that $X := \sigma W_{T/n} + (r - \sigma^2/2)\,T/n$ for some Brownian motion W. Note that

$$\begin{aligned} E(\exp(\gamma X)) &= \exp\left(\gamma\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n}\right)\exp\left(\frac{1}{2}\frac{T}{n}\sigma^2\gamma^2\right) \\ &= \exp\left(\gamma\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n}\right)\left(1 + \frac{1}{2}\frac{T}{n}\sigma^2\gamma^2 + \mathcal{O}\left(n^{-2}\right)\right). \end{aligned}$$

On the other hand

$$E(\exp(\gamma X_n)) = \exp\left(\gamma\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n}\right) M_Y\left(\gamma\sigma\sqrt{\frac{T}{n}}\right).$$

Taylor's expansion theorem gives that, for some $0 \leq \eta \leq \gamma\sigma\sqrt{T/n}$,

$$M_Y\left(\gamma\sigma\sqrt{\frac{T}{n}}\right) = 1 + M_Y'(0)\,\gamma\sigma\sqrt{\frac{T}{n}} + \frac{1}{2}M_Y''(0)\left(\gamma\sigma\sqrt{\frac{T}{n}}\right)^2 + \frac{1}{3!}M_Y'''(0)\left(\gamma\sigma\sqrt{\frac{T}{n}}\right)^3 + \frac{1}{4!}M_Y^{(4)}(\eta)\left(\gamma\sigma\sqrt{\frac{T}{n}}\right)^4.$$

Using $\sup_{0\leq\eta\leq1} M_Y^{(4)}(\eta) < \infty$ and condition **L1**, one gets

$$M_Y'(0) = 0, M_Y''(0) = 1, M_Y'''(0) = 0$$

and

$$M_Y\left(\gamma\sigma\sqrt{\frac{T}{n}}\right) = 1 + \frac{1}{2}\frac{T}{n}\sigma^2\gamma^2 + \mathcal{O}\left(n^{-2}\right).$$

Hence

$$\begin{aligned}|E(\exp(\gamma X_n)) - E(\exp(\gamma X))| &\leq \exp\left(\gamma\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n}\right)\mathcal{O}\left(n^{-2}\right)\\ &= \mathcal{O}\left(n^{-2}\right),\end{aligned}$$

showing that **A3** holds.

3. Speed of convergence of European options

3.1. **Speed of convergence for discontinuous payoffs.** Recall that $F_{S_{t_m}}$ and $F_{S_{t_m}^{(n)}}$ denote the cumulative distribution functions of S_{t_m} and $S_{t_m}^{(n)}$. Property **P1** can be extended to the following result:

Proposition 1. *If property **P1** holds and h belongs to $\mathcal{K}^{(0)}$ then, for any time step $0 < t_m \leq T$,*

$$\mathrm{Err}_{t_m}^n(h)(x) = \chi_0(h)\frac{1}{\sqrt{m}}\mathcal{O}(1). \tag{3.1}$$

Proof. Chose $0 < t_m \leq T$, $t_m \in T/n\mathbb{N}$ and let $c := \lim_{x\to\infty} h(x)$. Function h can be written as

$$h(x) = g(x) + \sum_{\ell=1}^{\infty}\Delta h(\beta_\ell)\,1_{[\beta_\ell,\infty)}(x) - \sum_{\ell=1}^{\infty}\Delta h(\beta_\ell) + c,$$

where g is a *continuous asymptotically vanishing at infinity* function belonging to $\mathcal{K}^{(0)}$. Note that clearly

$$\chi_0(g) \leq 4\chi_0(h).$$

Obviously,

$$\mathrm{Err}^n_{t_m}(h)(x) = \mathrm{Err}^n_{t_m}(g)(x) + \sum_{\ell=1}^{\infty} \Delta h(\beta_\ell)\,\mathrm{Err}^n_{t_m}\left(1_{[\beta_\ell,\infty)}\right)(x).$$

Property **P1** implies

$$\mathrm{Err}^n_{t_m}\left(1_{[\beta,\infty)}\right)(x) = \frac{1}{\sqrt{m}}\mathcal{O}(1),$$

where the $\mathcal{O}$ term is uniform in x and β. Hence

$$\begin{aligned}\sum_{\ell=1}^{\infty} \Delta h(\beta_\ell)\,\mathrm{Err}^n_{t_m}\left(1_{[\beta_\ell,\infty)}\right)(x) &= \sum_{\ell=1}^{\infty} \Delta h(\beta_\ell)\frac{1}{\sqrt{m}}\mathcal{O}(1)\\ &\le \chi_0(h)\frac{1}{\sqrt{m}}\mathcal{O}(1).\end{aligned}$$

