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EXERCISABILITY RANDOMIZATION OF THE AMERICAN OPTION

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Abstract. The valuation of American options is an optimal stopping time problem which typically leads to a free boundary problem. We introduce here the randomization of the exercisability of the option. This method considerably simplifies the problematic by transforming the free boundary problem into an evolution equation. This evolution equation can be transformed in a way that decomposes the value of the randomized option into a European option and the present value of continuously paid benefits. This yields a new binomial approximation for American options. We prove that the method is accurate and numerical results illustrate that it is computationally efficient.

1. Introduction

American options are a fundamental financial tool of modern financial markets and millions of them are traded every day. In the last two decades, the problem of valuing American options has become the subject of an extensive literature. While various enhancement of the traditional Black-Scholes model are at the center of abundant publications, the later remains a benchmark both for practitioners and theoreticians, and the new models are still facing the same difficulties inherent to the nature of American options. Such a core difficulty comes from the fact that the American option's value is the solution of a problem which involves finding the optimal boundary at which the option should be exercised, as part of the solution. Such problems are called free boundary problems.

One of the earliest successful method developed to address the valuation of American options in the Black-Scholes setting is the finite difference method with pioneer works by Brennan and Schwartz [1]. Still today, the most widely used valuation technique is certainly the

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binomial method due to Cox, Ross and Rubinstein [2]. Geske and Johnson [3] derived the *compound options* approach. MacMillan [4] and Barone-Adesi and Whaley [5] developed a quadratic approximation method. Kim [6], Jacka [7], Carr et al. [8] found integral-equation methods for the location of the optimal boundary which lead to further numerical methods in Huang, Subrahmanyam, and Yu [9] and approximation methods in Ju [10]. Another fundamental approach is the randomization of the option in Carr [11] which yields to simpler approximations, or the randomization of the underlying asset in Leisen [12], [13] yielding a smoother convergence of the numerical implementation.

The valuation of American options in the Black-Scholes setting is an optimal stopping time problem and, as such, has an interest that extends beyond the mere valuation of the derivative products. The question remains the subject of several publications. To name some recent papers, let us mention: Alobaidi and Mallier [14], Bunch and Johnson [15], AitShalia and Lai [16], Allegretto et al. [17], Alobaidi and Mallier [18], Longstaff and Schwartz [19], Zvan et al. [20], Allegretto et al. [21], Coleman et al. [22], Rogers [23], Wu and Ding [24], Zhu [25], Borici and Luthi [26], Cruz-Baez and Gonzalez-Rodriguez [27], Peskir [28], Schroder [29], Widdicks et al. [30], Camara and Chung [31], Gaudenzi and Lepellere [32], Zhu [33], Armadaa et al. [34], Chen and Chadam [35], Lin and Liang [36], Zhao et al. [37]. In a recent paper, Zhu [38] solved a "holy grail" of financial mathematics by finding a closed form solution to the American put option.

A central difficulty in research dealing with American options is that, to get the value of the option, one should also identify a free optimal boundary for which no simple form exists. This paper addresses that core issue. Its purpose is to develop a new approach for determining American option values, based on randomization, which eliminates the need for finding a free optimal boundary.

While a European option entitles his owner to exercise at maturity and an American option at any time up to maturity, a middle ground between these two extremes is an option which is exercisable at some but not all times up to maturity. A *randomly exercisable* American option gives the right to his owner to exercise at some *random* dates prior or up to maturity. In this paper the exercisable dates independently follow each other by waiting an exponentially distributed time. Note that these exercisable dates are completely independent of the stock process which is traded continuously.

The value of the exercisability randomized American option approximates the value of its American counterpart. For randomly exercisable option, the problem of finding the solution is simplified by the fact that the optimization problem that should be solved at time zero, to find the value of the option, has the same form as the one that should be solved at the time of the first exercisable date, thus yielding an ordinary evolution equation. Clearly, the shorter the average time between two exercisable dates is made, the better the approximation becomes. As the average waiting time between two exercisable dates approaches zero, the loss of opportunity vanishes and the value of the randomly exercisable option converges to the true American option value.

Randomization of the American option was introduced in Carr [11] where the maturity of the option is randomized. The maturity being the n^{th} jump of an independent Poisson process the solution is simplified by the memoryless property of exponential variables as, before the first jump, the horizon remains the same so that the optimal boundary remains constant. Randomization of the underlying asset as a mean of approaching the Black-Scholes setting was also studied in Follmer and Sondermann [39], Sondermann [40], Dengler and Jarrow [41] and Rogers and Stapleton [42]. But in the case of the American option, the complexity of identifying the optimal boundary remains unchanged. Leisen [12] studied the binomial asset pricing model with a random number of steps and obtained a smoother convergence.

In Kim [6], Jacka [7] and Carr, Jarrow and Myneni [8], natural decompositions of the American option into a European option and an early exercise premium expressed in terms of the optimal boundary not only reveals a fundamental feature of American options but is also at the heart of numerical techniques for valuing the option. In this paper, the randomization of the exercisability yields a natural break down of the option into a European option and the present value of some continuously paid *early exercise benefits*. At any given time this instantaneous benefit is equal to the excess of intrinsic value over the present value of the optimal deferred exercise.

The rigorous study of the convergence of some of the classical numerical method for valuing American options -among others Carr's maturity randomization- has been the specific subject of recent publications: as pointed out in Deng et al. [43], the convergence to the true American option value of many of the important classical methods for valuing American options had not been rigorously established and, they proved the convergence of Carr's method of lines and maturity randomization. In Dupuis and Wang [44] and in Bouchard, El Karoui and Touzi [45] this convergence is recognized as an important issue part of a broader problematic. Note that the non-triviality of proving the convergence in free boundary problem settings was pointed out in

Carr [11]. Here the convergence is established by breaking down the error into a discretization error, a randomization error and a law error which are shown to converge to zero. Numerical results are provided to illustrate the computational efficiency of the method.

The paper is organized as follows. In section 2 we collect the basic assumptions and notations used in this paper. Section 3 deals with the optimal stopping time for the randomized option. In section 4 we prove that the value of the randomly exercisable American option converges to the value of the *ordinary* American option. In section 5 we derive an evolution equation having the value of the randomized American option as its unique solution. We use Dynkin's generalized Feynman-Kac formula in Section 6 to transform this evolution equation in way that decomposes the value of the randomized American option into a European option and the present value of continuously paid *early exercise benefits.* This second evolution equation is the basis for a numerical implementation described in section 8. In Section 9 we show that our numerical method converges to the true value of the American option and in section 10, numerical results are provided to illustrate the efficiency of the method. As for section 7, it connects the ordinary American option to the randomly exercisable American option by exhibiting a suboptimal strategy for the owner of an American option which yields the value of the randomized option.

2. Basic settings

Throughout this paper we work with a strong Markov right process $\xi = (\xi, \mathcal{F}, \pi_{t,x})$ where $\mathcal{F}$ is the usual filtration and $\pi_{t,x}$ is the conditional expectation knowing that $\xi_t = x$. We assume that

$$\pi_{r,x}\left(f\left(\xi_t\right)\right) = \pi_{r,1}\left(f\left(x\xi_t\right)\right)$$

Indeed we are mostly interested in ξ a Geometric-Brownian motion with parameters ρ and σ, that is, for every $0 \leq r \leq t$ and every $x > 0$, $\pi_{r,x}\left(f\left(\xi_t\right)\right) = \pi\left(f\left(x\xi_{t-r}\right)\right)$ and

$$\pi\left(f\left(x\xi_{t-r}\right)\right) = \pi\left(x\exp\left(\sigma W_{t-r} + \left(\rho - \frac{1}{2}\sigma^2\right)(t-r)\right)\right)$$

where W is a Brownian motion and $\pi := \pi_{0,1}$.

A time $T > 0$, representing the time to maturity, is fixed throughout the paper.

A function $h(x) \geq 0$ representing the intrinsic value of the option is also fixed. We assume that $h(x)$ is Lipschitz and has compact support.

Hence there exists $\mathbb{K}, \mathbb{L} > 0$ such that

$$h(x) = h(x)1_{[0,\mathbb{K}]}(x)$$
$$|h(x) - h(y)| \leq \mathbb{L}\,|x - y|\,.$$

Note that the American put with strike K, is a particular case of this with $h(x) = \max(K - x, 0)$.

On the same space as ξ, an independent Poisson processes N^λ with parameter λ are realized, for $\lambda = 1, 2, 3, ...$ We denote by $N^\lambda(a, b]$ the number of jumps in interval $(a, b]$. We will refer to λ as the *exercisability intensity*.

