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# Folding and unfolding in periodic difference equations

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**Abstract**

Given a $p$-periodic difference equation $x_{n+1} = f_{n \bmod p}(x_n)$, where each $f_j$ is a continuous interval map, $j = 0, 1, \ldots, p-1$, we discuss the notion of folding and unfolding related to this type of non-autonomous equations. It is possible to glue certain maps of this equation to shorten its period, which we call folding. On the other hand, we can unfold the glued maps so the original structure can be recovered or understood. Here, we focus on the periodic structure under the effect of folding and unfolding. In particular, we analyze the relationship between the periods of periodic sequences of the $p$-periodic difference equation and the periods of the corresponding subsequences related to the folded systems.

**Keywords:** Non-autonomous difference equations, alternating systems, interval maps, periodic solutions, periods, cycles, folding, unfolding.
**Mathematics Subject Classification (2010):** 39A23; 37E05; 37E15

# 1 Introduction

Let $I = [a, b] \subset \mathbb{R}$ be a closed interval with $-\infty < a < b < \infty$, and denote by $\mathcal{C}(I)$ the space of continuous maps $f : I \to I$. Given a $p$-periodic sequence $\{f_n\}_{n\geq 0} \subset \mathcal{C}(I)$,

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that is, $f_{n+p} = f_n$ for all non-negative integers $n$, we consider the $p$-periodic difference equation

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{N} := \{0, 1, \ldots\}. \tag{1.1}$$

It is worth emphasizing here that we use period to denote the minimal period, unless mentioned otherwise. To stress the role of the maps involved in this $p$-periodic difference equation, we use the representation

$$[f_0, f_1, ..., f_{p-1}].$$

If $p = 1$, then the equation is autonomous and we simply denote it by $f_0$, that is, the alternated system is reduced to the classical discrete system $x_{n+1} = f_0(x_n)$, $n \geq 0$. Periodic difference equations of the form given in Eq. (1.1) appears in a natural way in technical and social sciences related to processes involving two or more interactions. For the knowledge of the behaviour of the general system, it is necessary to alternate different discrete dynamical systems corresponding to each period of the process. In this sense, it is interesting to stress that Eq. (1.1) can model certain populations in a periodically fluctuating environment [13, 10, 8, 17, 11]. For $p = 2$, we also find applications related to Physics [1, 14] and Economy [15, 16] in the context of Parrondo's paradox [12].

For an initial condition $x_0 \in I$, the solution or orbit through $x_0$ is given by

$$\begin{aligned} \mathcal{O}^+(x_0) :=& \{x_0, x_1, x_2, \ldots\} \\ =& \{x_0, f_0(x_0), f_1(f_0(x_0)), f_2(f_1(f_0(x_0))), \ldots\}. \end{aligned} \tag{1.2}$$

Characterizing periodic solutions of Eq. (1.1) has been a topic of growing interest in the past decade [2, 3, 5, 7, 9]. An orbit $\mathcal{O}^+(x_0)$ is called $r$-cycle if $r$ is the smallest positive integer for which $x_{n+r} = x_n$ for all $n \in \mathbb{N} := \{0, 1, 2, \ldots\}$. Notice that we use $r$-cycle rather than "$r$-periodic solution" to distinguish between talking about the periodicity of the system and periodicity of solutions. We also say $r$ is the period or order of $\mathcal{O}^+(x_0) = (x_n)$, which can be denoted by $\text{ord}_{[f_0,\ldots,f_{p-1}]}(x_0)$. By $\text{P}([f_0, ..., f_{p-1}])$ and $\text{Per}([f_0, \ldots, f_{p-1}])$ we denote the sets of periodic points and periods of $[f_0, \ldots, f_{p-1}]$, respectively. Note that in discrete autonomous systems if $x_0 \in \text{P}([f_0])$, then $x_n \in \text{P}([f_0])$ for all $n$, while in periodic nonautonomous systems systems [3], if $x_0 \in \text{P}([f_0, ..., f_{p-1}])$ then $x_{nd} \in \text{P}([f_0, ..., f_{p-1}])$, where $d$ is the greatest common divisor between $p$ and the order of $(x_n)$. Throughout this paper, we use $\gcd(p, q)$ and $\text{lcm}\,(p, q)$ to denote the greatest common divisor and least common multiple between $p$ and $q$, respectively. Eq. (1.1) can be of minimal period $p$ on the interval $I$, but reduces to an equation of shorter period on a nontrivial subinterval of $I$. In such case, it is possible to treat Eq. (1.1) based on the new shorter period and the partitioned domain. We refer the reader to [2] for more information about this scenario. However, we consider this scenario to be a degenerate one and avoid it throughout this paper.

In this paper, we focus on the notion of folding certain maps of Eq. (1.1) to shorten its period, while the unfolding is used to denote the reversed process. For instance, suppose that we have a 6-periodic system $[f_0, f_1, \dots, f_5]$. We can define the map $F := f_5 \circ f_4 \circ \dots \circ f_0$, then deal with the autonomous equation $x_{n+1} = F(x_n)$. Define the maps $F_0 := f_1 \circ f_0$, $F_1 := f_3 \circ f_2$, $F_2 := f_5 \circ f_4$, then deal with the periodic alternating system $[F_0, F_1, F_2]$. Or we can define the maps $F_0 := f_2 \circ f_1 \circ f_0$ and $F_1 := f_5 \circ f_4 \circ f_3$, then deal with the periodic alternating system $[F_0, F_1]$. One of the main objectives here is to characterize the periodic structures in all these possible scenarios. The notion of folding and unfolding was introduced by AlSalman and AlSharawi in [2]. However, we feel it has not blossomed yet; and therefore, we write this paper to develop further results that will be used by the authors in [6] to characterize forcing between cycles. It is worth mentioning here that the results of this paper are mostly based on the combinatorial structure of orbits, which do not need continuity. However, we assumed continuity of maps to keep the sittings within our long term goal which is the characterization of forcing between cycles. In Section 2, we discuss the notion of folding and its effect on periodic solutions. Given a $r$-cycle of Eq. (1.1), our main goal is to obtain information on the periods of the subsequences obtained by folding the initial system. Finally, the notion of unfolding and its concerning results are provided in Section 3. Here, from a $q$-cycle of a folded system we obtain information on the possible period of the corresponding unfolded cycle.

## 2 Folding in periodic difference equations

Consider the $p$-periodic equation in Eq. (1.1), and let $k$ be a positive integer. For $j = 0, 1, \dots,$ define the maps

$$F_j^{(k)} := f_{(jk+k-1) \bmod p} \circ f_{(jk+k-2) \bmod p} \circ \cdots \circ f_{(jk+1) \bmod p} \circ f_{jk \bmod p}. \tag{2.1}$$

We simply write $F_j$ when no confusion can arise with the value $k$. We obtain a periodic equation of the form $x_{n+1} = F_n^{(k)}(x_n)$ with a period that divides $\frac{p}{\gcd(p,k)}$ [2]. Here, we are more interested in the case in which $k$ is a divisor of $p$. Thus, the obtained equation has a period shorter than $p$, which can be more convenient to deal with in certain cases. For instance, if $k = p$, then we obtain the autonomous equation $x_{n+1} = F_0(x_n)$, with $F_0 = f_{p-1} \circ f_{p-2} \circ \cdots \circ f_0$. Thus, a $p$-cycle of Eq. (1.1) is a fixed point of $F_0$, and a $pr$-cycle of Eq. (1.1) is an $r$-cycle of $F_0$. We formalize this straightforward observation in the following result.

**Proposition 2.1.** *Let $k = p$ in Eq. (2.1). If $(x_n)$ is a $pr$–cycle of Eq. (1.1), then $(x_{kn})$ is an $r$–cycle of the map $F_0$ in Eq. (2.1).*

However, the case is not obvious when it comes to cycles of Eq. (1.1) with periods that are not multiples of $p$. We give the following example:

**Example 2.1.** *Consider the* 6*-periodic equation* $x_{n+1} = f_n(x_n)$*, where*

$$f_0(x) = x + 1, \quad f_2(x) = f_0(x) + (x-1)(x-3), \quad f_4(x) = f_0(x) - (x-1)(x-3),$$

$$f_1(x) = -x + 5, \quad f_3(x) = f_1(x) + (x-2)(x-4), \quad f_5(x) = f_1(x) - (x-2)(x-4).$$

*Then, it is straightforward to check that* $C_4 = \{1, 2, 3, 4\}$ *is a* 4*-cycle for the* 6*-periodic system.*

*(i) If* $k = 2$*, then* $F_0(x) = f_1(f_0(x))$*,* $F_1(x) = f_3(f_2(x))$ *and* $F_2(x) = f_5(f_4(x))$*. Thus,* $\widehat{C}_2 := \{1, 3\}$ *is a* 2*-cycle for the* 3*-periodic system* $x_{n+1} = F_n(x_n)$*.*

*(ii) If* $k = 3$*, then* $F_0(x) = f_2(f_1(f_0(x)))$ *and* $F_1(x) = f_5(f_4(f_3(x)))$*. Thus,* $\widehat{C}_4 := \{1, 4, 3, 2\}$ *is a* 4*-cycle for the* 2*-periodic system* $x_{n+1} = F_n(x_n)$*.*

**Remark 2.1.** *For simplicity, we are using polynomials defined on* $\mathbb{R}$ *in the concrete examples that we are giving. However, we can construct similar examples on compact intervals using the algorithm given in [2].*

Observe that the maps $F_j$ as defined in (2.1) are structured based on the fact that the starting time is $n = 0$; however, we can start at $n = m$ and define the maps

$$F_{j,m}^{(k)} := f_{((j+1)k-1+m) \bmod p} \circ f_{((j+1)k-2+m) \bmod p} \circ \cdots \circ f_{(jk+m) \bmod p}, \; j \geq 0. \tag{2.2}$$

Notice that $F_j^{(k)} = F_{j,0}^{(k)}$. When no confusion is possible with the choice of $k$, we will write $F_{j,m}^{(k)} = F_j^m$. In this way, Eq. (2.2) gives the periodic equations $x_{n+1} = F_n^m(x_n), m = 0, 1, \ldots, k-1$, of periods $p_m$ dividing $\frac{p}{k}$.