Now, to simplify the notation, let $F_m \overset{def}{=} F_{S_{t_m}}$ and $F_{m,n} \overset{def}{=} F_{S^{(n)}_{t_m}}$. Noting that $F_m(0) = 0$ and $\lim_{b\to\infty} g(b) = 0$, integration by parts gives

$$\begin{aligned}\mathcal{E}_{t_m}(g)(x) &= e^{-rt_m}\int_0^\infty g(u)\,F'_m(u)\,du\\ &= e^{-rt_m}\,F_m(u)\,g(u)\big|_0^\infty - e^{-rT}\int_0^\infty F_m(u)\,\mu_g(du)\\ &= -e^{-rt_m}\int_0^\infty F_m(u)\,\mu_g(du),\end{aligned}$$

where μ_g is the Borel measure such that $\mu_g(\alpha,\beta) = g(\beta) - g(\alpha)$. Although F_n may not be differentiable everywhere, integration by parts for distributions (see for instance [38, Exercise 5.10.17]) again gives

$$\mathcal{E}^n_{t_m}(g)(x) = -e^{-rT}F_{m,n}(0)\,g(0) - e^{-rT}\int_0^\infty F_{m,n}(u)\,\mu_g(du)$$

Invoking **P1**, it follows that

$$\begin{aligned}\left|Err^n_{t_m}(g)(x)\right| &\le \left|F_{m,n}(0) - F_m(0)\right|\left|g(0)\right|\\ &\quad + \left|\int_0^\infty \left\|F_{m,n} - F_m\right\|_\infty \mu_g(du)\right|\\ &\le 2\left\|F_{m,n} - F_m\right\|_\infty \left|g(0)\right|\\ &\le \chi_0(g)\frac{1}{\sqrt{m}}\mathcal{O}(1).\\ &= \chi_0(h)\frac{1}{\sqrt{m}}\mathcal{O}(1).\end{aligned}$$

□

3.2. **Speed of convergence for $C^{(1)}$ payoffs.** For geometric random walks satisfying properties **P0-P5** hold, a error formula was obtained in [32] for European options with payoff functions assumed to be polynomially bounded, in contrast to of bounded variation as it is the case in this paper. This bounded variation condition is used here to ensure a *uniform* rate of convergence of order $1/n$. Furthermore, in order to establish [32, Proposition 6.4], which is the analogue to Proposition 1, it is required in [32] that the payoff function φ be piecewise continuously differentiable. But when $\varphi(x) = x^2 h^{(2)}(x)$ this in turns requires h to be three times piecewise continuously differentiable, which is something that we want to avoid in this paper. Taking these changes into account, although stated slightly differently, Leduc [32] obtained the following result:

Theorem 4 (Convergence speed for $C^{(1)}$ payoffs)**.** *Assume that properties **P0-P5** hold and let h belong to $C^{(1)} \cap \mathcal{K}^{(2)}$. Then for every time step $0 < t_m \le T$ and for every $x \ge 0$,*

$$\mathrm{Err}_{t_m}^n h(x) = \frac{m}{n^2} \Upsilon_{t_m}^n(h, x) + \chi_2(h) m^{-\frac{1}{2}} \mathcal{O}\left(n^{-1}\right),$$

where

$$\begin{aligned}
\Upsilon_{t_m}^n(h, x) &= \sum_{k=2}^{4} \frac{n^2 \Delta_k^{(n)}}{k!} x^k \mathcal{E}_{t_m}^{(k)} h(x) \\
&= \left(\frac{1}{2} n^2 \Delta_2^{(n)} - \frac{1}{3} n^2 \Delta_3^{(n)} + \frac{1}{4} n^2 \Delta_4^{(n)}\right) e^{-r t_m} E_x\left(S_{t_m}^2 h''(S_{t_m})\right) \\
&+ \frac{1}{24} \frac{4n^2 \Delta_3^{(n)} - 5 n^2 \Delta_4^{(n)}}{\sigma \sqrt{t_m}} e^{-r t_m} E_x\left(S_{t_m}^2 h''(S_{t_m}) \eta_{t_m}\left(\frac{S_{t_m}}{x}\right)\right) \\
&\frac{1}{24} \frac{n^2 \Delta_4^{(n)}}{t_m \sigma^2} e^{-r t_m} E_x\left(S_{t_m}^2 h''(S_{t_m}) \left(\eta_{t_m}^2\left(\frac{S_{t_m}}{x}\right) - 1\right)\right),
\end{aligned}$$

and

$$\eta_{t_m}(z) = \frac{\ln(z) - \left(r - \frac{1}{2}\sigma^2\right) t_m}{\sqrt{t_m}\sigma}. \tag{3.2}$$

3.3. **Speed of convergence for put options.** Theorem 4 above immediately provides the speed of convergence for every option with payoff h in $C^{(1)} \cap \mathcal{K}^{(2)}$. Obviously, this is not the case for a put option. Let $K, T \ge 0$ represent the strike and maturity value of some put option. Let $P_{K,T}(x)$ denote the value of such option under model S when the spot price S_0 is $x \ge 0$, and let $P_{K,T}^{(n)}(x)$ denote the value of the same call option under model $S^{(n)}$. Obviously, $P_{K,0}(x) = \max(K - x, 0)$.