In our setting, an option bought at time t is exercisable at any of the jumps of N^λ in $(t, T]$ or, at the latest, at time T. We denote by

$$\widetilde{\mathbb{T}} := \widetilde{\mathbb{T}}^{\lambda,t,T} = \left\{\mathbb{T}_1^{\lambda,t,T}, \mathbb{T}_2^{\lambda,t,T}, ...\right\}$$

the set of all *Exercisable Dates* in the interval $(t, T]$, that is the times in this interval at which the option is exercisable (the time of the jumps of N^λ). Obviously,

$$\mathbb{T}_n := \mathbb{T}_n^{\lambda,t,T} \sim \left(t + \tau_1^\lambda + \tau_2^\lambda + ... + \tau_n^\lambda\right) \wedge T, \text{ for } n = 1, 2, ... \tag{2.1}$$

where the variables τ_i^λ, $i = 1, 2, ...$ are independent and identically exponentially distributed with parameter λ, and models the *"Waiting" Time* one has to wait in between two *Exercisable Dates*.

Clearly, developing a strategy to optimize the value of an American Option with random exercisable dates, involves the set of stopping times matching one of the values of $\widetilde{\mathbb{T}}$. We denote by $\mathcal{T}[t, T]$ the set of all t-stopping times bounded by T and by $\mathcal{T}^\lambda[t, T]$ the t-stopping times bounded by T whose value always coincide with one exercisable time $\mathbb{T}_n$. Recall that τ is a t-stopping times bounded by T if $t \leq \tau \leq T$ and if τ is a stopping time with respect to the filtration $(\mathcal{T}[t, s])_{s\in[t,T]}$.

Note that $\mathcal{T}[t, T]$ can be mapped into $\mathcal{T}^\lambda[t, T]$ through the formula

$$\tau \to \mathbb{T}(\tau, \widetilde{\mathbb{T}}) := \sum_{n=0}^{\infty} \mathbb{T}_{n+1} 1_{[\mathbb{T}_n, \mathbb{T}_{n+1})}(\tau) \tag{2.2}$$

where $\mathbb{T}_0 := t$. Hence $\mathbb{T}(\tau, \widetilde{\mathbb{T}})$ is the *first Exercisable Date after* τ.

We denote, for any value x of the underlying stock ξ and any time t up to maturity T, the value $v_{h,T}^\lambda(t, x)$ of the randomly exercisable American Option which is given by the formula

$$v_{h,T}^\lambda(t, x) := \sup_{\tau\in\mathcal{T}^\lambda[t,T]} \pi_{t,x}(e^{-\rho(\tau-t)}h(\xi_\tau))$$

We also denote the value of the European option by $v^{\mathcal{E}}_{h,T}(t,x)$ and the value of the American option by $v_{h,T}(t,x)$.

We are interested in the first *Exercisable Date* after time t for which the intrinsic value $h(\xi_s)$ is at least as big as the expected discounted value of the optimal future exercises of the option, $v^{\lambda}_{h,T}(s,\xi_s)$. That is we are interested in

$$\tau^{\lambda}_{*} = \inf\left\{ s \in \widetilde{\mathbb{T}} : h(\xi_s) \geq v^{\lambda}_{h,T}(s,\xi_s) \right\}. \tag{2.3}$$

We are also interested in the optimal exercise time after t of the American option

$$\tau_{*} = \inf\left\{ s \geq t : h(\xi_s) \geq v_{h,T}(s,\xi_s) \right\}.$$

As expected, τ^{λ}_{*} is the optimal exercise of the randomly exercisable option. This is established next.

3. Optimal stopping time

First, for τ^{λ}_{*} to be a stopping time we need $s \mapsto v^{\lambda}_{h,T}(s,\xi_s)$ to be right continuous. But this is guaranteed by the fact that if τ_n is a sequence of decreasing t-stopping times converging to τ_∞, then, clearly,

$$\lim_{n\to\infty} \pi_{t,x}\left(v^{\lambda}_{h,T}(\tau_n,\xi_{\tau_n})\right) = \pi_{t,x}\left(v^{\lambda}_{h,T}(\tau_\infty,\xi_{\tau_\infty})\right)$$

which guaranties almost sure right continuity according to [46, VI.48].

The fact that τ^{λ}_{*} is an optimal stopping time is common to all American style option. An adaptation of a similar argument in [47, Th. 4.4.5] is drawn here for the sake of completeness.

Let σ be a t-stopping time. Let us first show that exercising at $\sigma \wedge \tau^{\lambda}_{*}$ is at least as good as exercising as σ. To see this, first note that exercising at τ^{λ}_{*} is, by definition, at least as good as keeping the option, so

$$\begin{aligned}\pi_{t,x}\left(e^{-\rho\sigma\wedge\tau^{\lambda}_{*}} h\left(\xi_{\sigma\wedge\tau^{\lambda}_{*}}\right)\right) &\geq \pi_{t,x}\left(e^{-\rho\sigma} 1_{\sigma\leq\tau^{\lambda}_{*}} h\left(\xi_\sigma\right)\right) \\ &\quad + \pi_{t,x}\left(e^{-\rho\tau^{\lambda}_{*}} 1_{\sigma>\tau^{\lambda}_{*}} v^{\lambda}_{h,T}\left(\tau^{\lambda}_{*},\xi_{\tau^{\lambda}_{*}}\right)\right)\end{aligned} \tag{3.1}$$

But

$$v^{\lambda}_{h,T}\left(\tau^{\lambda}_{*},\xi_{\tau^{\lambda}_{*}}\right) \geq \pi_{\tau^{\lambda}_{*},\xi_{\tau^{\lambda}_{*}}}\left(e^{-\rho\left(\sigma\vee\tau^{\lambda}_{*}-\tau^{\lambda}_{*}\right)} h\left(\xi_{\sigma\vee\tau^{\lambda}_{*}}\right)\right)$$

thus, using the strong Markov property, we can continue (3.1) with

$$\begin{aligned}\pi_{t,x}\left(e^{-\rho\sigma\wedge\tau^{\lambda}_{*}} h\left(\xi_{\sigma\wedge\tau^{\lambda}_{*}}\right)\right) &\geq \pi_{t,x}\left(e^{-\rho\sigma}\left(1_{\sigma\leq\tau^{\lambda}_{*}} h\left(\xi_\sigma\right) + 1_{\sigma>\tau^{\lambda}_{*}} h\left(\xi_\sigma\right)\right)\right) \\ &= \pi_{t,x}\left(e^{-\rho\sigma} h\left(\xi_\sigma\right)\right)\end{aligned}$$

Thus no optimal strategy can impose us to wait longer than τ_*^λ. So let σ be a stopping time bounded by τ_*^λ. We need to show that

$$\pi_{t,x}\left(e^{-\rho\sigma}h\left(\xi_\sigma\right)\right) \leq \pi_{t,x}\left(e^{-\rho\tau_*^\lambda}h\left(\xi_{\tau_*^\lambda}\right)\right). \tag{3.2}$$

In order to do so, we will first show that, for every n, the optimal strategy for the option maturing at the n^{th} exercisable date $\mathbb{T}_n$ is to exercise at the first exercisable date $\widetilde{\tau}_*^{\lambda,n}$ where exercising h has more value than the expected discounted deferred optimal exercise, $\widetilde{v}_{h,T}^{\lambda,n}$. More precisely, we define

$$\begin{aligned}\widetilde{v}_{h,T}^{\lambda,n}(t,x) &= \sup_{\tau\in\mathcal{T}^\lambda[t,T]} \pi_{t,x}(e^{-\rho\tau\wedge\mathbb{T}_n}h(\xi_{\tau\wedge\mathbb{T}_n}))\\ \widetilde{\tau}_*^{\lambda,n} &= \inf\left\{\mathbb{T}_i\in\widetilde{\mathbb{T}} : h(\xi_{\mathbb{T}_i}) \geq v_{h,T}^{\lambda,n-i}(\mathbb{T}_i,\xi_{\mathbb{T}_i})\right\}\\ \sigma_n &= \sigma\wedge\widetilde{\tau}_*^{\lambda,n}.\end{aligned}$$

Note that $\widetilde{v}_{h,T}^{\lambda,n}(t,x) \nearrow v_{h,T}^\lambda(t,x)$ as n tends to infinity and, therefore, $\widetilde{\tau}_*^{\lambda,n} \nearrow \tau_*^\lambda$ and $\sigma_n \nearrow \sigma$ as n tends to infinity.