Next, define the set of positive integers

$$\mathcal{A}_{p,q} = \{m \in \mathbb{Z}^+ : \; \operatorname{lcm}(m, p) = p \cdot q\}. \tag{2.3}$$

We recall the following useful properties for positive integers $a$, $b$ and $c$:

$$a \cdot b = \operatorname{lcm}(a, b) \cdot \gcd(a, b), \tag{2.4}$$

$$\operatorname{lcm}(a, \operatorname{lcm}(b, c)) = \operatorname{lcm}(\operatorname{lcm}(a, b), c). \tag{2.5}$$

We write $a|b$ to denote that $a$ is a divisor of $b$. If $a|c$, we have

$$a \cdot \operatorname{lcm}(b, \frac{c}{a}) = \operatorname{lcm}(a \cdot b, c). \tag{2.6}$$

Thus, Eq. (2.4) allows us to rewrite

$$\mathcal{A}_{p,q} = \{m \in \mathbb{Z}^+ : \; m = \gcd(m, p) \cdot q\}. \tag{2.7}$$

It is worth mentioning that $\mathbb{Z}^+ = \cup_q \mathcal{A}_{p,q}$, and those clusters of positive integers were used in [2, 5] to characterize the forcing between periodic solutions of Eq. (1.1). The structure of the clusters $\mathcal{A}_{p,q}$ has been clarified in [5, 3]. However, the following proposition describes an arithmetical procedure for finding the elements of $\mathcal{A}_{p,q}$.

**Proposition 2.2.** *Let $p, q$, be positive integers.*

*(i) If $\gcd(p,q) > 1$ and $\{a_1, \ldots, a_n\}$ is precisely the set of primes appearing in both the decompositions of $p = a_1^{\alpha_1} \ldots a_n^{\alpha_n} b_1^{\beta_1} \ldots b_t^{\beta_t}$ and $q = a_1^{\delta_1} \ldots a_n^{\delta_n} c_1^{\gamma_1} \ldots c_\ell^{\gamma_\ell}$, then*

$$\begin{aligned} \mathcal{A}_{p,q} &= a_1^{\alpha_1+\delta_1} \ldots a_n^{\alpha_n+\delta_n} \cdot c_1^{\gamma_1} \ldots c_\ell^{\gamma_\ell} \cdot \{\textit{divisors of } b_1^{\beta_1} \ldots b_t^{\beta_t}\} \\ &= q \cdot a_1^{\alpha_1} \ldots a_n^{\alpha_n} \cdot \{\textit{divisors of } b_1^{\beta_1} \ldots b_t^{\beta_t}\} \end{aligned}$$

*(ii) If $\gcd(p,q) = 1$, and $p = b_1^{\beta_1} \ldots b_t^{\beta_t}$, $q = c_1^{\gamma_1} \ldots c_\ell^{\gamma_\ell}$, then*

$$\mathcal{A}_{p,q} = c_1^{\gamma_1} \ldots c_\ell^{\gamma_\ell} \cdot \{\textit{divisors of } b_1^{\beta_1} \ldots b_t^{\beta_t}\} = q \cdot \{\textit{divisors of } p\}.$$

The next proposition clarifies further the structure of $\mathcal{A}_{p,q}$.

**Proposition 2.3.** *Let $k$ be a positive integer. Each of the following holds true:*

*(i) $p \in \mathcal{A}_{k, \frac{p}{\gcd(k,p)}}$.*

*(ii) If $r_1 \neq r_2$, then $\mathcal{A}_{k,r_1}$ and $\mathcal{A}_{k,r_2}$ are disjoint.*

*(iii) Let $k|p$. $r \in \mathcal{A}_{p,q}$ if and only if $\frac{r}{\gcd(r,k)} \in \mathcal{A}_{\frac{p}{k},q}$.*

*(iv) Let $k|p$, and fix a positive integer $r$. Then*

$$\frac{\gcd(r,p)}{\gcd(r,k)} = \alpha$$

*is an integer given by*

$$\alpha = \gcd\left(\frac{p}{k}, \frac{r}{\gcd(r,k)}\right).$$

*Proof.* The first two properties are immediate consequences of the definition of $\mathcal{A}_{p,q}$. To prove Part (iii), notice that $\frac{r}{\gcd(r,k)} = \frac{\operatorname{lcm}(r,k)}{k}$ by Eq. (2.4), and by Eq. (2.6), we have

$$\begin{aligned} \frac{\operatorname{lcm}(\frac{r}{\gcd(r,k)}, \frac{p}{k})}{\frac{p}{k}} &= k\frac{\operatorname{lcm}(\frac{r}{\gcd(r,k)}, \frac{p}{k})}{p} = \frac{\operatorname{lcm}(\frac{rk}{\gcd(r,k)}, p)}{p} = \frac{\operatorname{lcm}(\operatorname{lcm}(r,k), p)}{p} \\ &= \frac{\operatorname{lcm}(r,p)}{p}, \end{aligned}$$

where we used Eq. (2.5) in the last equality. Finally, the definition of the sets $\mathcal{A}_{p,q}$ ends the proof of Part (iii). Next, we prove Part (iv). From Part (i), we have $r \in \mathcal{A}_{p,\frac{r}{\gcd(p,r)}}$. From Part (iii), we obtain $\frac{r}{\gcd(r,k)} \in \mathcal{A}_{\frac{p}{k},\frac{r}{\gcd(r,p)}}$, that is,

$$\frac{\frac{r}{\gcd(r,k)}}{\gcd(\frac{p}{k},\frac{r}{\gcd(r,k)})} = \frac{r}{\gcd(r,p)},$$

which simplifies to

$$\frac{\gcd(r,p)}{\gcd(r,k)} = \gcd(\frac{p}{k},\frac{r}{\gcd(r,k)}) =: \alpha. \tag{2.8}$$

□

Next, it is known in [5] that if we have an $r$-periodic sequence $(x_n)$ in the alternated system $[f_0, f_1, \ldots, f_{p-1}]$, then the starting point $x_0$ is a periodic point of the composition $f_{p-1} \circ \ldots \circ f_0$ of (minimal) period $\frac{r}{\gcd(r,p)}$. However, we extract the result here and provide an alternative proof.

**Lemma 2.1.** *Assume that $(x_n)_{n\geq 0}$ is an $r-$cycle of $[f_0, \ldots, f_{p-1}]$. Then $x_0 \in \mathrm{P}(f_{p-1} \circ \ldots \circ f_0)$ and $\mathrm{ord}_{f_{p-1}\circ\ldots\circ f_0}(x_0) = \frac{r}{\gcd(r,p)}$. Moreover, $x_j = (f_{j-1} \circ \ldots \circ f_0)(x_0)$ is a periodic point of $f_{j-1}f_{j-2} \circ \ldots \circ f_0 \circ f_{p-1} \ldots \circ f_j$ and $\mathrm{ord}_{f_{j-1}\circ\ldots\circ f_0 \circ f_{p-1}\ldots\circ f_j}(x_j) = \frac{r}{\gcd(r,p)}$ for all $j = 1, \ldots, p-1$.*

*Proof.* Since $x_0 = x_r = x_{r\frac{p}{\gcd(r,p)}}$ and $x_{r\frac{p}{\gcd(r,p)}} = (f_{p-1} \circ \ldots \circ f_0)^{\frac{r}{\gcd(r,p)}}(x_0)$, we deduce that $x_0$ is a periodic point of $f_{p-1} \circ \ldots \circ f_0$ and its order $t$ divides $\frac{r}{\gcd(r,p)}$. So $tp|\frac{rp}{\gcd(r,p)}$, or equivalently, $tp|\mathrm{lcm}(r,p)$. On the other hand, $x_{tp} = (f_{p-1} \circ \ldots \circ f_0)^t(x_0) = x_0$, $x_{tp+1} = f_0(x_0) = x_1, x_{tp+2} = f_1(x_1) = x_2, \ldots$, which implies $r|tp$ since $(x_n)$ has (minimal) period $r$. So, we find that $tp$ is a common multiple of $r$ and $p$. Hence, $\mathrm{lcm}(r,p)|tp$. Therefore, $tp = \mathrm{lcm}(r,p)$, and the proof is complete for the point $x_0$. A similar reasoning gives that the points $x_1, \ldots, x_{p-1}$ are periodic with the same period $\frac{r}{\gcd(r,p)}$ for the map $f_{p-1} \circ \ldots \circ f_1 \circ f_0$. □

Suppose that the $p$-periodic alternated system $[f_0, f_1, \ldots, f_{p-1}]$ has an $r$-cycle for some $r \in \mathcal{A}_{p,q}$. We are interested in knowing the elements of $\mathcal{A}_{\frac{p}{k},q}$ that are obtained as periods of the folded system $[F_0, F_1, \ldots, F_{\frac{p}{k}-1}]$. The rest of this section is devoted toward this task.