Theorem 5 (Convergence speed for put options)**.** *Assume that properties **P0-P5** hold. Then for any time step* $0 < t_m \leq T$*,*

$$P_{K,t_m}(x) = P^{(n)}_{K,t_m}(x) + K\sqrt{m^{-1}}\mathcal{O}\left(n^{-1/2}\right).$$

Proof. Set

$$\Delta_{\frac{T}{n}} P_K\left(x\right) \overset{def}{=} P_{K,\frac{T}{n}}(x) - P_{K,0}(x),$$

and note that

$$P_{K,T}(x) - P^{(n)}_{K,T}(x) = Err^n_T\left(P_{K,\frac{T}{n}}\right)(x) - Err^n_T\left(\Delta_{\frac{T}{n}} P_K\right)(x).$$

Because

$$\chi_0\left(\Delta_{T/n} P_K\right) = \left\|\Delta_{T/n} P_K\right\|_\infty = K\mathcal{O}\left(n^{-1/2}\right)$$

and $\Delta_{T/n}P_K$ is piecewise monotone continuous vanishing at infinity, it follows from Proposition 1 that, with $h\left(x\right) = \max\left(K - x, 0\right)$,

$$P_{K,t_m}(x) - P^{(n)}_{K,t_m}(x) = Err^n_{t_m}\left(\mathcal{E}_{\frac{T}{n}} h\right)(x) + K\sqrt{m^{-1}}\mathcal{O}\left(n^{-1/2}\right).$$

We just need to show that

$$Err^n_{t_m}\left(\mathcal{E}_{\frac{T}{n}} h\right)(x) = K\sqrt{m^{-1}}\mathcal{O}\left(n^{-1/2}\right).$$

But according to Theorem 4,

$$Err^n_{t_m}\left(\mathcal{E}_{\frac{T}{n}} h\right)(x) = \frac{m}{n^2}\sum_{k=2}^{4}\frac{n^2\Delta^{(n)}_k}{k!}x^k\mathcal{E}^{(k)}_{t_m} h\left(x\right) + \chi_2\left(\mathcal{E}_{\frac{T}{n}} h\right)\sqrt{m^{-1}}\mathcal{O}\left(n^{-1}\right).$$

Yet it is simple to see that

$$\chi_2\left(\mathcal{E}_{\frac{T}{n}} h\right) = K\sqrt{n}\mathcal{O}\left(1\right).$$

Hence

$$P_{K,t_m}(x) - P^{(n)}_{K,t_m}(x) = \frac{m}{n^2}\sum_{k=2}^{4}\frac{n^2\Delta^{(n)}_k}{k!}x^k\frac{\partial^k}{\partial x^k}P_{K,t_m}(x) + K\sqrt{m^{-1}}\mathcal{O}\left(n^{-1/2}\right).$$

Simple and direct calculations show that

$$\frac{m}{n^2}\sum_{k=2}^{4}\frac{n^2\Delta^{(n)}_k}{k!}x^k\frac{\partial^k}{\partial x^k}P_{K,t_m}(x)\left(x\right) = K\sqrt{m^{-1}}\mathcal{O}\left(n^{-1/2}\right),$$

yielding the desired result. ∎

3.4. **Speed of convergence for general payoffs.** Assume that h belongs to $\mathcal{K}^{(2)}$. It is not difficult to verify that h can be written as

$$h(x) = g(x) + \sum_{\ell=1}^{\infty} \Delta h(\beta_\ell) 1_{[\beta_\ell,\infty)}(x) + \sum_{\ell=1}^{\infty} \Delta h'(\beta_\ell) \max(\beta_\ell - x, 0), \tag{3.3}$$

where g is $C^{(1)}$ and belongs to $\mathcal{K}^{(2)}$. Hence, h can be split into a linear combination of digital options, put options, and a function which is continuously differentiable and in $\mathcal{K}^{(2)}$.

Theorem 6 (Convergence speed for general payoff)**.** *Assume that properties* ***P0-P5*** *hold and let h belong to $\mathcal{K}^{(2)}$. Then for any time step $0 < t_m \leq T$,*

$$\begin{aligned} Err_{t_m}^n(h)(x) = &\sum_{\ell=1}^{\infty} \Delta h(\beta_\ell) \sqrt{m^{-1}} \mathcal{O}(1) \\ &+ \sum_{\ell=1}^{\infty} \beta_\ell \Delta h'(\beta_\ell) \sqrt{m^{-1}} \mathcal{O}\left(n^{-1/2}\right) \\ &+ \left\| I^2 h'' \right\|_\infty \mathcal{O}\left(n^{-1}\right) \\ &+ \chi_2(h) \sqrt{m^{-1}} \mathcal{O}\left(n^{-1}\right). \end{aligned}$$