Obviously, for an option that matures at the first exercisable date (that is $\widetilde{v}_{h,T}^{\lambda,1}$) the optimal (and unique) strategy is to exercise at $\widetilde{\tau}_*^{\lambda,1} = \mathbb{T}_1$. Now assume that, for $k = 1, 2, .., n-1$, $\widetilde{\tau}_*^{\lambda,k}$ is an optimal strategy for the option maturing at the k^{th} exercisable dates. Consider an option maturing at the n^{th} exercisable date. We have

$$\begin{aligned}\pi_{t,x}\left(e^{-\rho\sigma_n}h\left(\xi_{\sigma_n}\right)\right) &= \pi_{t,x}\left(e^{-\rho\sigma_n}1_{\sigma_n=\widetilde{\tau}_*^{\lambda,n}}h\left(\xi_{\sigma_n}\right)\right)\\ &\quad+\sum_{i=1}^{n-1}\pi_{t,x}\left(e^{-\rho\sigma_n}1_{\sigma_n<\widetilde{\tau}_*^{\lambda,n}}1_{\sigma_n=\mathbb{T}_i}h\left(\xi_{\sigma_n}\right)\right)\end{aligned}$$

But on the set $\left\{\sigma_n < \widetilde{\tau}_*^{\lambda,n}\right\}\cap\{\sigma_n = \mathbb{T}_i\}$ only $n-i$ exercisable dates are left and since, (1) by induction hypothesis, it is optimal to exercise at the first transaction date $\mathbb{T}_j$ after $\mathbb{T}_i$, $j \in \{i+1, i+2, ..., n\}$, for which

$$h(\xi_{\mathbb{T}_j}) \geq v_{h,T}^{\lambda,n-j}(\mathbb{T}_j,\xi_{\mathbb{T}_j}), \tag{3.3}$$

since, (2) on that set, $\widetilde{\tau}_*^{\lambda,n} > \mathbb{T}_i$, and since, (3) $\widetilde{\tau}_*^{\lambda,n}$ is the *first* transaction date for which (3.3) occurs, it follows that $\widetilde{\tau}_*^{\lambda,n}$ is optimal. And this yields

$$\pi_{t,x}\left(e^{-\rho\sigma_n}h\left(\xi_{\sigma_n}\right)\right) \leq \pi_{t,x}\left(e^{-\rho\widetilde{\tau}_*^{\lambda,n}}h\left(\xi_{\widetilde{\tau}_*^{\lambda,n}}\right)\right).$$

Letting n tends to infinity we get (3.2), as desired.

4. Convergence of the model

It is intuitively clear that the value of the *exercisability randomized* American option with *exercisability intensity* λ converges to the value of the American option when the *exercisability intensity* λ blows up to infinity and virtually no time has to be waited in between exercisable dates.

First note that since the American option can be traded without restrictions before maturity we clearly have $v_{h,T}(t,x) \geq v^{\lambda}_{h,T}(t,x)$. On the other hand, recall that τ_* is the optimal stopping time of the American option, let $\widehat{\tau}^{\lambda}_* \in \mathcal{T}^{\lambda}[t,T]$ be the *first exercisable date after* τ_* as given by (2.2) and let

$$\widehat{v}^{\lambda}_{h,T}(t,x) := \pi_{t,x}\left(e^{-\rho(\widehat{\tau}^{\lambda}_*-t)}h(\xi_{\widehat{\tau}^{\lambda}_*})\right)$$

Then, obviously, by suboptimality,

$$\widehat{v}^{\lambda}_{h,T}(t,x) \leq v^{\lambda}_{h,T}(t,x) \leq v_{h,T}(t,x)$$

Now

$$t \leq \tau_* \leq \widehat{\tau}^{\lambda}_* \leq T$$

and

$$\begin{aligned}
0 &\leq v_{h,T}(t,x) - \widehat{v}^{\lambda}_{h,T}(t,x) \\
&= \pi_{t,x}\left(e^{-\rho(\tau_*-t)}h(\xi_{\tau_*}) - e^{-\rho(\tau_*-t)}h(\xi_{\widehat{\tau}^{\lambda}_*})\right) \\
&+ \pi_{t,x}\left(\left(e^{-\rho(\tau_*-t)} - e^{-\rho(\widehat{\tau}^{\lambda}_*-t)}\right)h(\xi_{\widehat{\tau}^{\lambda}_*})\right).
\end{aligned}$$

But by the memoryless property of exponential variables, the time to wait for the first exercisable date after τ_*, that is $\widehat{\tau}^{\lambda}_*-\tau_*$, is exponentially distributed with average $\frac{1}{\lambda}$, so we have

$$\begin{aligned}
\pi_{t,x}\left(\left(e^{-\rho(\tau_*-t)} - e^{-\rho(\widehat{\tau}^{\lambda}_*-t)}\right)h(\xi_{\widehat{\tau}^{\lambda}_*})\right) &\leq \pi_{t,x}\left(\rho\left(\widehat{\tau}^{\lambda}_* - \tau_*\right)\right)\|h\|_{\infty} \\
&\leq \frac{\rho\|h\|_{\infty}}{\lambda}
\end{aligned}$$

and therefore

$$\begin{aligned}
0 &\leq v_{h,T}(t,x) - \widehat{v}^{\lambda}_{h,T}(t,x) \\
&\leq \pi_{t,x}\left(e^{-\rho(\tau_*-t)}\left(h(\xi_{\tau_*}) - \pi_{\tau_*,\xi_{\tau_*}}h(\xi_{\widehat{\tau}^{\lambda}_*})\right)\right) + \frac{\rho\|h\|_{\infty}}{\lambda} \\
&\leq \pi_{t,x}\left|h(\xi_{\tau_*}) - \int_0^{\infty} S^{\tau_*}_{(\tau_*+u)\wedge T}(h)(\xi_{\tau_*})e^{-\lambda u}\lambda du\right| + \frac{\rho\|h\|_{\infty}}{\lambda} \qquad (4.1)
\end{aligned}$$

which converges to zero as λ goes to infinity, where $S_{t_2}^{t_1}$ is the semigroup of ξ, that is

$$S_{t_2}^{t_1}(h)(x) := \pi_{t_1,x} h(\xi_{t_2}), \text{ for } 0 \le t_1 \le t_2, \text{ and } x \ge 0.$$

Finally, since

$$0 \le v_{h,T} - v_{h,T}^{\lambda} \le v_{h,T} - \widehat{v}_{h,T}^{\lambda}, \tag{4.2}$$

our claim that $v_{h,T}^{\lambda} \to v_{h,T}$ as $\lambda \to \infty$ follows.

5. Evolution equation of the option value $v_{h,T}^{\lambda}$

The benefit of randomizing the exercisability of an American option is that it transforms the free boundary problem solved by its value $v_{h,T}$ into an ordinary evolution equation. And it is considerably easier to work with such integral equations than with the original problem. As we will see now, it is simple to derive this evolution equation.

Let $\mathbb{T}_1$ be the first exercisable date of the option, as defined by (2.1). Then

$$\begin{aligned}
v_{h,T}^{\lambda}(t,x) &= \pi_{t,x}\left(e^{-\rho\tau_*^{\lambda}} h\left(\xi_{\tau_*^{\lambda}}\right)\right) \\
&= \pi_{t,x}\left(1_{\tau_*^{\lambda}=\mathbb{T}_1} e^{-\rho\tau_*^{\lambda}} h\left(\xi_{\tau_*^{\lambda}}\right) + 1_{\tau_*^{\lambda}>\mathbb{T}_1} e^{-\rho\tau_*^{\lambda}} h\left(\xi_{\tau_*^{\lambda}}\right)\right) \\
&= \pi_{t,x}\left(1_{\tau_*^{\lambda}=\mathbb{T}_1} e^{-\rho\mathbb{T}_1} h\left(\xi_{\mathbb{T}_1}\right)\right) \\
&+ \pi_{t,x}\left(1_{\tau_*^{\lambda}>\mathbb{T}_1} e^{-\rho\mathbb{T}_1} \pi_{\mathbb{T}_1,\xi_{\mathbb{T}_1}}\left(e^{-\rho\left(\tau_*^{\lambda}-\mathbb{T}_1\right)} h\left(\xi_{\tau_*^{\lambda}}\right)\right)\right) \\
&= \pi_{t,x}\left(1_{\tau_*^{\lambda}=\mathbb{T}_1} e^{-\rho\mathbb{T}_1} h\left(\xi_{\mathbb{T}_1}\right) + 1_{\tau_*^{\lambda}>\mathbb{T}_1} e^{-\rho\mathbb{T}_1} v_{h,T}^{\lambda}(\mathbb{T}_1, \xi_{\mathbb{T}_1})\right).
\end{aligned}$$