**Theorem 2.1.** *Let $k$ be a divisor of $p$ and define the maps $F_j$ as in Eq. (2.1). Let $(x_n)_{n\geq 0}$ be an $r$-cycle of the alternated system $[f_0, f_1, \ldots, f_{p-1}]$. Then $(x_{km})_{m\geq 0}$ is an $r^*$-cycle of $[F_0, F_1, \ldots, F_{\frac{p}{k}-1}]$ for some $r^* \in \mathcal{A}_{\frac{p}{k},q}$, with $q = \frac{r}{\gcd(r,p)}$. Moreover, $\frac{r}{\gcd(r,p)} \leq r^* \leq \frac{r}{\gcd(r,k)}$, with $r^*|\frac{r}{\gcd(r,k)}$ and $\frac{r}{\gcd(r,p)}|r^*$. Furthermore, if we define the maps $F_j^m$, $j = 0, 1, ..., k-1$, as in (2.2), then each of the periodic equations $x_{n+1} = F_n^m(x_n)$ has a cycle of period $r_j^*$ for some $r_j^*$ that satisfies the same conditions as $r^*$.*[1]

[1]Note that $r_0^*$ is $r^*$.

*Proof.* Since $k|p$ and $(x_n)$ is $r$-periodic, taking into account

$$x_{kj} = x_{r+kj} = x_{r\frac{k}{\gcd(r,k)}+kj} = x_{k(\frac{r}{\gcd(r,k)}+j)},$$

we find that $(x_{km})_{m\geq 0}$ is periodic for $[F_0, F_1, \ldots, F_{\frac{p}{k}-1}]$ and its period $r^*$ is a divisor of $\frac{r}{\gcd(r,k)}$. Even more, since $x_0 = x_{kr^*} = x_{\frac{p}{k}kr^*} = x_{pr^*} = (f_{p-1} \circ \ldots \circ f_0)^{r^*}(x_0)$, Lemma 2.1 yields $\frac{r}{\gcd(r,p)}|r^*$. To prove that $r^* \in \mathcal{A}_{\frac{p}{k},q}$, use Lemma 2.1 to obtain

$$\mathrm{ord}_{F_{\frac{p}{k}-1}\circ\ldots\circ F_0}(x_0) = \frac{r^*}{\gcd(r^*, \frac{p}{k})},$$

and realize that $\mathrm{ord}_{F_{\frac{p}{k}-1}\circ\ldots\circ F_0}(x_0) = \mathrm{ord}_{f_{p-1}\circ\ldots\circ f_0}(x_0) = \frac{r}{\gcd(r,p)} = q$. Thus, $\frac{r^*}{\gcd(r^*,\frac{p}{k})} = q$, that is, $r^* \in \mathcal{A}_{\frac{p}{k},q}$ according to Eq. (2.7). The same type of argument applies to the equations $x_{n+1} = F_n^j(x_n)$ and the sequences $(x_{kn+j})_n$, with $j = 1, \ldots, k-1$. $\square$

Another way to state the above result is as follows: denote

$$r_{\max} = \frac{r}{\gcd(r,k)},$$

and write $div(r_{\max}) = \{m \in \mathbb{N} : m \text{ divides } r_{\max}\}$ to denote the set of divisors of $r_{\max}$, then define

$$div^*(r_{\max}) := \left\{ m \in div(r_{\max}) : m \geq \frac{r}{\gcd(r,p)} \right\}.$$

Now, Theorem 2.1 establishes that if $k$ divides $p$ and $(x_n)$ is an $r$–cycle for $x_{n+1} = f_n(x_n)$, then for $j = 0, 1, ..., k-1$, the subsequence $(x_{j+kn})$ is an $r^*$-cycle for some

$$r^* \in div^*(r_{\max}) \cap \mathcal{A}_{\frac{p}{k},q},$$

where $q = \frac{r}{\gcd(r,p)}$. Additionally, suppose that $s$ is an element of $div^*(r_{\max})$ such that

$$\frac{s}{\gcd(s, \frac{p}{k})} \geq \frac{r}{\gcd(r,p)}. \tag{2.9}$$

We prove that all elements of $div^*(r_{\max})$ holding this condition belong to $\mathcal{A}_{\frac{p}{k},q}$.

**Proposition 2.4.** *If $s \in div^*(r_{\max})$ satisfies Condition (2.9), then $s \in \mathcal{A}_{\frac{p}{k},q}$, where $q = \frac{r}{\gcd(r,p)}$.*

*Proof.* First, note that $s \in \mathcal{A}_{\frac{p}{k},\frac{r}{\gcd(r,p)}}$ iff $\mathrm{lcm}(s, \frac{p}{k}) = \frac{p}{k}\frac{r}{\gcd(r,p)}$, and since

$$\frac{p}{k}\frac{r}{\gcd(r,p)} = \frac{\mathrm{lcm}(r,p)}{k},$$

we obtain $s \in \mathcal{A}_{\frac{p}{k},\frac{r}{\gcd(r,p)}}$ iff $k\mathrm{lcm}(s, \frac{p}{k}) = \mathrm{lcm}(r, p)$ iff $\mathrm{lcm}(sk, p) = \mathrm{lcm}(r, p)$. Next, let $r^* \in div^*(r_{\max})$ be such that $\frac{r^*}{\gcd(r^*,\frac{p}{k})} \geq \frac{r}{\gcd(r,p)}$. Then

$$\frac{\frac{p}{k}r^*}{\gcd(r^*, \frac{p}{k})} \geq \frac{pr}{k\gcd(r,p)} = \frac{\mathrm{lcm}(r,p)}{k}.$$

Since

$$\frac{\frac{p}{k}r^*}{\gcd(r^*, \frac{p}{k})} = \mathrm{lcm}(r^*, \frac{p}{k}),$$

we conclude that $k\mathrm{lcm}(r^*, \frac{p}{k}) \geq \mathrm{lcm}(r, p)$, that is, $\mathrm{lcm}(r^*k, p) \geq \mathrm{lcm}(r, p)$ according to Eq. (2.6). Note that $r^*$ divides $r_{\max}$ and $r_{\max}$ divides $r$, which implies that $r^*$ divides $r$ and $r^*k|\frac{rk}{\gcd(r,k)} = \mathrm{lcm}(r, k)$.
Finally, by the fact that $\mathrm{lcm}(r, k)$ divides $\mathrm{lcm}(r, p)$ and that $\mathrm{lcm}(r, p)$ is a common multiple of $r^*k$ and $p$, we deduce that $\mathrm{lcm}(r^*k, p) \leq \mathrm{lcm}(r, p)$. Therefore, $\mathrm{lcm}(r^*k, p) = \mathrm{lcm}(r, p)$, and consequently, $r^* \in \mathcal{A}_{\frac{p}{k},q}$ as we desire, where $q = \frac{r}{\gcd(r,p)}$. □

**Remark 2.2.** *From the above result we deduce that the set $div^*(r_{\max})$ can be partitioned in two classes. If $s \in div^*(r_{\max})$ then either $\frac{s}{\gcd(s,p/k)} = \frac{r}{\gcd(r,p)}$, and consequently $s \in \mathcal{A}_{\frac{p}{k},\frac{r}{\gcd(r,p)}}$, or $\frac{s}{\gcd(s,p/k)} < \frac{r}{\gcd(r,p)}$. The latter case implies that $s$ is not included in $\mathcal{A}_{\frac{p}{k},\frac{r}{\gcd(r,p)}}$. Example 2.2 shows that both possibilities can occur. Although the sets $div^*(r_{\max})$ and $\mathcal{A}_{\frac{p}{k},\frac{r}{\gcd(r,p)}}$ are not equal in general, according to Proposition 2.3(iii) at least their intersection always contains the number $r_{\max}$.*

Next, we show some examples of $p$-periodic systems $[f_0, f_1, ...f_{p-1}]$ having an $r$-cycle, and the possible periods of subsequences after the folding process.

**Example 2.2.** *Consider $p = 144 = 2^4 \cdot 3^2$, $k = 6 = 2 \cdot 3$ and $r = 270 = 2 \cdot 3^3 \cdot 5$. Then $270 \in \mathcal{A}_{144,15} = 15 \cdot 3^2 \cdot \{1, 2, 4, 8, 16\}$ while $p/k = 24 = 2^3 \cdot 3$,*

$$r_{\max} = \frac{r}{\gcd(r,k)} = \frac{2 \cdot 3^3 \cdot 5}{\gcd(2 \cdot 3^3 \cdot 5, 2 \cdot 3)} = 45,$$

$$\frac{r}{\gcd(r,p)} = \frac{2 \cdot 3^3 \cdot 5}{\gcd(2 \cdot 3^3 \cdot 5, 2^4 \cdot 3^2)} = 15$$

*and $\mathcal{A}_{24,15} = 15 \cdot 3 \cdot \{1, 2, 4, 8\}$. Since $div^*(r_{\max}) = div^*(45) = \{15, 45\}$, we have only two possibilities. But because $15 \notin \mathcal{A}_{24,15}$, then the only possible period is 45. Thus, a 270-cycle of the p-periodic system $[f_0, \ldots, f_{p-1}]$ gives a 45-cycle for the folded system $[F_0, \ldots, F_{\frac{p}{k}-1}]$.*