Proof. We use the decomposition (3.3) of h into a function $g \in C^{(1)} \cap \mathcal{K}^{(2)}$ and digital and put payoff functions. Let φ be the pdf of a standard normal random variable. Recall the definition of η_{t_m} from (3.2). Note that if

$$\zeta_{t_m}(z) \stackrel{def}{=} \exp\left(\sigma\sqrt{t_m} z + \left(r - \frac{1}{2}\sigma^2\right) t_m\right), \tag{3.4}$$

then, for every function f,

$$E_x(f(S_{t_m})) = \int_{-\infty}^{\infty} f(x\zeta_{t_m}(z)) \varphi(z) \, dz, \tag{3.5}$$

and additionally

$$\eta_{t_m}(\zeta_{t_m}(z)) = z.$$

Hence, for some constant $Q>0$,

$$\begin{aligned}\left|E_x\left(S_{t_m}^2 g''\left(S_{t_m}\right)\right)\right| &\le \left\|I^2 g''\right\|_\infty ,\\ \left|E_x\left(S_{t_m}^2 g''\left(S_{t_m}\right)\eta_{t_m}\left(\frac{S_{t_m}}{x}\right)\right)\right| &\le \left\|I^2 g''\right\|_\infty \int_{-\infty}^{\infty}|z|\,\varphi(z)\,dz\\ &\le \left\|I^2 g''\right\|_\infty Q,\end{aligned}$$

and similarly,

$$E_x\left(S_{t_m}^2 g''\left(S_{t_m}\right)\left(\eta_{t_m}^2\left(\frac{S_{t_m}}{x}\right)-1\right)\right)\le \left\|I^2 g''\right\|_\infty Q.$$

Recalling $\Upsilon_{t_m}^{(n)}(g,x)$ from Theorem 4, and noticing that $h''=g''$, it follows that

$$\left|\Upsilon_{t_m}^{(n)}(g,x)\right| = \frac{\|I^2 g''\|_\infty}{t_m}\mathcal{O}(1) = \frac{\|I^2 h''\|_\infty}{t_m}\mathcal{O}(1),$$

and

$$\left|\mathrm{Err}_{t_m}^n\, g(x)\right| \le \frac{m}{n^2}\frac{\|I^2 h''\|_\infty}{t_m}\mathcal{O}(1)+\chi_2(g)\frac{1}{\sqrt{m}}\mathcal{O}\left(n^{-1}\right).$$

Hence

$$\mathrm{Err}_{t_m}^n\, g(x) = \left\|I^2 h''\right\|_\infty \mathcal{O}\left(n^{-1}\right)+\chi_2(g)\frac{1}{\sqrt{m}}\mathcal{O}\left(n^{-1}\right). \tag{3.6}$$

It is not difficult to see that

$$\chi_2(g)\le 7\chi_2(h)$$

and therefore

$$\mathrm{Err}_{t_m}^n\, g(x) = \left\|I^2 h''\right\|_\infty \mathcal{O}\left(n^{-1}\right)+\chi_2(h)\frac{1}{\sqrt{m}}\mathcal{O}\left(n^{-1}\right).$$

It remains to tackle the digital and put options in the decomposition of h. But these options are treated respectively using property **P1** and Theorem 5, completing the proof. □

3.5. **Proof of Theorem 1.** Let $h\in\mathcal{K}^{(2)}$ and fix constants T_1,T_2 such that $0<T_1\le T_2\le T$. Let T_1^* be the largest time step t_k such that $t_k\le T_1$. Assume that n is large enough to guarantee that $T_1^*\ge T_1/2>0$. For any $t\in[T_1,T_2]$, let $t_{\mathfrak{m}}$ be the largest time step t_k such that $T_1^*\le t_k\le t$.

Because $S_t^{(n)}$ is constant in between two time steps,

$$\sup_{T_1^*\le t\le T_2}\ \sup_{0\le x}\left|\mathcal{E}_t^n h(x)-\mathcal{E}_{t_{\mathfrak{m}}}^n h(x)\right| = \|h\|_\infty \mathcal{O}\left(n^{-1}\right). \tag{3.7}$$

Recalling (3.4) and (3.5) we get

$$\frac{\partial}{\partial t}E_x\left(h\left(S_t\right)\right)=\frac{\sigma}{\sqrt{t}}\int_{-\infty}^{\infty}zx\zeta_t\left(z\right)h'\left(x\zeta_{t_m}\left(z\right)\right)\varphi\left(z\right)dz$$
$$+\left(r-\frac{1}{2}\sigma^2\right)\int_{-\infty}^{\infty}x\zeta_t\left(z\right)h'\left(x\zeta_{t_m}\left(z\right)\right)\varphi\left(z\right)dz.$$

Because $xh'\left(x\right)$ is bounded, it clearly follows that

$$\sup_{T_1^*\leq t\leq T_2}\sup_{0\leq x}\left|\frac{\partial}{\partial t}E_x\left(h\left(S_t\right)\right)\right|=\left\|Ih'\right\|_\infty\mathcal{O}\left(1\right).$$