Since, $\tau_*^{\lambda} = \mathbb{T}_1$ iff it is optimal to exercise at $\mathbb{T}_1$, we obtain

$$v_{h,T}^{\lambda}(t,x) = \pi_{t,x}\left(e^{-\rho\mathbb{T}_1} \max(h\left(\xi_{\mathbb{T}_1}\right), v_{h,T}^{\lambda}(\mathbb{T}_1, \xi_{\mathbb{T}_1}))\right).$$

Recall from (2.1) that the first exercisable date $\mathbb{T}_1$ after t is defined by $\mathbb{T}_1 = \left(t + \tau_1^{\lambda}\right) \wedge T$ where τ_1^{λ} is and independent exponential variable with mean $\frac{1}{\lambda}$. Therefore

$$\begin{aligned}
v_{h,T}^{\lambda}(t,x) &= \pi_{t,x}\left(e^{-(\lambda+\rho)(T-t)} h(\xi_T)\right) \\
&+ \pi_{t,x} \int_t^T e^{-(\lambda+\rho)(s-t)} \max(h(\xi_s), v_{h,T}^{\lambda}(s,\xi_s)) \lambda ds.
\end{aligned} \tag{5.1}$$

Note that, rewriting the first member of the right-hand side in terms of the European option $v^{\mathcal{E}}_{h,T}$, this is obviously the same as

$$
\begin{aligned}
(5.2)\qquad v^{\lambda}_{h,T}(t,x) &= e^{-\lambda t} v^{\mathcal{E}}_{h,T}(t,x)\\
&\quad + \pi_{t,x}\int_t^T e^{-(\lambda+\rho)(s-t)} 1_{\left\{h(\xi_s)\geq v^{\lambda}_{h,T}(s,\xi_s)\right\}} h(\xi_s)\lambda ds\\
&\quad + \pi_{t,x}\int_t^T e^{-(\lambda+\rho)(s-t)} 1_{\left\{h(\xi_s)< v^{\lambda}_{h,T}(s,\xi_s)\right\}} v^{\lambda}_{h,T}(s,\xi_s)\lambda ds.
\end{aligned}
$$

6. Decomposition into a European option and an early exercise premium

A powerful tool for analyzing evolution equations is Dynkin's generalized Feynman-Kac formula (see [48, Th. 4.1.1- 4.1.4]). We will use this to get a natural relation between the exercisability randomized American option and the European option. It breaks down the randomly exercisable option into a European option and the expected present value of the continuously paid excess of its intrinsic value over the expected present value of the optimal deferred exercise. More precisely, we show in this section that

$$
\begin{aligned}
(6.1)\qquad v^{\lambda}_{h,T}(t,x) &= v^{\mathcal{E}}_{h,T}(t,x)\\
&\quad + \pi_{t,x}\int_t^T e^{-\rho(s-t)} \max\left(h(\xi_s) - v^{\lambda}_{h,T}(s,\xi_s), 0\right)\lambda ds.
\end{aligned}
$$

where $v^{\mathcal{E}}_{h,T}$ denote the European option with maturity T and intrinsic value h.

Recall that a random non-negative measure $A(ds)$ is called an additive functional of ξ if, for every time interval (α,β), $A(.,(\alpha,\beta))$ is measurable with respect to the completion of $\mathcal{F}(\alpha,\beta)$ with respect to $\pi_{\alpha,x}$ for every x. A particular case of Dynkin's generalized Feynman-Kac formula is stated in Theorem 1 below. For f a measurable non-negative function and some fixed time T, it is typical (such as in [48] or [49] for instance) to use this formula with additive functionals of ξ having the form $C(I) := \int_I f(s,\xi_s)ds$ or $B(I) := 1_I(T)f(T,\xi_T)$, for I a time interval.

Theorem 1. *Let C be a non-negative additive functional of ξ. Put $H(I) := \exp C(I)$, for any interval I. Let B be a signed additive functional of ξ. Assume that*

$$\pi_{r,x}\left|B\right|(r,t] < \infty$$

The function

$$g(r,x) = \pi_{r,x} \int_{(r,t]} H(r,s)^{-1} B(ds) \tag{6.2}$$

satisfies the equation

$$g(r,x) = \pi_{r,x} B(r,t] - \pi_{r,x} \int_r^t g(s,\xi_s) C(ds) \tag{6.3}$$

If g is finite and if the left side of (6.3) is well-defined then (6.3) implies (6.2).

We will use this result to find a explicit solution to (5.2) using this result. To do this, let the additive functional $B(ds)$ be defined, for any time interval I, by

$$\begin{aligned} B(I) := 1_I(T) h(\xi_T) + \int_I 1_{\left\{h(\xi_s) \geq v^\lambda_{h,T}(s,\xi_s)\right\}} h(\xi_s) \lambda ds \\ + \int_I 1_{\left\{h(\xi_s) < v^\lambda_{h,T}(s,\xi_s)\right\}} v^\lambda_{h,T}(s,\xi_s) \lambda ds \end{aligned}$$

Then (5.2) precisely says that

$$v^\lambda_{h,T}(t,x) = \pi_{t,x} \int_t^T e^{-(\lambda+\rho)(s-t)} B(ds)$$

Therefore, according to Theorem 1,

$$v^\lambda_{h,T}(t,x) = \pi_{t,x} \left(B(t,T)\right) - \pi_{t,x} \int_t^T v^\lambda_{h,T}(s,\xi_s) \left(\lambda + \rho\right) ds$$

Or, said otherwise,

$$\begin{aligned} v^\lambda_{h,T}(t,x) = \pi_{t,x} \left(h(\xi_T)\right) + \pi_{t,x} \int_t^T 1_{\left\{h(\xi_s) \geq v^\lambda_{h,T}(s,\xi_s)\right\}} h(\xi_s) \lambda ds \\ - \pi_{t,x} \int_t^T v^\lambda_{h,T}(s,\xi_s) \rho ds \\ - \pi_{t,x} \int_t^T 1_{\left\{h(\xi_s) \geq v^\lambda_{h,T}(s,\xi_s)\right\}} v^\lambda_{h,T}(s,\xi_s) \lambda ds \end{aligned} \tag{6.4}$$

Which is the same as

$$\begin{aligned} v^\lambda_{h,T}(t,x) = \pi_{t,x} \left(h(\xi_T)\right) + \pi_{t,x} \int_t^T \max\left(h(\xi_s) - v^\lambda_{h,T}(s,\xi_s), 0\right) \lambda ds \\ - \pi_{t,x} \int_t^T v^\lambda_{h,T}(s,\xi_s) \rho ds. \end{aligned}$$

Involving the generalized Feynman-Kac formula with $B(ds)$ and $C(ds)$ defined, for any time interval I, by

$$B(I) := 1_I(T)h(\xi_T) + \int_I \max\left(h(\xi_s) - v^\lambda_{h,T}(s,\xi_s), 0\right)\lambda ds$$

$$C(I) := \int_I \rho ds$$

we obtain (6.1) as desired.

7. Connection with the *ordinary* American option

We put here in an aside a representation of $v^\lambda_{h,T}$ as the value of an American option under a suboptimal strategy, thus obtaining another formula for the value of $v^\lambda_{h,T}$. A simple lattice calculation of $v^\lambda_{h,T}$ using this representation is possible yielding, we believe, a somehow exotic implementation of the option price calculation!