**Example 2.3.** *Consider $p = 18$, $k = 6$ and $r = 20$. Here*

$$\frac{r}{\gcd(r,p)} = \frac{20}{\gcd(20,18)} = 10$$

*and*

$$r_{\max} = \frac{r}{\gcd(r,k)} = \frac{20}{\gcd(20,6)} = 10.$$

*By Theorem 2.1, $r_{\max}$ is the period of the corresponding folded sequence. Note that $\mathcal{A}_{3,10} = \{10, 30\}$ and div$^*(10) = \{10\}$; so div$^*(r_{\max})$ and $\mathcal{A}_{\frac{p}{k}, \frac{r}{\gcd(r,p)}}$ are not equal.* □

An interesting question remains even unsolved: in the general conditions of this section, can we ensure the existence of an $r_{\max}$-cycle in the folded system $[F_0^m, \ldots, F_{\frac{p}{k}-1}^m]$ for some $m \in \{0, 1, \ldots, k-1\}$? In some cases, we can give a positive answer. As an immediate application of Theorem 2.1, we obtain the following two results:

**Corollary 2.1.** *Let $k$ be a divisor of $p$ and define the maps $F_j$ as in Eq. (2.1). If the $p$-periodic difference equation in Eq. (1.1) has an $r$-cycle for some $r \in \mathcal{A}_{p,q}$, and one of the following situations holds:*

*(i)* $\gcd(r,k) = \gcd(r,p)$,

*(ii)* $r_{\max}$ *is the smallest element of* $\mathcal{A}_{\frac{p}{k},q}$,

*then for $m \in \{0, 1, \ldots, k-1\}$, each of the periodic equations $x_{n+1} = F_n^m(x_n)$ has a $\frac{r}{\gcd(r,k)}$-cycle.*

**Corollary 2.2.** *In the conditions of Theorem 2.1, if $\gcd(r,p) = 1$ then $r_{\max} = r \in \mathcal{A}_{\frac{p}{k},q} \cap \mathrm{Per}[F_0^m, F_1^m, \ldots, F_{\frac{p}{k}-1}^m]$ for all $m \in \{0, 1, \ldots, k-1\}$.*

Before addressing the general case, we give some examples that will shed some light on our posed question about the existence of $r_{\max}$-cycles in the folded systems.

**Example 2.4.** *Consider $p = 4$, $k = 2$ and $r = 4$. Then*

$$\frac{r}{\gcd(r,p)} = \frac{4}{\gcd(4,4)} = 1,$$

$$r_{\max} = \frac{r}{\gcd(r,k)} = \frac{4}{\gcd(4,2)} = 2,$$

*$r \in \mathcal{A}_{4,1} = \{1, 2, 4\}$ and $\mathcal{A}_{2,1} = \{1, 2\}$. So we have $r^* = 1$ or $2$ as possible periods for the subsequences. In the scheme*

$$\begin{array}{cccc} f_0 & f_1 & f_2 & f_3 \\ x_0 & x_1 & x_2 & x_3 \end{array}$$

*with $x_i \neq x_j$ if $i \neq j$, all the folding sequences have period two, whereas*

$$\begin{array}{cccc} f_0 & f_1 & f_2 & f_3 \\ x_0 & A & x_2 & A \end{array}$$

*shows that 2 and 1 are possible periods of the folding sequences if we define $f_0(x_0) = f_2(x_2) = A$, $f_1(A) = x_2$, $f_3(A) = x_0$ and $x_0 \neq x_2$.*

**Example 2.5.** *Consider $p = 4$, $k = 2$, and let*

$$f_0(x) = \frac{1}{2}(-3x^2 + 20x - 13), \qquad f_1(x) = \frac{1}{16}(3x^2 - 32x + 100),$$
$$f_2(x) = \frac{1}{2}(3x^2 - 16x + 29), \qquad f_3(x) = \frac{-1}{2}x + 7.$$

*We have $C_{12} := \{1, 2, 3, 4, 5, 6, 1, 8, 3, 10, 5, 12\}$ is a 12-cycle of the 4-periodic system $x_{n+1} = f_n(x_n)$. Keep in mind that $12 \in \mathcal{A}_{4,3} = \{3, 6, 12\}$. By Theorem 2.1, $r^* \in \mathcal{A}_{\frac{p}{k},3} = \mathcal{A}_{2,3} = \{3, 6\}$. Indeed, $x_{n+1} = F_n^0(x_n)$ has the three cycle $C_3 := \{1, 3, 5\}$, while $x_{n+1} = F_n^1(x_n)$ has the six cycle $C_6 := \{2, 4, 6, 8, 10, 12\}$. Finally, observe that* $\operatorname{lcm}(3, 6) = 6 = r_{\max}$.

Up to this end, the given examples exhibit $r_{\max}$ as the period of some subsequences obtained from folding an $r$-periodic sequence. However, the next example show that $r_{\max}$ need not be the period of a subsequence obtained from folding an $r$-cycle of Eq. (1.1).

**Example 2.6.** *Consider $k = 2$ and $p = 12$. Take distinct $x_i, y_i \in [0, 1]$, $i = 0, 1$, $j = 0, 1, 2$, and define the constant interval maps $f_0(x) = y_0$, $f_1(x) = x_1$ for all $x \in [0, 1]$. Also, define the continuous functions $f_n$, $n > 1$, to be non-constant and satisfy*

$$f_{(2m) \bmod 12}(x_{m \bmod 2}) = y_{m \bmod 3},$$

$$f_{(2m+1) \bmod 12}(y_{m \bmod 3}) = x_{(m+1) \bmod 2},$$

*for all $m = 1, 2, \ldots$. Obviously, the alternated 12-periodic system $[f_0, f_1, \ldots, f_{11}]$ has a globally attracting $r$-cycle, $r = 12$, given by*

$$\{x_0, y_0, x_1, y_1, x_0, y_2, x_1, y_0, x_0, y_1, x_1, y_2\}.$$

*On the other hand, we have $r_{\max} = \frac{r}{\gcd(r,k)} = 6$, $\frac{r}{\gcd(r,p)} = 1$, and therefore $div^*(r_{max}) = \mathcal{A}_{6,1} = \{1, 2, 3, 6\}$. However, none of the folded systems $[F_0^i, F_1^i, \ldots, F_5^i]$, $i = 0, 1$, has a 6-cycle. In fact, $[F_0^0, F_1^0, \ldots, F_5^0]$ has the globally attracting 2-cycle $\{x_0, x_1\}$ while $[F_0^1, F_1^1, \ldots, F_5^1]$ has the globally attracting 3-cycle $\{y_0, y_1, y_2\}$. Notice that* $\operatorname{lcm}(2, 3) = 6 = r_{\max}$.

In the above example $r_{\max}$ is not reached as a period of the corresponding folded subsequences, however we have seen that the least common multiple of the periods $r^*$ of the subsequences is precisely $r_{\max}$. The following result confirms this heuristic fact, and establishes that the least common multiple of the values $r^*$ has to be $r_{\max}$.

**Theorem 2.2.** *Let $k$ be a divisor of $p$ and define the maps $F_j$ as in Eq. (2.1). Let $(x_n)_{n\geq 0}$ be an $r$-cycle of the alternated system $[f_0, f_1, \ldots, f_{p-1}]$. For each $m = 0, 1, \ldots, k-1$, the subsequence $(x_{kn+m})_{n\geq 0}$ is an $r_m^*$-cycle of the folded system $x_{n+1} = F_n^m(x_n)$ and* $\operatorname{lcm}(r_0^*, \ldots, r_{k-1}^*) = \frac{r}{\gcd(r,k)}$.

*Proof.* From Theorem 2.1, $(x_{kn+m})_{n\geq 0}$ is an $r_m^*$-cycle of the folded system $x_{n+1} = F_n^m(x_n)$ for some $r_m^* \in \mathcal{A}_{\frac{p}{k},q}$, where $q = \frac{r}{\gcd(r,p)}$. Thus, we proceed to show that $\operatorname{lcm}(r_0^*, ..., r_{k-1}^*) = \frac{r}{\gcd(r,k)}$. From Eq. (2.8), we can write $\gcd(r,p) = \alpha \gcd(r,k)$ for some $\alpha \geq 1$, that is,

$$\frac{r}{\gcd(r,k)} = \alpha \frac{r}{\gcd(r,p)}. \tag{2.10}$$

Realize that $\alpha$ divides both $r$ and $p$. Set $\ell := \operatorname{lcm}(r_0^*, ..., r_{k-1}^*)$. For each $j = 0, 1, \ldots, k-1$, $r_j^*$ verifies the conditions of Theorem 2.1, and therefore, the common multiple $\ell$ satisfies $\frac{r}{\gcd(r,p)}|\ell$ and $\ell|\frac{r}{\gcd(r,k)}$. Because $\frac{r}{\gcd(r,p)}|\ell$ and $\ell|\frac{r}{\gcd(r,k)}$, there exist positive integers $\alpha_1$ and $\beta$ such that $\alpha = \alpha_1 \cdot \beta$ and $\ell = \beta \frac{r}{\gcd(r,p)}$. Now, Eq. (2.10) implies

$$\frac{r}{\gcd(r,k)} = \alpha_1 \left(\beta \frac{r}{\gcd(r,p)}\right) = \alpha_1 \ell = \frac{\alpha}{\beta}\ell. \tag{2.11}$$

Thus, $\frac{\alpha}{\beta}$ divides $\frac{r}{\gcd(r,k)}$, and consequently

$$\frac{\alpha}{\beta}\gcd(r,k)|r. \tag{2.12}$$

On the other hand, since $\ell$ is a common multiple of $r_j^*$'s and each subsequence $(x_{kn+j})_n$ has a period $r_j^*$, $j = 0, 1, \ldots, k-1$, we deduce that $x_{k\ell+s} = x_s$ for all $s \geq 0$. Since $(x_n)$ is an $r$-cycle, we must have $r|k\ell$. Based on Eq. (2.8), we have $\alpha$ and $\frac{\alpha}{\beta}$ divide $r$. Use this fact together with Eq. (2.11) to obtain

$$k\ell = \frac{k}{\gcd(r,k)}\frac{r}{\frac{\alpha}{\beta}}. \tag{2.13}$$

Since $r|k\ell$, Eq. (2.13) gives us $\frac{k\ell}{r}\gcd(r,k)\frac{\alpha}{\beta} = k$, and consequently

$$\frac{\alpha}{\beta}\gcd(r,k)|k. \tag{2.14}$$

From Eq. (2.12), Eq. (2.14) and the definition of $\gcd(r,k)$, we conclude that $\frac{\alpha}{\beta}\gcd(r,k) \leq \gcd(r,k)$, which implies $\alpha = \beta$. Therefore,

$$\ell = \operatorname{lcm}(r_0^*, r_1^*, \ldots, r_{k-1}^*) = \alpha \frac{r}{\gcd(p,r)} = \frac{r}{\gcd(r,k)}.$$

$\square$

In the following results, we use $\operatorname{Card}(\cdot)$ to denote the cardinality of a set. The results are straightforward consequences of Theorem 2.2 and Theorem 2.1.