Therefore, from Taylor's theorem,

$$\sup_{T_1^*\leq t\leq T_2}\sup_{0\leq x}\left|\mathcal{E}_th\left(x\right)-\mathcal{E}_{t_m}h\left(x\right)\right|=\left\|Ih'\right\|_\infty\mathcal{O}\left(n^{-1}\right).\tag{3.8}$$

Thanks to Theorem 6,

$$\sup_{T_1^*\leq t_k\leq T_2}\sup_{0\leq x}\left|\mathcal{E}_{t_k}^nh\left(x\right)-\mathcal{E}_{t_k}h\left(x\right)\right|=\chi_2\left(h\right)\mathcal{O}\left(n^{-1}\right).\tag{3.9}$$

Putting (3.7), (3.8) and (3.9) together completes the proof.

4. A general framework

Recall $X\left[t_{k-1},t_k\right]$ and $X_n\left[t_{k-1},t_k\right]$ from (1.1) and (1.3). Recall also that if $S_0=1$ then $X=\ln\left(S_{T/n}\right)$, $X_n=\ln\left(S_{T/n}^{(n)}\right)$, $A=S_{T/n}$ and $A_n=S_{T/n}^{(n)}$. Furthermore, for any real number γ, any integer $j=0,...,n$, and $t_j=jT/n$,

$$\mathcal{E}_{t_j}\left(I^\gamma\right)\left(x\right)=e^{-rt_j}E_x\left(S_{t_j}\right)=e^{-rt_j}x^\gamma\left(E\left(A^\gamma\right)\right)^j=e^{-rt_j}x^\gamma E^j\left(A^\gamma\right).$$

The same statement is true with $\mathcal{E},A,S$ replaced by $\mathcal{E}^n,A_n,S^{(n)}$. Note the following restatements of **A3**:

$$E(A_n^\gamma)=E(A^\gamma)+\mathcal{O}\left(n^{-2}\right).$$

This section is devoted to prove that under Assumption 1, properties **P1-P5** hold.

4.1. Speed of convergence for digital option.

Here we establish property **P1**. This requires the following lemma:

Lemma 1. *Under Assumption 1, for any constant integer $M \geq 0$, the following holds:*

$$E\left(|X_n|^M\right) = \mathcal{O}\left(n^{-\frac{M}{2}}\right), \tag{4.1}$$

$$\frac{\rho_n}{\sigma_n^3} \stackrel{def}{=} \frac{E\left(|X_n - \mu_n|^3\right)}{\sigma_n^3} = \mathcal{O}(1), \tag{4.2}$$

$$\frac{1}{\sqrt{n}\sigma_n} = \frac{1}{\sigma\sqrt{T}} + \mathcal{O}\left(n^{-\frac{1}{2}}\right), \tag{4.3}$$

$$\frac{\sqrt{n}\mu_n}{\sigma_n} = \frac{\sqrt{T}\left(r - \frac{1}{2}\sigma^2\right)}{\sigma} + \mathcal{O}\left(n^{-\frac{1}{2}}\right). \tag{4.4}$$

Proof. For every integer $M > 0$, we have

$$\begin{aligned} E\left(\left|\sqrt{\frac{n}{T}}X_n\right|^M\right) &\leq E\left(e^{M\sqrt{\frac{n}{T}}X_n}\right) + E\left(e^{-M\sqrt{\frac{n}{T}}X_n}\right) \\ &= \mathcal{O}(1), \end{aligned}$$

where the last equality follows from **A4** and proves (4.1).

To establish (4.2), let

$$\rho_n = E\left(|X_n - \mu_n|^3\right)$$

and note that

$$\begin{aligned} \frac{\rho_n}{\sigma_n^3} &= \frac{E\left(\left|\sqrt{\frac{n}{T}}X_n - \sqrt{\frac{n}{T}}\mu_n\right|^3\right)}{\left(\sqrt{\frac{n}{T}}\sigma_n\right)^3} \\ &\leq \frac{\sum_{k=0}^{3}\left(\sqrt{\frac{n}{T}}\mu_n\right)^{3-k}\binom{3}{k}E\left(\left|\sqrt{\frac{n}{T}}X_n\right|^k\right)}{\left(\sqrt{\frac{n}{T}}\sigma_n\right)^3}. \end{aligned}$$

It follows from **A1**, **A2** and (4.1) that

$$\sup_n \frac{\rho_n}{\sigma_n^3} < \infty.$$

As for equation (4.3) and (4.4), they are obtained through simple algebra. □

A European digital put option is an option that pays one at maturity if the asset value is below some strike (denoted in this section by x) while a European digital call option pays one at maturity if the asset value exceeds the strike. The theorem below establishes that under Assumption 1, digital option values with maturity t_m converge at a speed of $\mathcal{O}\left(m^{-1/2}\right)$ to the value of the option under the Black-Scholes model.