This representation is obtained by making use of Dynkin's generalized Feynman-Kac formula in order to get

$$v^\lambda_{h,T}(t,x) = \pi_{t,x}\left(e^{-\rho(T-t)}e^{-C(t,T)}h(\xi_T)\right) + \pi_{t,x}\int_t^T e^{-\rho(s-t)}e^{-C(t,s)}h(\xi_s)C(ds) \tag{7.1}$$

where $C(ds)$ is the continuous additive functional defined for any time subinterval I of $[0,t]$ by

$$C(I) = \int_I 1_{\left\{h(\xi_u)\geq v^\lambda_{h,T}(u,\xi_u)\right\}}\lambda du$$

Now if τ_1 is an independent variable which is exponentially distributed with mean 1 and if we define at time t the variable

$$\tau = \inf\left\{s \geq t : C(t,s) \geq \tau_1\right\}$$

then (7.1) precisely says that

$$v^\lambda_{h,T}(t,x) = \pi_{t,x}\left(e^{-\rho\tau}h(\xi_\tau)\right)$$

Multiplying τ and C by $\frac{1}{\lambda}$, this can be interpreted as saying that, starting at time t, if the owner of an *ordinary* American option waits, before exercising, that the total time after t, $\int_t^s 1_{\left\{h(\xi_u)\geq v^\lambda_{h,T}(u,\xi_u)\right\}}du$ spent within the *exercise region* of $v^\lambda_{h,T}$, is as long as the time until the first exercisable date $\mathbb{T}_1$, then, this *suboptimal* strategy is equivalent to holding the *randomly exercisable* option. Note that this strategy is unavailable to the owner of the randomized option since his option may not be exercisable at that time.

8. An Early Exercise Premium implementation

In this section we show how equation (6.1) can be solved numerically. But first let us write

$$b^{\lambda}_{h,T}(t,x) = h(x) - v^{\lambda}_{h,T}(t,x) \tag{8.1}$$

$$a^{\lambda}_{h,T}(t,x) = \pi_{t,x}\int_t^T e^{-\rho(s-t)}\max\left(b^{\lambda}_{h,T}(s,\xi_s),0\right)\lambda ds$$

and for $0<\Delta<T-t$

$$_{\Delta}|a^{\lambda}_{h,T}(t,x) = e^{-\rho\Delta}\pi_{t,x}a^{\lambda}_{h,T}(t+\Delta,\xi_{t+\Delta})$$

$$a^{\lambda}_{h,T}(t,x)_{\overline{\Delta|}} = \pi_{t,x}\int_t^{t+\Delta} e^{-\rho(s-t)}\max\left(b^{\lambda}_{h,T}(s,\xi_s),0\right)\lambda ds$$

These quantities have clear financial meanings: $\max\left(b^{\lambda}_{h,T}(t,x),0\right)$ is the *instantaneous early exercise benefit* resulting from exercising at time t (as opposed to keeping the option); $a^{\lambda}_{h,T}(t,x)$ is the *present value of these continuously paid benefits until maturity*; $_{\Delta}|a^{\lambda}_{h,T}(t,x)$ is the *present value of these benefits when the payments are deferred by* Δ and $a^{\lambda}_{h,T}(t,x)_{\overline{\Delta|}}$ is the *present value of these benefits paid during a temporary period of* Δ.

Note that (6.1) can be rewritten as

$$v^{\lambda}_{h,T}(t,x) = v^{\mathcal{E}}_{h,T}(t,x) + a^{\lambda}_{h,T}(t,x) \tag{8.2}$$

and for any Markov process ξ and any time step $0<\Delta<t$, the continuous payments $a^{\lambda}_{h,T}(t,x)$ can be decomposed into the *temporary* payments $a^{\lambda}_{h,T}(t,x)_{\overline{\Delta|}}$ and the *deferred* ones $_{\Delta}|a^{\lambda}_{h,T}(t,x)$ yielding

$$a^{\lambda}_{h,T}(t,x) = a^{\lambda}_{h,T}(t,x)_{\overline{\Delta|}} + {}_{\Delta}|a^{\lambda}_{h,T}(t,x). \tag{8.3}$$

Now in order to evaluate the integral $a^{\lambda}_{h,T}(t,x)$, we need to discretize ξ. The simplest way to do so is to replace ξ by the binomial asset price process ξ^n. That is we divide the time into steps of size $\Delta_n = \frac{T}{n}$ and, starting at time $t_0=0$ at position $\xi^n_0 = x$, process ξ^n jumps at each time $t_i = i\Delta_n$, $i=1,2,...$ from its current state ξ^n_{i-1} to the state $\xi^n_i = u\xi^n_{i-1}$ with probability p_u and to the state $\xi^n_i = d\xi^n_{i-1}$ with probability p_d where

$$u = \exp(\sigma\sqrt{\Delta_n}), d = \frac{1}{u}, p_u = \frac{\exp(\rho\Delta_n)-d}{u-d}, p_d = 1-p_u.$$

This yields

$$v^{n,\lambda}_{h,T}(t,x) = v^{n,\mathcal{E}}_{h,T}(t,x) + a^{n,\lambda}_{h,T}(t,x) \tag{8.4}$$

where the superscript n indicates that ξ has been replaced by the binomial asset process ξ^n in (8.2).

In order to recursively estimate the members of (8.4) define for every x,

$$\begin{aligned}\widehat{a}^{n,\lambda}_{h,T}(T,x) &= 0\\ \widehat{b}^{n,\lambda}_{h,T}(T,x) &= 0\\ \widehat{v}^{n,\lambda}_{h,T}(T,x) &= h(x),\end{aligned}$$

and, for $0 \le t < T$,

$$\widehat{b}^{n,\lambda}_{h,T}(t,x) = \sum_{i=0}^{n-1} 1_{[t_i,t_{i+1})}(t)\left(h(x) - \widehat{v}^{n,\lambda}_{h,T}(t_i,x)\right) \tag{8.5}$$

$$\widehat{a}^{n,\lambda}_{h,T}(t,x) = \pi_{t,x}\int_t^T e^{-\rho(s-t)}\max\left(\widehat{b}^{n,\lambda}_{h,T}(s,\xi^n_s),0\right)\lambda ds$$

$$\widehat{v}^{n,\lambda}_{h,T}(t,x) = v^{n,\mathcal{E}}_{h,T}(t,x) + \widehat{a}^{n,\lambda}_{h,T}(t,x). \tag{8.6}$$

Again $\widehat{a}^{n,\lambda}_{h,T}$ can be decomposed into the temporary and deferred payments

$$\begin{aligned}\widehat{a}^{n,\lambda}_{h,T}(t,x)_{\overline{\Delta|}} &= \pi_{t,x}\int_t^{t+\Delta} e^{-\rho(s-t)}\max\left(\widehat{b}^{n,\lambda}_{h,T}(s,\xi^n_s),0\right)\lambda ds\\ {}_{\Delta|}\widehat{a}^{n,\lambda}_{h,T}(t,x) &= e^{-\rho\Delta}\pi_{t,x}\widehat{a}^{n,\lambda}_{h,T}\left(t+\Delta,\xi^n_{t+\Delta}\right)\end{aligned}$$

so that

$$\widehat{a}^{n,\lambda}_{h,T}(t,x) = \widehat{a}^{n,\lambda}_{h,T}(t,x)_{\overline{\Delta|}} + {}_{\Delta|}\widehat{a}^{n,\lambda}_{h,T}(t,x). \tag{8.7}$$

From (8.5), (8.6) and (8.7), we obtain the following equation for $i = 0,...,n-1$

$$\begin{aligned}\widehat{v}^{n,\lambda}_{h,T}(t_i,x) &= h(x) - \widehat{b}^{\lambda}_{h,T}(t_i,x)\\ &= v^{n,\mathcal{E}}_{h,T}(t_i,x) + \widehat{a}^{n,\lambda}_{h,T}(t_i,x)\\ &= v^{n,\mathcal{E}}_{h,T}(t_i,x) + \widehat{a}^{n,\lambda}_{h,T}(t_i,x)_{\overline{\Delta_n|}} + {}_{\Delta_n|}\widehat{a}^{n,\lambda}_{h,T}(t_i,x).\end{aligned}$$

But obviously

$$\begin{aligned}{}_{\Delta_n|}\widehat{a}^{n,\lambda}_{h,T}(t_i,x) &= \pi_{t_i,x}\left(e^{-\rho\Delta_n}\widehat{a}^{n,\lambda}_{h,T}(t_{i+1},\xi_{t_{i+1}})\right)\\ &= e^{-\rho\Delta_n}\left(\widehat{a}^{n,\lambda}_{h,T}(t_{i+1},xu)p_u + \widehat{a}^{n,\lambda}_{h,T}(t_{i+1},xd)p_d\right) \qquad (8.8)\end{aligned}$$

and since ξ_s and $\widehat{b}^{n,\lambda}_{h,T}$ are constant on each time interval $[t_i, t_{i+1})$ we also have

$$\widehat{a}^{n,\lambda}_{h,T}(t_i,x)\overline{_{\Delta_n|}} = \pi_{t_i,x}\int_{t_i}^{t_{i+1}} e^{-\rho(s-t)}\max\left(\widehat{b}^{\lambda}_{h,T}(t_i,x),0\right)\lambda ds$$
$$= \max\left(\widehat{b}^{\lambda}_{h,T}(t_i,x),0\right) f^{\lambda}_{\Delta_n,\rho}$$

where

$$f^{\lambda}_{\Delta_n,\rho} = \frac{1-\exp\left(-\rho\Delta_n\right)}{\rho}\lambda.$$