**Corollary 2.3.** *Under the hypothesis of Theorem 2.2, if*

$$\operatorname{Card}(div^*(r_{\max}) \cap \mathcal{A}_{\frac{p}{k},\frac{r}{\gcd(r,p)}}) \leq 2,$$

*then $\frac{r}{\gcd(r,k)}$ is a period of a cycle for some of the periodic difference equations $x_{n+1} = F_n^m(x_n)$, $m \geq 0$.*

*Proof.* From Part (iii) of Proposition 2.3, we have

$$\frac{r}{\gcd(r,k)} = r_{\max} \in \mathcal{A}_{\frac{p}{k},\frac{r}{\gcd(r,p)}}.$$

Also, it is obvious from the structure of $div^*(r_{\max})$ that $r_{\max} \in div^*(r_{\max})$. Now, if the intersection of the two sets contains only one element, then the element must be $r_{\max}$, and consequently, it must be the period of a cycle for some of the periodic equations $x_{n+1} = F_n^m(x_n)$, $m \geq 0$. On the other hand, if the intersection has exactly two elements, i.e.,

$$\mathrm{Card}(div^*(r_{\max}) \cap \mathcal{A}_{\frac{p}{k},\frac{r}{\gcd(r,p)}}) = \{r_{\max}, \tilde{r}\}$$

with $\tilde{r}$ dividing $r_{\max}$, then the conclusion of Theorem 2.2 is not satisfied unless $r_{\max}$ is again the period of a cycle for some of the periodic equations $x_{n+1} = F_n^m(x_n)$, $m \geq 0$. □

The situation described in the last corollary appears, for instance, in Examples 2.3, 2.4 and 2.5. In these examples, $\mathrm{Card}(\mathcal{A}_{\frac{p}{k},\frac{r}{\gcd(r,p)}}) \leq 2$. Next, using the idea of Corollary 2.3 and (ii) of Corollary 2.1, we give the following corollary.

**Corollary 2.4.** *Under the hypothesis of Theorem 2.2, consider*

$$\mathcal{D} := \{s \in \mathcal{A}_{\frac{p}{k},\frac{r}{\gcd(r,p)}} : s \leq r_{\max}\}.$$

*If* $\mathrm{Card}(\mathcal{D}) \leq 2$*, then* $\frac{r}{\gcd(r,k)}$ *is a period of a cycle for some of the periodic difference equations* $x_{n+1} = F_n^m(x_n)$*,* $m \geq 0$*.*

Another case for which $r_{\max}$ is allowed to be a period of a folded subsequence appears when $p$ is a power prime, i.e., $p = q^s$ for some prime number $q$ and positive integer $s$.

**Corollary 2.5.** *Under the hypothesis of Theorem 2.2, if* $p$ *is a power prime, then* $\frac{r}{\gcd(r,k)}$ *is a period of a cycle for some of the periodic difference equations* $x_{n+1} = F_n^m(x_n)$*,* $m \geq 0$*.*

*Proof.* Let $p = q^s$ for some prime number $q$ and positive integer $s$. Also, let $k = q^{s_1}$ for some $0 \leq s_1 \leq s$. From the structure of the clusters given in Proposition 2.2, we have

$$\mathcal{A}_{\frac{p}{k},\frac{r}{\gcd(r,p)}} = \frac{r}{\gcd(r,p)} \cdot \left\{q^{s-s_1}\right\} \tag{2.15}$$

when $\gcd(\frac{p}{k}, \frac{r}{\gcd(r,p)}) > 1$, and

$$\mathcal{A}_{\frac{p}{k},\frac{r}{\gcd(r,p)}} = \frac{r}{\gcd(r,p)} \cdot \left\{q^j : \ j = 0, 1, \ldots, s - s_1\right\} \tag{2.16}$$

when $\gcd(\frac{p}{k}, \frac{r}{\gcd(r,p)}) = 1$. The first case is obvious (realize that the cluster has a unique element), so we clarify the second case. From Theorem 2.2, the folded cycles are of periods

$$r_0^*, \ldots, r_{k-1}^* \in \mathcal{A}_{\frac{p}{k},\frac{r}{\gcd(r,p)}} \quad \text{and} \quad \mathrm{lcm}\,(r_0^*, r_1^*, \ldots, r_{k-1}^*) = \frac{r}{\gcd(r,k)}.$$

However, from the structure of $r_j^*$ as given in Eq. (2.16), we obtain

$$\operatorname{lcm}(r_0^*, r_1^*, \ldots, r_{k-1}^*) = \max\{r_0^*, r_1^*, \ldots, r_{k-1}^*\}.$$

Thus, $\frac{r}{\gcd(r,k)} = \max\{r_0^*, r_1^*, \ldots, r_{k-1}^*\}$ is a period of a cycle for one of the folded systems. $\square$

We can also ensure that $r_{\max}$ is the period of some folded subsequence when the set $\mathcal{A}_{\frac{p}{k}, \frac{r}{\gcd(r,p)}}$ has a prime number of elements:

**Corollary 2.6.** *Under the hypothesis of Theorem 2.2, if* $\operatorname{Card}(\mathcal{A}_{\frac{p}{k}, \frac{r}{\gcd(r,p)}})$ *is a prime number, then* $r_{\max} = \frac{r}{\gcd(r,k)}$ *is a period of a cycle for some of the periodic difference equations* $x_{n+1} = F_n^m(x_n)$, $m \geq 0$.

*Proof.* From Proposition 2.2 we have

$$\mathcal{A}_{\frac{p}{k}, \frac{r}{\gcd(r,p)}} = \frac{r}{\gcd(r,p)} \cdot c \cdot \{ \text{ divisors of } b_1^{\beta_1} \ldots b_t^{\beta_t} \},$$

where $c$ is related to the common divisors of $\frac{p}{k}$ and $\frac{r}{\gcd(r,p)}$. Also, for each $j = 1, \ldots, t$, $b_j$ is not a divisor of $\frac{r}{\gcd(r,p)}$. Since the number of elements of $\mathcal{A}_{\frac{p}{k}, \frac{r}{\gcd(r,p)}}$ is a prime number, say $m$, and the number of divisors of $b_1^{\beta_1} \ldots b_t^{\beta_t}$ is $\prod_{j=1}^{t}(\beta_j + 1)$, we must have $t = 1$, and consequently, $\beta_1 = m - 1$. Now, use the same reasoning as in Corollary 2.5 to obtain that $r_{\max}$ is the period of a cycle for one of the folded systems. $\square$

# 3 Unfolding in periodic difference equations

Let $k$ be a divisor of $p$, and consider the maps $F_j$, $j = 0, 1, \ldots, \frac{p}{k} - 1$ defined in Eq. (2.1). Our goal here is to use the periodic orbits of the folded $\frac{p}{k}$-periodic system

$$x_{n+1} = F_n(x_n) \tag{3.1}$$

to describe the periodic orbits of Eq. (1.1) through the unfolding of the maps $F_j$. Before starting the development of the discussion of unfolding systems, let us recall the following result explaining the relationship between an $m$-periodic sequence $(x_n)$ of $[f_0, \ldots, f_{p-1}]$ and the folded subsequences corresponding to $k = p$, when $m$ and $p$ are relatively prime. This was given in [3], but apparently has not taken its roots in the literature; so we present it here together with an alternative proof within the context of this work.

**Proposition 3.1.** *Fix an integer* $m \geq 1$ *with* $\gcd(m, p) = 1$. *Then* $m \in \operatorname{Per}[f_0, \cdots, f_{p-1}]$ *if and only if* $f_0, \ldots, f_{p-1}$ *have a common periodic orbit* $(x_0, \ldots, x_{m-1}, x_0, \ldots)$ *of period* $m$ *such that* $f_j(x_i) = f_0(x_i)$, $i = 0, \ldots, m-1$, $j = 0, \ldots, p-1$.