Theorem 7 (Convergence Speed for Digital Options)**.** *Under Assumption 1, for any time step* $0 < t_m \leq T$*,*

$$\sup_x \left| F_{S^{(n)}_{t_m}}(x) - F_{S_{t_m}}(x) \right| = \sqrt{m^{-1}}\mathcal{O}(1). \tag{4.5}$$

Proof. Let Φ and φ denote respectively the cumulative distribution function and probability density function of a standard normal random variable. Note that, if $\mathcal{N}$ a standard normal random variable, then

$$\begin{aligned} F_{S_{t_m}}(x) &= P\left(S_0 \exp\left(\sigma\sqrt{t_m}\mathcal{N} + \left(r - \frac{1}{2}\sigma^2\right)t_m\right) \leq x\right) \\ &= \Phi(x_m^*), \end{aligned}$$

where $x_m^* := x_m^*(x)$ is defined by

$$x_m^* \overset{def}{=} \frac{\ln x - \ln S_0 - \left(r - \frac{1}{2}\sigma^2\right)t_m}{\sigma\sqrt{t_m}}.$$

Let $\mu_n \overset{def}{=} E(X_n)$ and $\sigma_n \overset{def}{=} \sqrt{Var(X_n)}$. Similarly as above,

$$\begin{aligned} F_{S^{(n)}_{t_m}}(x) &= P\left(S_0 \exp\left(\sum_{k=1}^m X_n[t_{k-1}, t_k]\right) \leq x\right) \\ &= P\left(\mathcal{N}^*_{n,m} \leq x^*_{n,m}\right) \end{aligned}$$

where

$$\begin{aligned} \mathcal{N}^*_{n,m} &\overset{def}{=} \frac{1}{\sqrt{m}\sigma_n}\sum_{k=1}^m \left(X_n[t_{k-1}, t_k] - \mu_n\right), \\ x^*_{n,m} &\overset{def}{=} \frac{\ln x - \ln S_0 - m\mu_n}{\sqrt{m}\sigma_n}. \end{aligned}$$

The Berry-Esseen theorem (see [13, Theorem 1, p. 542]) guarantees that

$$\sup_x \left| P\left(\mathcal{N}^*_{n,m} \leq x^*_{n,m}\right) - \Phi\left(x^*_{n,m}\right) \right| \leq 3\frac{\rho_n}{\sigma_n^3\sqrt{m}},$$

where

$$\frac{\rho_n}{\sigma_n^3} = \frac{E\left(|X_n - \mu_n|^3\right)}{(\sigma_n)^3}.$$

According to Lemma 1,

$$\sup_n \frac{\rho_n}{\sigma_n^3} < \infty,$$

thus

$$\sup_x \left| P\left(\mathcal{N}^*_{n,m} \leq x^*_{n,m}\right) - \Phi\left(x^*_{n,m}\right) \right| \leq \sqrt{m^{-1}}\mathcal{O}(1).$$

Clearly,

$$\sup_x \left| F_{S^{(n)}_{t_m}} - F_{S_{t_m}}(x) \right| \leq \sup_x \left| P\left(\mathcal{N}^*_{n,m} \leq x^*_{n,m}\right) - \Phi\left(x^*_{n,m}\right) \right| + \sup_x \left| \Phi\left(x^*_{n,m}\right) - \Phi\left(x^*_m\right) \right|.$$

Hence it remains to show that

$$\sup_x \left| \Phi\left(x^*_{n,m}\right) - \Phi\left(x^*_m\right) \right| = \sqrt{m^{-1}}\mathcal{O}(1).$$

Let $\alpha := \alpha(m)$ and $\beta := \beta(m, x)$ be defined as

$$\alpha \stackrel{def}{=} -\sqrt{t_m}\frac{\left(r - \frac{1}{2}\sigma^2\right)}{\sigma},$$
$$\beta \stackrel{def}{=} \sqrt{t_m^{-1}}\frac{(\ln x - \ln S_0)}{\sigma}.$$

Using (4.3) and (4.4), one verifies that

$$x^*_m = \alpha + \beta,$$
$$x^*_{n,m} = \alpha + \beta + \mathcal{O}\left(n^{-\frac{1}{2}}\right) + \beta\mathcal{O}\left(n^{-\frac{1}{2}}\right).$$