Hence

$$h(x) - \widehat{b}^{\lambda}_{h,T}(t_i,x) = \max\left(\widehat{b}^{\lambda}_{h,T}(t_i,x),0\right) f^{\lambda}_{\Delta_n,\rho} + v^{n,\mathcal{E}}_{h,T}(t,x) + {}_{\Delta_n|}\widehat{a}^{n,\lambda}_{h,T}(t_i,x)$$

Which is solved by

$$\widehat{b}^{\lambda}_{h,T}(t_i,x) = \frac{1}{1+f^{\lambda}_{\Delta_n,\rho}}\max(h(x) - v^{n,\mathcal{E}}_{h,T}(t,x) - {}_{\Delta_n|}\widehat{a}^{n,\lambda}_{h,T}(t_i,x),0) \tag{8.9}$$

from which we can set

$$\widehat{a}^{n,\lambda}_{h,T}(t_i,x) = \max\left(\widehat{b}^{\lambda}_{h,T}(t_i,x),0\right) f^{\lambda}_{\Delta_n,\rho} + {}_{\Delta_n|}\widehat{a}^{n,\lambda}_{h,T}(t_i,x). \tag{8.10}$$

The method for solving $\widehat{b}^{n,\lambda}_{h,T}$, $\widehat{a}^{n,\lambda}_{h,T}$ and $\widehat{v}^{n,\lambda}_{h,T}$ recursively is now simple to describe. Let $\xi^n_{i,j}$ be the values that can take ξ^n at time t_i, for $i = 0, ..., n$ and $j = 0, ..., i$. The present value of the benefits, $\widehat{a}^{n,\lambda}_{h,T}(t_i, \xi^n_{i,j})$, are calculated recursively first by $\widehat{a}^{n,\lambda}_{h,T}(t_n, \xi^n_{n,j}) = 0$ for $j = 0, ..., n$ and, assuming $\widehat{a}^{n,\lambda}_{h,T}(t_i, \xi^n_{i,j})$ has already been calculated for $j = 0, ..., i$, the values ${}_{\Delta_n|}\widehat{a}^{n,\lambda}_{h}(t_{i-1}, \xi^n_{i-1,j})$ and $\widehat{b}^{\lambda}_{h,T}(t_{i-1}, \xi^n_{i-1,j})$, for $j = 0, ..., i-1$, are obtained through (8.8) and (8.9). $\widehat{a}^{n,\lambda}_{h,T}(t_{i-1}, \xi^n_{i-1,j})$ and $\widehat{v}^{n,\lambda}_{h,T}(t_{i-1}, \xi^n_{i-1,j})$ are then given by (8.10) and (8.6). Finally, at $i = 0$, $(t_0, \xi^n_{0,0}) = (0, x)$, yielding the desired estimate $\widehat{v}^{n,\lambda}_{h,T}(0, x)$ of the randomized American option value. Taking $\lambda = n$ makes $\widehat{v}^{n,n}_{h,T}$ converge to the value $v_{h,T}$ of the American option, as shown in the next section.

9. Convergence of the numerical method

All that remains to be shown now is that $\widehat{v}^{n,n}_{h,T}(0, x)$ converges to $v_{h,T}(0, x)$. This is going to be done by breaking down the approximation error into three part, a law error $\mathcal{L}^n$, a randomization error $\mathcal{R}^{n,n}$ and a discretization error $\mathcal{D}^{n,n}$. The latter is itself bounded by a modulus of continuity $\omega^{n,n}$ defined in the next section.

In addition to the previously defined notation for $v_{h,T}^{n,\lambda}$ and $v_{h,T}^{\lambda}$ we will use $v_{h,T}^{n,\infty}$ to denote the value of the American option when the underlying stock process ξ^n is the binomial asset pricing model. We will also use $v_{h,T}^{\infty,\infty}$ to denote the value of the American option with underlying ξ, that is $v_{h,T}^{\infty,\infty} = v_{h,T}$.

9.1. **Modulus of continuity over one time step $\omega^{n,\lambda}$.** We define the following modulus of continuity over one time step by

$$\omega^{n,\lambda} = \max_{i=0,\dots,n-1} \sup_{\delta\in[0,\frac{T}{n})} \sup_{x\geq 0} \left| v_{h,T}^{n,\lambda}(t_i,x) - v_{h,T}^{n,\lambda}(t_i+\delta,x) \right|$$

where $t_i = i\frac{T}{n}$, $i=1,...,n$. In this section we show that

$$\omega^{n,\lambda} \leq \left(1 - e^{-\rho\frac{T}{n}}\right) \|h\|_\infty . \tag{9.1}$$

To see this, take t in $\{t_1,...,t_n\}$ and δ in $[0,\frac{T}{n})$. Since ξ^n is constant on the interval $[t,t+\frac{T}{n})$, it follows that, for any s in $[t+\delta,T)$,

$$\pi_{t+\delta,x}\left(f(\xi_s)\right) = \pi_{t,x}\left(f(\xi_s)\right)$$

and therefore

$$v_{h,T}^{n,\lambda}(t+\delta,x) = \sup_{\sigma\in[t,T]} e^{\rho\delta}\pi_{t,x}\left(e^{-\rho\sigma\vee(t+\delta)}h(\xi^n_{\sigma\vee(t+\delta)})\right).$$

Note that

$$\begin{aligned} 0 &\leq v_{h,T}^{n,\lambda}(t,x) - v_{h,T}^{n,\lambda}(t+\delta,x) \\ &\leq v_{h,T}^{n,\lambda}(t,x) - e^{-\rho\delta}v_{h,T}^{n,\lambda}(t+\delta,x) \\ &\leq \sup_{\sigma\in[t,T]} e^{\rho t}\pi_{t,x}\left(e^{-\rho\sigma}h(\xi^n_\sigma) - e^{-\rho\sigma\vee(t+\delta)}h(\xi^n_{\sigma\vee(t+\delta)})\right). \end{aligned}$$

Now on the set $\{\sigma > t+\delta\}$,

$$e^{-\rho\sigma}h(\xi^n_\sigma) - e^{-\rho\sigma\vee(t+\delta)}h(\xi^n_{\sigma\vee(t+\delta)}) = 0,$$

and therfore we can continue with

$$= \sup_{\sigma\in[t,T]} e^{\rho t}\pi_{t,x}\left(e^{-\rho\sigma\wedge(t+\delta)}h(\xi^n_{\sigma\wedge(t+\delta)}) - e^{-\rho(t+\delta)}h(\xi^n_{(t+\delta)})\right).$$

But, because t is in $\{t_1,...,t_n\}$ and ξ^n is constant on the interval $[t,t+\frac{T}{n})$, we can continue with

$$= \sup_{\sigma\in[t,T]} e^{\rho t}\pi_{t,x}\left(e^{-\rho\sigma\wedge(t+\delta)} - e^{-\rho(t+\delta)}\right)h(x).$$

Now $\sigma \wedge (t+\delta) \geq t$, so we can continue with

$$\begin{aligned} &\leq e^{\rho t}\left(e^{-\rho t} - e^{-\rho(t+\delta)}\right)\|h\|_\infty \\ &= \left(1 - e^{-\rho\delta}\right)\|h\|_\infty \end{aligned}$$

and (9.1) follows.