*Proof.* The implication "$\Leftarrow$" is immediate. We prove the implication "$\Rightarrow$". To this end, assume that $m \in \mathrm{Per}[f_1, \cdots, f_p]$, with $\gcd(m,p) = 1$. Let $(x_0, x_1, \ldots, x_{m-1}, \ldots)$ be an $m$-cycle of $[f_0, \ldots, f_{p-1}]$. Since $m \in \mathcal{A}_{p,m}$ (apply Proposition 2.2(ii)), Theorem 2.1 (use $k = p$) yields the periodicity of $x_0$ under $f_{p-1} \circ \cdots \circ f_0$ and $m = \mathrm{ord}_{f_{p-1}\circ\cdots\circ f_0}(x_0)$. Next, we show that $f_0(x_i) = f_1(x_i) = \ldots = f_{p-1}(x_i)$ for $i = 0, \ldots, m-1$ and $x_0 \in \mathrm{Per}(f_j)$ with $\mathrm{ord}_{f_j}(x_0) = m$ for $j = 0, \ldots, p-1$, that is, $f_0, f_1, \ldots, f_{p-1}$ share the same periodic orbit of order $m$. Since $\gcd(m,p) = 1$, there is a bijective map $\varphi : \{0, 1, \ldots, p-1\} \to \{0, 1, \ldots, p-1\}$ such that $\varphi(0) = 0$ and $j \cdot m = p \cdot k_j + \varphi(j)$ for some non-negative integer $k_j$, $1 \le j < p$. Indeed, if not, then there are $j_1 < j_2$ in $\{0, 1, \ldots, p-1\}$ such that $\varphi(j_1) = \varphi(j_2)$, which implies $j_i \cdot m = p \cdot k_{j_i} + \varphi(j_i)$, $i = 1, 2$. Thus, we deduce that $(j_2 - j_1) \cdot m = p \cdot (k_{j_2} - k_{j_1})$, and as a consequence of $\gcd(m,p) = 1$, we obtain $p|(j_2 - j_1)$, which is not possible because $j_2 - j_1 < p$. Now, set $\nu = \varphi^{-1}$, we can write $\nu(r) \cdot m = p \cdot q_r + r$ for $r \in \{0, 1, ..., p-1\}$ and the corresponding non-negative integers $q_r$'s. Then for any $j \in \{0, 1, \ldots, p-1\}$,

$$x_{\nu(j)\cdot m+1} = (f_j \circ \ldots \circ f_0 \circ (f_{p-1} \circ \ldots \circ f_0)^{q_j}(x_0)) = x_1 \tag{3.2}$$

and

$$x_{\nu(j)\cdot m} = (f_{j-1} \circ \ldots \circ f_0 \circ (f_{p-1} \circ \ldots \circ f_0)^{q_j}(x_0)) = x_0 \tag{3.3}$$

(notice that $j = 0$ in the last equality means $f_{j-1} \circ \ldots \circ f_0 = \text{Identity}$). So, combining (3.2) and (3.3) gives us $x_1 = f_j(x_0)$ for $j = 0, \ldots, p-1$. Since $(x_0, \ldots, x_{m-1}, \ldots)$ is a periodic sequence, the role played by $x_0$ in the above reasoning can be played by $x_1, \ldots, x_{m-1}$, and so we have $f_j(x_i) = f_0(x_i)$ for $j = 0, \ldots, p-1$ and $i = 0, \ldots, m-1$. Hence, the initial sequence $(x_0, \ldots, x_{m-1}, \ldots)$ can be written as

$$(x_0, f_j(x_0), \ldots, f_j^{m-1}(x_0), \ldots)$$

for $j = 0, \ldots, p-1$, and therefore, $x_0 \in \mathrm{P}(f_j)$ with $\mathrm{ord}_{f_j}(x_0) = m$, and the proof is complete. $\square$

**Remark 3.1.** *Note that Proposition 3.1 also proves that $f_j^i(x_1) = f_k^i(x_1)$ for all $i \ge 1$, $j, k \in \{0, 1, \ldots, p-1\}$.*

Concerning our discussion of unfolding, we extract two particular results from [2] before proceeding to the general case.

**Proposition 3.2.** *Let $k = p$. If Eq. (3.1) has a $q$-cycle, then Eq. (1.1) has a $qs$-cycle for some positive integer $s$ that divides $p$.*

*Proof.* Consider $k = p$ and suppose that Eq. (3.1) has a $q$-cycle. Eq. (3.1) is 1-periodic, i.e., we just have the autonomous equation

$$x_{n+1} = F_0(x_n) = (f_{p-1} \circ \ldots \circ f_1 \circ f_0)(x_n).$$

Unfold the maps to obtain a $\widetilde{m}$-cycle of Eq. (1.1) for some $\widetilde{m} \in \mathcal{A}_{p,\widetilde{q}}$. From Theorem 2.1 and our hypothesis $k = p$, the corresponding folded subsequence has period $\widetilde{m}^* = q$ with $\widetilde{m}^* = q \in \mathcal{A}_{1,\widetilde{q}}$. Hence, $\widetilde{q} = q$ and $\widetilde{m} \in \mathcal{A}_{p,q}$. From the definition of $\mathcal{A}_{p,q}$ (see Proposition 2.2), we obtain $\widetilde{m} = qs$ for some positive integer $s$ that divides $p$. $\square$

**Proposition 3.3.** *Let $p$ be a prime number, and consider $k = p$. If the folded system $x_{n+1} = F_0(x_n)$ has an $r$-cycle $C_r$, then the unfolded system in Eq. (1.1) either has an $rp$-cycle, or all maps $f_j$ share the same cycle $C_r$.*

*Proof.* Certainly, the $r$-cycle of the folded system $x_{n+1} = F_0(x_n)$ gives an $r^*$-cycle for the unfolded system in Eq. (1.1), and reasoning as in Proposition 3.2 we find $r^* \in \mathcal{A}_{p,r}$. Now, since $p$ is a prime number, $r$ can be a multiple of $p$ or relatively prime with $p$. If $r$ is a multiple of $p$, we obtain $\mathcal{A}_{p,r} = \{rp\}$, and consequently $rp = r^*$. On the other hand, if $r$ is relatively prime with $p$, then $\mathcal{A}_{p,r} = \{p, rp\}$. If $r^* \neq rp$, then $r^* = r$, and according to Proposition 3.1 all autonomous equations $x_{n+1} = f_j(x_n)$, $j = 0, \ldots, p-1$, must share the same $r$-cycle. $\square$

A similar result is obtained when $p$ is a power prime, say $p = p_1^n$ for some prime number $p_1$. In this case (see Proposition 2.2), $\mathcal{A}_{p,r}$ is equal to the set $\{r, rp_1, \ldots, rp_1^n\}$ when $r$ is relatively prime with $p_1$, and equal to $\{rp_1^n\}$ when $\gcd(r,p) > 1$. Thus, we state the following result.

**Proposition 3.4.** *Let $p$ be a power prime, say $p = p_1^n$, and consider $k = p$. If the folded system $x_{n+1} = F_0(x_n)$ has an $r$-cycle $C_r$, then the unfolded system in Eq. (1.1) has an $rp_1^s$-cycle for some $0 < s \leq n$, or all maps $f_j$ share the same cycle $C_r$.*

Next, we proceed to investigate the general case of unfolding difference equations when $p$ is an arbitrary positive integer and $k$ an arbitrary divisor of $p$. Let

$$C_q^0 = \{c_0^0, c_1^0, c_2^0, \ldots, c_{q-1}^0\}$$

be a $q$-cycle of Eq. (3.1). When the unfolding process takes place, each element of $C_q^0$ will be extended into a segment of $k$ elements, and they may not be distinct. To visualize this situation and put the ideas within the framework of the previous section, we give the lattice in Figure 1.

**Lemma 3.1.** *Let $k$ be a divisor of $p$, and consider the folded maps $F_j^m$ as defined in Eq. (2.2), $m \in \{0, 1, \ldots, k-1\}$. If for some fixed $m_0 \in \{0, 1, \ldots, k-1\}$, the equation $x_{n+1} = F_n^{m_0}(x_n)$ is $\frac{p}{k}$-periodic and has a $q$-cycle, then each of the following holds true.*

*(i) There exists an $r$-cycle of Eq. (1.1) for some $r \in \mathcal{A}_{p,\frac{q}{\gcd(q,\frac{p}{k})}}$.*

*(ii) Each equation $x_{n+1} = F_n^m(x_n)$ has an $q_m$-cycle for some $q_m \in \mathcal{A}_{\frac{p}{k},\frac{q}{\gcd(q,\frac{p}{k})}}$.*