From Taylor's theorem we get that

$$\left| \Phi\left(x^*_{n,m}\right) - \Phi\left(x^*_m\right) \right| \leq \mathcal{O}\left(n^{-\frac{1}{2}}\right) + \left|\beta\varphi(\alpha + \beta_n)\right| \mathcal{O}\left(n^{-\frac{1}{2}}\right),$$

where $\beta_n := \beta\left(1 + \eta\mathcal{O}\left(n^{-1/2}\right)\right)$ for some $0 \leq \eta \leq 1$. Note that $-L \leq \alpha \leq L$, where

$$L := \sqrt{T}\frac{\left|r - \frac{1}{2}\sigma^2\right|}{\sigma}.$$

For n large enough, note that $\beta_n = 0$ if and only if $\beta = 0$. Since

$$\sup_{-L \leq \alpha \leq L} \sup_{\beta_n \in \mathbb{R}} \left|\beta_n\varphi(\alpha + \beta_n)\right| < \infty,$$

and $\beta/\beta_n = \mathcal{O}(1)$, it follows that

$$\left| \Phi\left(x^*_{n,m}\right) - \Phi\left(x^*_m\right) \right| = \mathcal{O}\left(n^{-\frac{1}{2}}\right).$$

□

4.2. **Speed of convergence for polynomial payoff functions.** We prove that **A3** is equivalent to **P4d**. In other words we show that convergence of options with polynomial payoff occurs at rate of n^{-1} if and only if condition **A3** holds. More generally we have:

Lemma 2. *Let $0 \leq \beta < 1$. Conditions* ***A3***[β] *and* ***P4d***[β] *below are equivalent:*

$$E\left(\exp\left(\gamma X_n\right)\right) = E\left(\exp\left(\gamma X\right)\right) + \mathcal{O}\left(n^{-(2-\beta)}\right) \tag{A3[β]}$$

$$\max_{j=0,\ldots,n}\left|\mathrm{Err}^n_{\frac{jT}{n}}\left(I^\gamma\right)(x)\right| = x^\gamma \mathcal{O}\left(n^{-(1-\beta)}\right) \tag{P4d[β]}$$

Proof. Fix γ and $0 \leq \beta < 1$. ($\Longrightarrow$) For $j = 0, ..., n$, thanks to **A3**[β],

$$\begin{aligned}\left|\mathcal{E}^n_{\frac{jT}{n}}\left(I^\gamma\right)(x) - \mathcal{E}_{\frac{jT}{n}}\left(I^\gamma\right)(x)\right| &= x^\gamma\left|E^j\left(A_n^\gamma\right) - E^j\left(A^\gamma\right)\right| \\ &= x^\gamma\left|\left(E\left(A^\gamma\right) + \mathcal{O}\left(n^{-(2-\beta)}\right)\right)^j - E^j\left(A^\gamma\right)\right| \\ &= x^\gamma E^j\left(A^\gamma\right)\left|\left(1 + \mathcal{O}\left(n^{-(2-\beta)}\right)\right)^j - 1\right|.\end{aligned}$$

Now

$$\max_{j=0,\ldots,n} E^j\left(A^\gamma\right) = \max_{j=0,\ldots,n} E\left(S^\gamma_{t_j}\right) = \mathcal{O}(1),$$

and

$$\begin{aligned}\left|\left(1 + \mathcal{O}\left(n^{-(2-\beta)}\right)\right)^j - 1\right| &\leq \left(\left(1 + \left|\mathcal{O}\left(n^{-(2-\beta)}\right)\right|\right)^n - 1\right) \\ &\quad + \left|\mathcal{O}\left(n^{-(2-\beta)}\right)\right| \\ &= \mathcal{O}\left(n^{-(1-\beta)}\right)\end{aligned}$$

yields **P4d**[β].

($\Longleftarrow$) Suppose that **A3**[β] fails. Then *for every* $b > 0$, there exists infinitely many n such that either

$$E\left(A_n^\gamma\right) > E\left(A^\gamma\right) + b/n^{(2-\beta)} \tag{4.6}$$

or

$$E\left(A_n^\gamma\right) < E\left(A^\gamma\right) - b/n^{(2-\beta)}. \tag{4.7}$$

Assume that (4.6) holds; the other case is treated similarly. Let

$$\mathcal{L} := \overline{\lim_{n\to\infty}}\, n\left(E_1\left(\left(S_T^{(n)}\right)^\gamma\right) - E_1\left(\left(S_T\right)^\gamma\right)\right).$$

Recall that $E^n\left(A^\gamma\right) = E\left(\left(S_T\right)^\gamma\right)$ and $E\left(A^\gamma\right) = 1 + o(1)$. We get

$$\begin{aligned}\mathcal{L} &= \overline{\lim_{n\to\infty}}\, n^{(1-\beta)}\left(E^n\left(A_n^\gamma\right) - E^n\left(A^\gamma\right)\right) \\ &> \overline{\lim_{n\to\infty}}\, n^{(1-\beta)}\left(\left(E\left(A^\gamma\right) + \frac{b}{n^{(2-\beta)}}\right)^n - E^n\left(A^\gamma\right)\right) \\ &= \overline{\lim_{n\to\infty}}\, E^n\left(A^\gamma\right) n^{(1-\beta)}\left(\left(1 + \frac{b}{E\left(A^\gamma\right) n^{(2-\beta)}}\right)^n - 1\right) \\ &= E_1\left(\left(S_T\right)^\gamma\right) b.\end{aligned}$$

Because $b > 0$ is arbitrary large, this shows that

$$E_1\left(\left(S_T^{(n)}\right)^\gamma\right) \neq E_1\left((S_T)^\gamma\right) + \mathcal{O}\left(n^{-(1-\beta)}\right).$$

Hence **P4d**[β] fails. □

4.3. **Properties P1-P5.** We have shown in sections 4.1 and 4.2 that under Assumption 1, conditions **P1** and **Pd4** hold. We establish here that the remaining properties of **P1-P5** also hold.