9.2. **Randomization error $\mathcal{R}^{n,\lambda}$.** We estimate here the randomization error $\mathcal{R}^{n,\lambda}$ resulting from approximating the value $v^{n,\infty}_{h,T}(t,x)$ of an *ordinary* American option with underlying process ξ^n by the value $v^{n,\lambda}_{h,T}(t,x)$ of its *randomized* counterpart. More formally, we define the Randomization error $\mathcal{R}^{n,\lambda}$ by

$$\mathcal{R}^{n,\lambda} = \left| v^{n,\lambda}_{h,T}(t,x) - v^{n,\infty}_{h,T}(t,x) \right| \tag{9.2}$$

and, in this section, we show that

$$\lim_{(n,\lambda)\to\infty} \mathcal{R}^{n,\lambda} = 0. \tag{9.3}$$

Recall from (4.1) and (4.2) that

$$\begin{aligned} 0 &\leq v^{n,\infty}_{h,T}(t,x) - v^{n,\lambda}_{h,T} \\ &\leq \pi_{t,x}\left| h(\xi_{\tau_*}) - \int_0^\infty {}^nS^{\tau_*}_{(\tau_*+u)\wedge T}(h)(\xi^n_{\tau_*}) e^{-\lambda u}\lambda du \right| + \frac{\rho\|h\|_\infty}{\lambda} \\ &= \pi_{t,x}\left| \int_0^\infty \left(h(\xi_{\tau_*}) - {}^nS^{\tau_*}_{(\tau_*+u)\wedge T}(h)(\xi^n_{\tau_*}) \right) e^{-\lambda u}\lambda du \right| + \frac{\rho\|h\|_\infty}{\lambda} \end{aligned}$$

where ${}^nS^{t_1}_{t_2}$ is the semigroup of ξ^n, that is

$${}^nS^{t_1}_{t_2}(h)(x) := \pi_{t_1,x} h(\xi^n_{t_2}), \text{ for } 0 \leq t_1 \leq t_2, \text{ and } x \geq 0.$$

Now, since τ_* is optimal, $\xi^n_{\tau_*}$ must be "in the money" (that is $h(\xi^n_{\tau_*}) > 0$) or $\tau_* = T$. In the later case

$$h(\xi^n_{\tau_*}) - \int_0^\infty {}^nS^{\tau_*}_{(\tau_*+u)\wedge T}(h)(\xi^n_{\tau_*}) e^{-\lambda u}\lambda du = 0,$$

so we just have to treat the case where $h(\xi^n_{\tau_*}) > 0$. To do that, fix ω, let $x_* = \xi^n_{\tau_*(\omega)}(\omega), t_* = \tau_*(\omega)$, let $u \in [0, T - t_*]$ and let $\Delta_n = \frac{T}{n}$ and $u^n_* = u + frac(\frac{t_*}{\Delta_n})$ where $frac(z)$ is the fractional part of z. Note that

$${}^nS^{t_*}_{t_*+u}(h)(x_*) = {}^nS^0_{u^n_*}(h)(x_*).$$

Furthermore, since the support of h is contained in $[0, \mathbb{K}]$ and x_* is "in the money" then $0 \leq x_* \leq \mathbb{K}$. Denoting $\pi := \pi_{0,1}$ and using the fact that h is Lipschitz with constant $\mathbb{L}$, we get

$$\begin{aligned}\left|h(x_*) - {}^nS^0_{u^n_*}(h)(x_*)\right| &\leq \pi\left(\left|h\left(x_*\right) - h\left(x_*\xi^n_{u^n_*}\right)\right|\right)\\ &\leq \mathbb{L}x_*\pi\left(\left|1-\xi^n_{u^n_*}\right| \wedge \|h\|_\infty\right)\\ &\leq \mathbb{L}\mathbb{K}\pi(\sup_{0\leq s\leq u^n_*}\xi^n_s \wedge (1+\|h\|_\infty) - \inf_{0\leq s\leq u^n_*}\xi^n_s).\end{aligned}$$

Obviously,

$$u^n_* \leq u + \frac{T}{n}$$

and therefore

$$\left|h(x_*) - {}^nS^0_{u^n_*}(h)(x_*)\right| \leq \mathbb{L}\mathbb{K}\int_0^\infty \left({}^n\overline{S}_{u\wedge T} - {}^n\underline{S}_{u\wedge T}\right)e^{-\lambda u}\lambda du$$

where

$$\begin{aligned}{}^n\overline{S}_u &= \pi(\sup_{0\leq s\leq u+\frac{T}{n}}\xi^n_s \wedge (1+\|h\|_\infty))\\ {}^n\underline{S}_u &= \pi(\inf_{0\leq s\leq u+\frac{T}{n}}\xi^n_s).\end{aligned}$$

Hence,

$$\left\|v^{n,\lambda}_{h,T} - v^{n,\infty}_{h,T}\right\|_\infty \leq \mathbb{L}\mathbb{K}\int_0^\infty \left({}^n\overline{S}_{u\wedge T} - {}^n\underline{S}_{u\wedge T}\right)e^{-\lambda u}\lambda du + \frac{\rho\|h\|_\infty}{\lambda}. \tag{9.4}$$

Now it is well known that the laws of ξ^n in the Skorokhod space of càdlàg trajectories converge weakly to the law of the geometric Brownian motion ξ, see Coquet and Toldo [50]. By mean of taking a subsequence, the convergence can be assumed to be uniform almost surely. This implies that the laws of

$$\begin{aligned}\overline{\xi}^n_u &= \sup_{0\leq s\leq u+\frac{T}{n}}\xi^n_s\\ \underline{\xi}^n_u &= \inf_{0\leq s\leq u+\frac{T}{n}}\xi^n_s\end{aligned}$$

converge to the laws of

$$\begin{aligned}\overline{\xi}_u &= \sup_{0\leq s\leq u}\xi_s\\ \underline{\xi}_u &= \inf_{0\leq s\leq u}\xi_s\end{aligned}$$

and thus, if

$$\begin{aligned}{}^n\overline{S}^*_u &= \sup_{k\geq n}\pi\left(\overline{\xi}^k_u \wedge (1+\|h\|_\infty)\right)\\ {}^n\underline{S}^*_u &= \inf_{k\geq n}\pi\left(\underline{\xi}^k_u\right)\end{aligned}$$

then, from (9.4), we have

$$\lim_{(n,\lambda)\to\infty} \mathcal{R}^{n,\lambda} = \lim_{(n,\lambda)\to\infty} \mathbb{LK} \int_0^\infty \left({}^n\overline{S}_u^* - {}^n\underline{S}_u^* \right) e^{-\lambda u} \lambda du + \frac{\rho \|h\|_\infty}{\lambda} = 0.$$

9.3. **Law error $\mathcal{L}^n$.** We define the law error $\mathcal{L}^n$ by

$$\mathcal{L}^n = \left| v^{n,\infty}(t,x) - v^{\infty,\infty}(t,x) \right|. \tag{9.5}$$

Since, the laws of ξ^n on the Skorokhod converge weakly to the law of ξ, it follows immediately that

$$\lim_{n\to\infty} \mathcal{L}^n = 0 \tag{9.6}$$

see for instance Coquet and Toldo [50].

9.4. **Discretization error $\mathcal{D}^{n,\lambda}$.** We define the discretization error as

$$\mathcal{D}^{n,\lambda} = \max_{i=0,\dots,n-1} \sup_{x\geq 0} \left| v_{h,T}^{n,\lambda}(t_i,x) - \widehat{v}_{h,T}^{n,\lambda}(t_i,x) \right| \tag{9.7}$$

and, in this section, we show that

$$\lim_{n\to\infty} \mathcal{D}^{n,n} = 0. \tag{9.8}$$

In order to shorten the expressions, we will use the following notation, for functions f and g, and numbers α, β:

$$\begin{aligned} \left(\alpha f - \beta h\right)(t,x) &= \alpha f(t,x) - \beta h(t,x) \\ \left|\alpha f - \beta h\right|(t,x) &= \left|\alpha f(t,x) - \beta h(t,x)\right|. \end{aligned}$$

Take some $t = i\frac{T}{n} < T$, $i = 0, ..., n-1$, and write

$$\begin{aligned} v_{h,T}^{n,\lambda}(t,x) - \widehat{v}_{h,T}^{n,\lambda}(t,x) &= a_{h,T}^{n,\lambda}(t,x) - \widehat{a}_{h,T}^{n,\lambda}(t,x) \\ &= {}_{\frac{T}{n}}|a_{h,T}^{n,\lambda}(t,x) - {}_{\frac{T}{n}}|\widehat{a}_{h,T}^{n,\lambda}(t,x) \\ &+ a_{h,T}^{\lambda,n}(t,x)_{\overline{\frac{T}{n}|}} - \widehat{a}_{h,T}^{\lambda,n}(t,x)_{\overline{\frac{T}{n}|}}. \end{aligned} \tag{9.9}$$