$$
\begin{tikzcd}
c_0^0 \arrow[r, "f_0"] \arrow[d, "F_0^0"] & c_0^1 \arrow[r, "f_1"] \arrow[d, "F_0^1"] & c_0^2 \arrow[r, "f_2"] \arrow[d, "F_0^2"] & c_0^3 \arrow[r] \arrow[d, "F_0^3"] & \cdots \arrow[r, "f_{k-2}"] \arrow[d, "\cdots"] & c_0^{k-1} \arrow[r, "f_{k-1}"] \arrow[d, "F_0^{k-1}"] & {} \\
c_1^0 \arrow[r, "f_k"] \arrow[d, "F_1^0"] & c_1^1 \arrow[r, "f_{k+1}"] \arrow[d, "F_1^1"] & c_1^2 \arrow[r, "f_{k+2}"] \arrow[d, "F_1^2"] & c_1^3 \arrow[r] \arrow[d, "F_1^3"] & \cdots \arrow[r, "f_{2k-2}"] \arrow[d, "\cdots"] & c_1^{k-1} \arrow[r, "f_{2k-1}"] \arrow[d, "F_1^{k-1}"] & {} \\
c_2^0 \arrow[r, "f_{2k}"] \arrow[d, "F_2^0"] & c_2^1 \arrow[r, "f_{2k+1}"] \arrow[d, "F_2^1"] & c_2^2 \arrow[r, "f_{2k+2}"] \arrow[d, "F_2^2"] & c_2^3 \arrow[r] \arrow[d, "F_2^3"] & \cdots \arrow[r, "f_{3k-2}"] \arrow[d, "\cdots"] & c_2^{k-1} \arrow[r, "f_{3k-1}"] \arrow[d, "F_2^{k-1}"] & {} \\
\vdots \arrow[r] \arrow[d, "F_{q-2}^0"] & \vdots \arrow[r] \arrow[d, "F_{q-2}^1"] & \vdots \arrow[r] \arrow[d, "F_{q-2}^2"] & \vdots \arrow[r] \arrow[d, "F_{q-2}^3"] & \cdots \arrow[r] \arrow[d, "\cdots"] & \vdots \arrow[r] \arrow[d, "F_{q-2}^{k-1}"] & {} \\
c_{q-1}^0 \arrow[r, "f_{(q-1)k}"] & c_{q-1}^1 \arrow[r, "f_{(q-1)k+1}"] & c_{q-1}^2 \arrow[r, "f_{(q-1)k+2}"] & c_{q-1}^3 \arrow[r] & \cdots \arrow[r, "f_{qk-2}"] & c_{q-1}^{k-1} \arrow[r, "f_{qk-1}", "\circlearrowleft"'] & {}
\end{tikzcd}
$$

Figure 1: This lattice shows the relationship between the orbits of the folded systems $x_{n+1} = F_n^m(x_n)$ and the unfolded system $x_{n+1} = f_n(x_n)$. The maps $F_j^m$ are the maps $F_{j,m}^{(k)}$ as defined in Eq. (2.2). In particular, the folded system in Eq. (3.1) shows up in the first column of this figure.

*Proof.* Without loss of generality, we consider $m_0 = 0$ and $m = 1$. Let $C_q^0 := \{c_0^0, \ldots, c_{q-1}^0\}$ be a $q$-cycle of $x_{n+1} = F_n^0(x_n)$. Since this equation is $\frac{p}{k}$-periodic, $q$ must be in the cluster $\mathcal{A}_{\frac{p}{k}, \frac{q}{\gcd(q, \frac{p}{k})}}$. Now, we investigate the action of $x_{n+1} = F_n^1(x_n)$ on $f_0(c_0^0) =: c_0^1$. We have

$$F_0^1(f_0(c_0^0)) = f_k \circ f_{k-1} \circ \cdots \circ f_1 \circ f_0(c_0^0) = f_k(F_0^0(c_0^0)) = f_k(c_1^0).$$

Similarly, $F_j^1(f_{jk(\bmod p)}(c_j^0)) = f_{(j+1)k(\bmod p)}(c_{j+1}^0)$ for all positive integers $j$. Now, define $c_j^1 := f_{jk(\bmod p)}(c_j^0)$, $j \geq 1$. Observe that we have periodicity in the sequence of maps $f_{jk(\bmod p)}$ and periodicity in the sequence of points $c_j^0$. In particular, the periodicity of $\{jk : \ j = 0, 1, \ldots\}$ in $(\mathbb{Z}_p, +, \cdot)$ is $\frac{p}{k}$. The periodicity of $f_{jk(\bmod p)}$ is $\frac{p}{k}$ by assumption; however, the periodicity of $f_{jk(\bmod p)+i}$, $1 \leq i \leq k-1$ is a divisor of $\frac{p}{k}$. By hypothesis, the periodicity of $c_j^0$ is $q$, and the obtained orbit

$$C_{q_1}^1 := \{f_0(c_0^0), f_k(c_1^0), f_{2k}(c_2^0), \ldots\}$$

forms a $q_1$-cycle in the folded system $x_{n+1} = F_n^1(x_n)$ for some positive integer $q_1$ that divides $\operatorname{lcm}(\frac{p}{k}, q)$. Similarly, we find that the given $q$-cycle generates a $q_i$-cycle (say $C_{q_i}^i$) for each folded system $x_{n+1} = F_n^i(x_n)$, $2 \leq i \leq k-1$. Up to this end, the initial point $c_0^0$ generates a periodic solution (say $(x_n)$ with period $r$) for the unfolded system in Eq. (1.1). According to Theorem 2.1 we know that the period $q$ of the folded sequence $(x_{kn})_n$, whose associated orbit is precisely $C_q^0$, must verify $q \in \mathcal{A}_{\frac{p}{k}, \frac{r}{\gcd(r,p)}}$, and at the same time (see Proposition 2.3), $q \in \mathcal{A}_{\frac{p}{k}, \frac{q}{\gcd(\frac{p}{k},q)}}$. Therefore,

$$\frac{r}{\gcd(r,p)} = \frac{q}{\gcd(\frac{p}{k},q)}, \tag{3.4}$$

and consequently, $r \in \mathcal{A}_{p, \frac{q}{\gcd(\frac{p}{k},q)}}$. This completes the prove of Part (i).

To prove part (ii), we go back to the formed $q_i$-cycles in part (i). Again by Theorem 2.1 and Eq. (3.4), the period $q_i$ of the folded sequence $C^i_{q_i}$ must satisfy $q_i \in \mathcal{A}_{\frac{p}{k}, \frac{q}{\gcd(\frac{p}{k},q)}}$. We conclude that the equation $x_{n+1} = F^i_n(x_n)$ has a $q_i$-cycle and $q_i \in \mathcal{A}_{\frac{p}{k}, \frac{q}{\gcd(q,\frac{p}{k})}}$. Observe that this periodic solution is represented by $\{c^i_0, c^i_1, c^i_2, \ldots\}$ in the $i^{th}$ column of Figure 1. This completes the proof of Part (ii).

For the general case, realize that we can start at some intermediate column rather than the first column; however, the reasoning is similar to that in the case $m_0 = 0$. $\square$

**Theorem 3.1.** *Let $k$ be a divisor of $p$, and consider the folded maps $F^m_j$ as defined in Eq. (2.2). If for some fixed $m_0$, the equation $x_{n+1} = F^{m_0}_n(x_n)$ is $\frac{p}{k}$-periodic and has a $q$-cycle, then the unfolded system $x_{n+1} = f_n(x_n)$ has an $r$-cycle for some*

$$r \in \mathcal{A}_{k,q^*} \cap \mathcal{A}_{p, \frac{q}{\gcd(q,\frac{p}{k})}},$$

*where $q^* = \frac{sq}{\gcd(q,\frac{p}{k})}$ for some integer $s$ that divides $\frac{p}{k}$. Furthermore, the relationship between $s$ and $r$ is $\gcd(r,p) = s\gcd(r,k)$, i.e., $s = \alpha$ as given in Eq. (2.8).*

*Proof.* A $q$-cycle of the equation $x_{n+1} = F^{m_0}_n(x_n)$ means we have a $q$-periodic sequence along one of the columns in Figure 1. Unfold the maps as shown in Figure 1 to obtain periodic sequences of periods $q_1, q_2, \ldots, q_k$ along the columns. By tracking the orbit row-by-row, we obtain a periodic solution of the unfolded system $x_{n+1} = f_n(x_n)$ with a certain period $r$. By Lemma 3.1, the obtained $r$-cycle must be in the cluster $\mathcal{A}_{p, \frac{q}{\gcd(q,\frac{p}{k})}}$. On the other hand, it must be $r \in \mathcal{A}_{k,\widetilde{q}}$ for some integer $\widetilde{q} \geq 1$ since $\mathbb{Z}^+ = \bigcup_{t\geq 1} \mathcal{A}_{k,t}$. Now, by the definition of the clusters and Proposition 2.3(iv),

$$\widetilde{q} = \frac{r}{\gcd(r,k)} = \alpha \frac{r}{\gcd(r,p)}, \tag{3.5}$$

where $\alpha = \gcd(\frac{p}{k}, \frac{r}{\gcd(r,k)})$. Moreover, since $r \in \mathcal{A}_{p, \frac{q}{\gcd(\frac{p}{k},q)}}$, we use the definition of the clusters to obtain

$$\frac{r}{\gcd(r,p)} = \frac{q}{\gcd(\frac{p}{k},q)}. \tag{3.6}$$

From (3.5) and (3.6),

$$\widetilde{q} = \alpha \frac{q}{\gcd(\frac{p}{k},q)}.$$

Finally, from Proposition 2.3(ii), $\mathcal{A}_{k,q^*} = \mathcal{A}_{k,\widetilde{q}}$ implies that $\widetilde{q} = q^*$, which gives $s = \alpha$ and ends the proof. $\square$

**Remark 3.2.** *$q$ has to be a divisor of $\frac{r}{\gcd(r,k)} =: r_{\max}$ according to Theorem 2.1. Also, notice that $kq \in \mathcal{A}_{p,\frac{q}{\gcd(q,\frac{p}{k})}}$ since $\frac{kq}{\gcd(kq,p)} = \frac{kq}{k\gcd(q,\frac{p}{k})} = \frac{q}{\gcd(q,\frac{p}{k})}$. Therefore, if the cluster $\mathcal{A}_{p,\frac{q}{\gcd(q,\frac{p}{k})}}$ is a singleton, then necessarily $r = kq$.*

Two obvious cases of Theorem 3.1 are when $k = 1$ and $k = p$. At $k = 1$, the folded and unfolded systems are the same, and the $q$-cycle is the same as the $r$-cycle in the theorem. At $k = p$, the folding process changes the $p$-periodic system into a 1-periodic system, i.e., an autonomous system represented by the $p$-fold map $F = f_{p-1} \circ f_{p-2} \circ \cdots , f_0$. In this case, the clusters $\mathcal{A}_{k,q^*}$ and $\mathcal{A}_{p,\frac{q}{\gcd(q,\frac{p}{k})}}$ are equal.