Lemma 3. *Under Assumption 1, conditions* ***P1-P5*** *hold.*

Proof. In order to establish **P2**, note first that

$$\begin{aligned} E\left|\exp\left(X_n\right)-1\right|^M &\leq E\left(\left|X_n^M\right|\left(\exp\left(MX_n\right)+\exp\left(-MX_n\right)\right)\right) \\ &\leq \sqrt{E\left(X_n^{2M}\right)}\sqrt{E\left(\exp\left(MX_n\right)+\exp\left(-MX_n\right)\right)^2} \end{aligned}$$

It follows from (4.1) and **A4** that

$$\begin{aligned} \sqrt{E\left(X_n^{2M}\right)} &= \mathcal{O}\left(n^{-\frac{M}{2}}\right), \\ \sqrt{E\left(\exp\left(MX_n\right)+\exp\left(-MX_n\right)\right)^2} &= \mathcal{O}\left(1\right), \end{aligned}$$

establishing **P2**.

As for **P3**, first note that in the case $0 \leq M \leq 4$, using **A3**, one writes

$$\begin{aligned} \mathrm{Err}^n_{\frac{T}{n}}\left((I-1)^M\right)(1) &= \sum_{k=0}^{M}(-1)^{n-k}\binom{M}{k}\mathrm{Err}^n_{\frac{T}{n}}\left(I^k\right)(1) \\ &= \mathcal{O}\left(n^{-2}\right), \end{aligned}$$

showing **P3** in the case $0 \leq M \leq 4$. In the case $M \geq 5$, using **P2** we get

$$E\left(\left(S^{(n)}_{\frac{T}{n}}-1\right)^M\right) \leq \sqrt{E\left(\left(S^{(n)}_{\frac{T}{n}}-1\right)^{2M}\right)} = \mathcal{O}\left(n^{-\frac{M}{2}}\right).$$

The same is true with $S^{(n)}_{\frac{T}{n}}$ replaced by $S_{\frac{T}{n}}$, and this yields

$$\begin{aligned} \left|\mathrm{Err}^n_{\frac{T}{n}}\left((I-1)^M\right)(1)\right| &\leq E\left(\left(S_{\frac{T}{n}}-1\right)^M\right)+E\left(\left(S^{(n)}_{\frac{T}{n}}-1\right)^M\right) \\ &= \mathcal{O}\left(n^{-\frac{M}{2}}\right) \end{aligned}$$

proving **P3**.

As for **P4a**, it follows from **A1**-**A2** that

$$E\left(|X_n|\right) \leq \sqrt{E\left(|X_n|^2\right)} \leq \sqrt{Var\left(X_n\right) + E^2\left(X_n\right)} = \mathcal{O}\left(n^{-\frac{1}{2}}\right).$$

Equation **P4b** is a restatement of **A3**.

As noted in [32], **P4c**, **P4d** and **P4e** follow from **P4a** and **P4b**.

It remains to prove **P5**. Using Schwarz's inequality twice,

$$\begin{aligned}&\mathcal{E}^n_{\frac{T}{n}}(\left|\int_1^I u^\gamma \left(I-u\right)^M du\right|)\left(1\right)\\ &\leq \sqrt{\mathcal{E}^n_{\frac{T}{n}}(\left|\int_1^I u^{2\gamma} du)\right|)\left(1\right)}\sqrt{\mathcal{E}^n_{\frac{T}{n}}(\left|\int_1^I \left(I-u\right)^{2M} du\right|)\left(1\right)}.\end{aligned}$$

But

$$\begin{aligned}\mathcal{E}^n_{\frac{T}{n}}(\left|\int_1^I \left(I-u\right)^{2M} du\right|)\left(1\right) &= \frac{1}{2M+1}\mathcal{E}^n_{\frac{T}{n}}(\left|I-1\right|^{2M+1})\left(1\right)\\ &= \mathcal{O}\left(n^{-\frac{2M+1}{2}}\right).\end{aligned}$$

Moreover, it follows from **P4c** when $\gamma \neq -1/2$ and from (4.1) when $\gamma = -1/2$, that

$$\mathcal{E}^n_{\frac{T}{n}}(\left|\int_1^I u^{2\gamma} du\right|)(1) = \mathcal{O}\left(n^{-\frac{1}{2}}\right).$$

This yields **P5**. □

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