Assume that during its whole trajectory in between t and $t+\frac{T}{n}$, process $v_{h,T}^{n,\lambda}(s,\xi_s^n) = v_{h,T}^{n,\lambda}(s,x)$ doesn't cross the value $\widehat{v}_{h,T}^{n,\lambda}(t,x)$. Then the quantities $v_{h,T}^{n,\lambda}(t,x) - \widehat{v}_{h,T}^{n,\lambda}(t,x)$ and $a_{h,T}^{\lambda,n}(t,x)_{\overline{\frac{T}{n}|}} - \widehat{a}_{h,T}^{\lambda,n}(t,x)_{\overline{\frac{T}{n}|}}$ have opposite signs. If indeed, to fix the ideas, $v_{h,T}^{n,\lambda}(t,x) - \widehat{v}_{h,T}^{n,\lambda}(t,x) > 0$ and $v_{h,T}^{n,\lambda}(s,x) > \widehat{v}_{h,T}^{n,\lambda}(t,x)$ for every s in $[t, t+\frac{T}{n})$ then

$$\max\left(h(x) - v_{h,T}^{n,\lambda}(s,x), 0\right) < \max\left(h(x) - \widehat{v}_{h,T}^{n,\lambda}(t,x), 0\right)$$

for every s in $[t, t+\frac{T}{n})$ form which

$$a_{h,T}^{\lambda,n}(t,x)_{\overline{\frac{T}{n}|}} < \widehat{a}_{h,T}^{\lambda,n}(t,x)_{\overline{\frac{T}{n}|}}$$

and the claim follows. We conclude that in this case

$$\left|v_{h,T}^{n,\lambda} - \widehat{v}_{h,T}^{n,\lambda}\right|(t,x) \le \left|{}_{\frac{T}{n}}|a_{h,T}^{n,\lambda} - {}_{\frac{T}{n}}|\widehat{a}_{h,T}^{n,\lambda}\right|(t,x).$$

On the other hand suppose now that, indeed, $v_{h,T}^{n,\lambda}(s,\xi_s) = v_{h,T}^{n,\lambda}(s,x)$ crosses $\widehat{v}_{h,T}^{n,\lambda}(t,x)$ for some s in $[t, t+\frac{T}{n})$. That means that $v_{h,T}^{n,\lambda}(t,x)$ is within one modulus of continuity $\omega^{n,\lambda}$ of $\widehat{v}_{h,T}^{n,\lambda}(t,x)$, in other words,

$$\left|v_{h,T}^{n,\lambda}(t,x) - \widehat{v}_{h,T}^{n,\lambda}(t,x)\right| \le \omega^{n,\lambda}.$$

Therefore

$$\left|v_{h,T}^{n,\lambda} - \widehat{v}_{h,T}^{n,\lambda}\right|(t,x) \le \max(\left|{}_{\frac{T}{n}}|a_{h,T}^{n,\lambda} - {}_{\frac{T}{n}}|\widehat{a}_{h,T}^{n,\lambda}\right|(t,x), \omega^{n,\lambda}). \tag{9.10}$$

Now

$$\begin{aligned}
{}_{\frac{T}{n}}|a_{h,T}^{n,\lambda}(t,x) &= e^{-\rho\frac{T}{n}}\left(a_{h,T}^{n,\lambda}\cdot p_u + a_{h,T}^{n,\lambda}\cdot p_d\right)(t+\frac{T}{n}, xu)\\
{}_{\frac{T}{n}}|\widehat{a}_{h,T}^{n,\lambda}(t,x) &= e^{-\rho\frac{T}{n}}\left(\widehat{a}_{h,T}^{n,\lambda}\cdot p_u + \widehat{a}_{h,T}^{n,\lambda}\cdot p_d\right)(t+\frac{T}{n}, xu)
\end{aligned}$$

hence

$$\begin{aligned}
\left({}_{\frac{T}{n}}|a_{h,T}^{n,\lambda} - {}_{\frac{T}{n}}|\widehat{a}_{h,T}^{n,\lambda}\right)(t,x) = {}& e^{-\rho\frac{T}{n}}p_u\left(a_{h,T}^{n,\lambda} - \widehat{a}_{h,T}^{n,\lambda}\right)(t+\frac{T}{n}, xu)\\
&+ e^{-\rho\frac{T}{n}}p_d\left(a_{h,T}^{n,\lambda} - \widehat{a}_{h,T}^{n,\lambda}\right)(t+\frac{T}{n}, xd)
\end{aligned}$$

and, from (9.9) we get, taking the absolute value,

$$\begin{aligned}
\left|{}_{\frac{T}{n}}|a_{h,T}^{n,\lambda} - {}_{\frac{T}{n}}|\widehat{a}_{h,T}^{n,\lambda}\right|(t,x) \le {}& e^{-\rho\frac{T}{n}}p_u\left|v_{h,T}^{n,\lambda} - \widehat{v}_{h,T}^{n,\lambda}\right|(t+\frac{T}{n}, xu)\\
&+ e^{-\rho\frac{T}{n}}p_d\left|v_{h,T}^{n,\lambda} - \widehat{v}_{h,T}^{n,\lambda}\right|(t+\frac{T}{n}, xd).
\end{aligned} \tag{9.11}$$

What (9.10) and (9.11) are saying is that if $\left|v_{h,T}^{n,\lambda} - \widehat{v}_{h,T}^{n,\lambda}\right|(t,x)$ is not bounded by $\omega^{n,\lambda}$, it is then bounded by the expected present value of terms which are themselves in the form $\left|v_{h,T}^{n,\lambda} - \widehat{v}_{h,T}^{n,\lambda}\right|$, but with a time to maturity shortened by $\frac{T}{n}$. Each of these terms can be worked out recursively in the same manner and, ultimately, the terms have to be bounded by $\omega^{n,\lambda}$ since, if maturity is reached, that is if $t+\frac{T}{n} = T$ in (9.11), then the right hand side of (9.11) is zero. Hence,

$$\left|v_{h,T}^{n,n}(t,x) - \widehat{v}_{h,T}^{n,n}(t,x)\right| \le \mathcal{D}^{n,n} \le \omega^{n,n}$$

and (9.8) follows.

9.5. **Convergence.** Obviously, the error of estimating the value $v_{h,T}^{\infty,\infty}$ of the American option by $\widehat{v}_{h,T}^{n,n}$ can be broken down into three parts: a law error resulting from approximating $v_{h,T}^{\infty,\infty}$ by $v_{h,T}^{n,\infty}$, a randomization error coming from approximating $v_{h,T}^{n,\infty}$ by $v_{h,T}^{n,n}$ and a discretization error resulting from approximating $v_{h,T}^{n,n}$ by $\widehat{v}_{h,T}^{n,n}$. Recalling (9.7), (9.2) and (9.5), we get

$$\left|\widehat{v}_{h,T}^{n,n}(t,x) - v_{h,T}(t,x)\right| \leq \mathcal{D}^{n,n} + \mathcal{R}^{n,n} + \mathcal{L}^{n}$$

and thus from (9.8), (9.3) and (9.6) we have

$$\lim_{n\to\infty} \widehat{v}_{h,T}^{n,n}(t,x) = v_{h,T}(t,x)$$

as wanted.

10. Numerical results

The efficiency of the method described in section 8 is illustrated here for the put option

$$h(x) = \max(100 - x, 0)$$

with

$$\begin{aligned} T &= 1 \\ \rho &= 0.1 \\ \sigma &= 0.15 \\ \xi_0 &= 100. \end{aligned}$$

We define the error ε^n by

$$\varepsilon^n := v_{h,T}^{\infty,\infty}(0,100) - \widehat{v}_{h,T}^{n,n}(0,100)$$

based on the value

$$v_{h,T}^{\infty,\infty}(0,100) = 3.150699687$$

which we calculated with the classical binomial asset pricing model value with $n = 15000$ time steps.

Number of steps n	Error ε^n
$25 \leq n \leq 100$	$0.022340715 \leq \varepsilon^n \leq 0.110579023$
$100 < n \leq 250$	$0.008185433 \leq \varepsilon^n \leq 0.028444563$
$250 < n \leq 500$	$0.004122794 \leq \varepsilon^n \leq 0.011175892$
$500 < n \leq 750$	$0.002773273 \leq \varepsilon^n \leq 0.005573383$
$750 < n \leq 1000$	$0.002090356 \leq \varepsilon^n \leq 0.003734639$
$1000 < n \leq 2500$	$0.000798636 \leq \varepsilon^n \leq 0.002776534$
$2500 < n \leq 5000$	$0.000379865 \leq \varepsilon^n \leq 0.001073829$
$5000 < n \leq 7500$	$0.000237743 \leq \varepsilon^n \leq 0.00051667$
$7500 < n \leq 10000$	$0.000165286 \leq \varepsilon^n \leq 0.000328836$
$10000 < n$	$\varepsilon^n \leq 0.000233548$

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