When it is not possible to precisely identify the exact period of the cycle arising from the unfolding process, it is desirable to limit the search to certain values. We close this section by illustrating this assertion with several corollaries and some illustrative examples.

**Corollary 3.1.** *Let $k$ be a divisor of $p$, and consider the folded maps $F_j$ as defined in Eq. (2.1). If the folded system $x_{n+1} = F_n(x_n)$ has a $q$-cycle for some $q = q_1\frac{p}{k}$, then the unfolded system in Eq. (1.1) has an $r$-cycle for some $r \in \mathcal{A}_{k,q}$.*

*Proof.* According to Theorem 3.1, we have $r \in \mathcal{A}_{p,\frac{q}{\gcd(q,\frac{p}{k})}} \cap \mathcal{A}_{k,q^*}$, where $q^* = \frac{sq}{\gcd(q,\frac{p}{k})}$ for some integer $s$ that divides $\frac{p}{k}$. Thus, the task will be achieved by proving that $q = q^*$. Indeed, From Proposition 2.3(iv), we have $\gcd(r,p) = \alpha \gcd(r,k)$ with $\alpha = \gcd(\frac{p}{k}, \frac{r}{\gcd(r,k)})$. So, $\alpha | \frac{p}{k}$. On the other hand, $r \in \mathcal{A}_{p,\frac{q}{\gcd(q,\frac{p}{k})}}$ yields

$$\frac{r}{\gcd(r,p)} = \frac{q}{\gcd(q,\frac{p}{k})} = \frac{q_1\frac{p}{k}}{\gcd(q_1\frac{p}{k},\frac{p}{k})} = q_1.$$

So $\frac{r}{\gcd(r,k)} = \alpha\frac{r}{\gcd(r,p)} = \alpha q_1$. From Theorem 2.1, $q = q_1\frac{p}{k}$ has to divide $r_{\max} = \frac{r}{\gcd(r,k)} = \alpha q_1$, that is, $\frac{p}{k}$ must divide $\alpha$. Hence, $\alpha = \frac{p}{k}$. Since $\alpha = s$ (see Theorem 3.1), we finally deduce

$$q^* = \frac{sq}{\gcd(q,\frac{p}{k})} = \frac{\alpha q_1\frac{p}{k}}{\gcd(q_1\frac{p}{k},\frac{p}{k})} = \frac{\alpha q_1\frac{p}{k}}{\frac{p}{k}} = \alpha q_1 = \frac{p}{k}q_1 = q,$$

as desired. □

**Corollary 3.2.** *Let $k$ be a divisor of $p$, and consider the folded maps $F_j$ as defined in Eq. (2.1). If the folded system $x_{n+1} = F_n(x_n)$ has a $q$-cycle, where $q$ is a divisor of $\frac{p}{k}$, then the unfolded system in Eq. (1.1) has an $r$-cycle for some $r$ that divides $p$.*

*Proof.* This is a straightforward consequence of Theorem 3.1. We obtain $r \in \mathcal{A}_{p,1}$, which is the set of divisors of $p$. □

**Corollary 3.3.** *Let $k$ be a divisor of $p$, and consider the folded maps $F_j$ as defined in Eq. (2.1). If the folded system $x_{n+1} = F_n(x_n)$ has a $q$-cycle, where $\gcd(p, q) = 1$, then the unfolded system in Eq. (1.1) has an $r$-cycle for some $r \in \mathcal{A}_{p,q} \cap \mathcal{A}_{k,\alpha q}$, where $\alpha$ as given in Eq. (2.8). Moreover, $r = qz$ for some divisor $z$ of $p$ such that $\alpha | z$ and $z \in \mathcal{A}_{k,\alpha}$.*

*Proof.* By Theorem 3.1, it suffices to consider that $\gcd(\frac{p}{k}, q) = 1$ to obtain $r \in \mathcal{A}_{p,q} \cap \mathcal{A}_{k,\alpha q}$. Now, apply Proposition 2.2(ii) to obtain $r = qz$ for some divisor $z$ of $p$. Since $r = qz \in \mathcal{A}_{k,\alpha q}$ by the definition of the clusters, we have $\frac{qz}{\gcd(qz,k)} = \frac{qz}{\gcd(z,k)} = \alpha q$. Therefore, $\alpha = \frac{z}{\gcd(z,k)}$ and $z \in \mathcal{A}_{k,\alpha}$. □

**Example 3.1.** *Consider $p = 6$ and $k = 2$ (so, $F_0^0 = f_1 \circ f_0, F_1^0 = f_3 \circ f_2, F_2^0 = f_5 \circ f_4$, and $F_0^1 = f_2 \circ f_1, F_1^1 = f_4 \circ f_3, F_2^1 = f_0 \circ f_5$) in each of the following:*

*(i) Suppose the 3-periodic system $x_{n+1} = F_n^0(x_n)$ has a 6-cycle. From Lemma 3.1(ii), since $\mathcal{A}_{3,2} = \{2, 6\}$, the folded system $x_{n+1} = F_n^1(x_n)$ has a 2-cycle or a 6-cycle. From Theorem 3.1, $\mathcal{A}_{k,q^*}$ is either $\mathcal{A}_{2,2} = \{4\}$ or $\mathcal{A}_{2,6} = \{12\}$, while $\mathcal{A}_{p,\frac{q}{\gcd(q,\frac{p}{k})}} = \mathcal{A}_{6,2} = \{4, 12\}$. Taking into account that from Theorem 2.1, $q = 6$ is a divisor of $r_{\max} = \frac{r}{\gcd(r,k)}$, and that for $r = 4$ we find $r_{\max} = \frac{4}{\gcd(4,2)} = 2$, we discard the value $r = 4$ and conclude that the unfolded 6-periodic system $x_{n+1} = f_n(x_n)$ has a 12-cycle.*

*(ii) Suppose the 3-periodic system $x_{n+1} = F_n^0(x_n)$ has a 15-cycle. From Lemma 3.1(ii), since $\mathcal{A}_{3,5} = \{5, 15\}$, the folded system $x_{n+1} = F_n^1(x_n)$ has a 5-cycle or a 15-cycle. From Theorem 3.1, $\mathcal{A}_{k,q^*}$ is either $\mathcal{A}_{2,5} = \{5, 10\}$ or $\mathcal{A}_{2,15} = \{15, 30\}$, while $\mathcal{A}_{p,\frac{q}{\gcd(q,\frac{p}{k})}} = \mathcal{A}_{6,5} = \{5, 10, 15, 30\}$. From Corollary 3.1, $q^* = q = 15$, so $r \in \mathcal{A}_{2,15}$, and we conclude that the unfolded 6-periodic system $x_{n+1} = f_n(x_n)$ has a 15-cycle or a 30-cycle.*

As observed in Part (i) of Example 3.1, we can use the period of a given cycle in a folded system to identify the exact period arises in the unfolded system. In Part (ii), we can just identify certain possibilities. Finding the exact period needs some information about the combinatorial structure of the given cycle and the action of the other folded systems on this cycle, i.e., we need to know the structure of the other columns in Figure 1. To illustrate this idea, we go back to Part (ii) of Example 3.1 and show that the two possibilities are indeed feasible. Define

$$f_0(x) = x + 1$$

$$f_1(x) = f_0(x) + \prod_{j=0}^{4}(x - 3j - 1)$$

$$f_2(x) = \frac{1}{648}(14 - x)(5x^3 - 60x^2 + 315x - 268)$$

and

$$f_3(x) = f_4(x) = f_0(x), \quad f_5(x) = f_2(x).$$

Take $F_0 = f_1 \circ f_0$, $F_1 = f_3 \circ f_2$ and $F_2 = f_5 \circ f_4$, then the folded system $[F_0, F_1, F_2]$ has the 15-cycle

$$C_{15} = \{0, 2, 4, \ldots, 14, 1, 3, 5, \ldots, 13\},$$

which gives the 15-cycle $C^*_{15} = \{0, 1, 2, 3, \ldots, 14\}$ for the unfolded system $[f_0, f_1, \ldots, f_5]$. On the other hand, if we define

$$\begin{aligned}
f_0(x) =& \frac{-1}{486} x(5x^3 - 120x^2 + 900x - 2241)\\
f_1(x) =& \frac{1}{8}(15x^4 - 110x^3 + 225x^2 - 82x + 8)\\
f_2(x) =& \frac{1}{162}(x - 10)(5x^2 - 55x + 32)\\
f_3(x) =& \frac{1}{2}(-5x^3 + 30x^2 - 43x + 22)\\
f_4(x) =& \frac{1}{486}(x - 5)(5x^3 - 135x^2 + 1065x - 1954)\\
f_5(x) =& \frac{-1}{2}(x - 4)(5x^3 - 20x^2 + 20x + 3),
\end{aligned}$$

then the folded system $[F_0, F_1, F_2]$ has the 15-cycle $C_{15} = \{0, 1, 2, 3, 4, \ldots, 14\}$, which gives the 30-cycle

$$C_{30} = \{0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 0, 6, 1, 7, 2, 8, 3, 9, 4, 10, 0, 11, 1, 12, 2, 13, 3, 14, 4\}$$

for the unfolded system $[f_0, f_1, \ldots, f_5]$.

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