OPTIMIZATION OF P-CHART FOR PROCESSES WITH MULTIPLE ASSIGNABLE CAUSES AND RANDOM SHIFT

by

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Declaration of Authorship

I declare that this thesis is my own work and, to the best of my knowledge and belief, it does not contain material published or written by a third party, except where permission has been obtained and/or appropriately cited through full and accurate referencing.

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Dedication

To my beloved wife Asmaa,

for her continuous support and encouragement throughout.

Abstract

Attribute control charts are used extensively in many industries to detect assignable causes for many processes. They are particularly useful in the service industries due to difficulty of evaluating services using variable scale. In addition, several critical-to-quality characteristics in manufacturing and service can be combined to determine whether to accept or reject the product. The optimization of fraction non-conforming *p*-chart has been mainly addressed from either statistical or economic prospective or considering only single assignable cause. In this research, we propose an economic-statistical model that considers the process history of the nonconforming units to design a *p*-chart for processes with multiple assignable causes. The method is demonstrated using a drinking water bottling case and shows improved results compared to existing methods. When comparing the results of the proposed method with traditional methods, the proposed method is expected to reduce poor quality cost by 0.86% per unit. For a mass production company such as the water bottling company with half a million bottles filled every day, the proposed method is expected to provide significant monetary savings along with improved reputation.

Keywords: ***nonconforming chart, p-chart, optimization***

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ARL	Average Run Length
ASI	Adaptive Sampling Interval
ASS	Additive Sampling Size
ATS	Average Time to Signal
CPU	Cost Per Unit Produced
LCL	Lower Control Limit
MOESD	Multi-Objective Economic Statistical Design
SPC	Statistical Quality Control
UCL	Upper Control Limit
VSI	Variable Sample Interval
VSS	Variable Sampling Size

Chapter 1. Introduction

Control chart is one of the most powerful tools used in Statistical Process Control (SPC). It is one of the seven basic tools which include histogram, check sheet, Pareto chart, cause-and-effect diagram, defect concentration diagram and the scatter diagram. SPC tools are mainly used to detect abnormal behavior of processes caused by assignable causes, and to improve process capability through the reduction of variability.

In almost any production process, a certain level of inherent natural variability will always exist, which results from the cumulative effect of small unavoidable causes. One the other hand, there is another type of variability that arises mainly from three sources: improperly adjusted machines, operator errors, or non-conforming raw material. SPC aims at detecting these assignable causes of variation, so that corrective actions are taken before many non-conforming units are manufactured [1].

The concept of control charts was first introduced by Walter Shewhart in 1931. He proposed a variable control chart the monitors the mean of the quality characteristic [2]. Later, many researchers have proposed different types of control charts, which can be classified into two categories: variable control charts and attribute control charts.

A typical control chart is shown in Figure 1. It consists of three parameters: a center line (CL), an upper control limit (UCL), and a lower control limit (LCL).

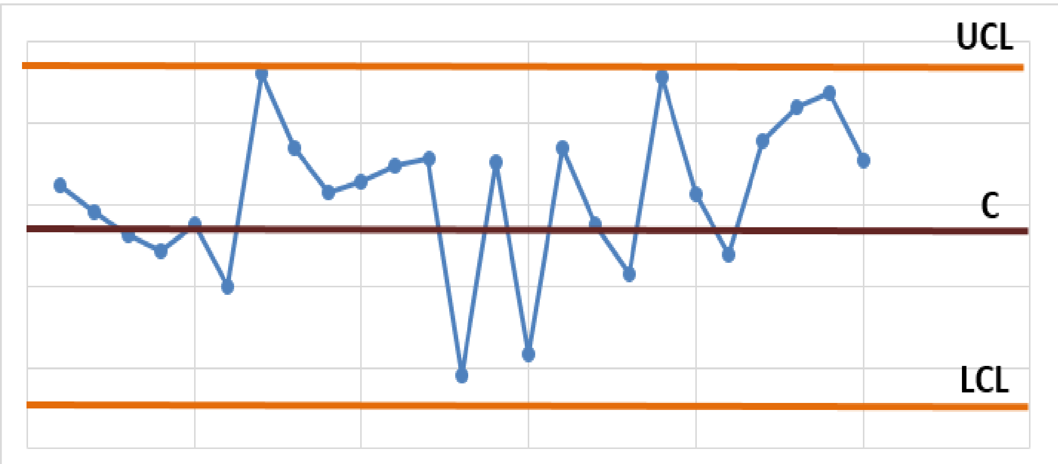

Figure 1: Typical Control Chart

If a point exceeds the control limit, an investigation should be carried out, and correction must be taken if a true assignable cause is found. Otherwise, the process is considered to be in control. Moreover, a heuristic introduced by Shewhart known as Western Electric Rules, is used to indicate the possibility of a shift from the sequence and pattern of the plotted points.

1.1. Variable and Attribute Control Charts

There are many types of variables control charts used in manufacturing and other business processes. Some of them are suitable for detecting moderate to high processes shifts like the $\bar{X}$-*R* chart, $\bar{X}$-*S* chart and *X-MR*. Moreover, other are applied when it is desired to detect small shifts, like the Exponentially Weighted Moving Average (EWMA) chart and the Cumulative Sum (CUSUM) control charts.

On the other hand, there are different types of attribute charts that are used when it is required classify the units as conforming and non-conforming like the *p*-chart and *np*-chart. Moreover, for processes that monitor multiple non-conformities per unit, the *c*-chart and *u*-charts are applied.

1.2. Fraction Non-conforming *p*-chart

The fraction non-conforming or percentage of defective is defined as the ratio of the number of non-conforming items in a population to the total number of items in that population. If the items do not conform to the standard of one or more pre-specified quality characteristics during the inspection, it is classified as non-conforming.

The fraction non-conforming *p*-chart is one of the most used types of attribute charts. It is particularly useful in the service industries and in transactional business processes because many of the characteristics in those fields are not easily measured on a numerical scale.

The statistical principles underlying the *p*-chart is based on the binomial distribution. Its structure consists of a center line, lower and upper limits. Figure 2 shows an example of a *p*-chart.

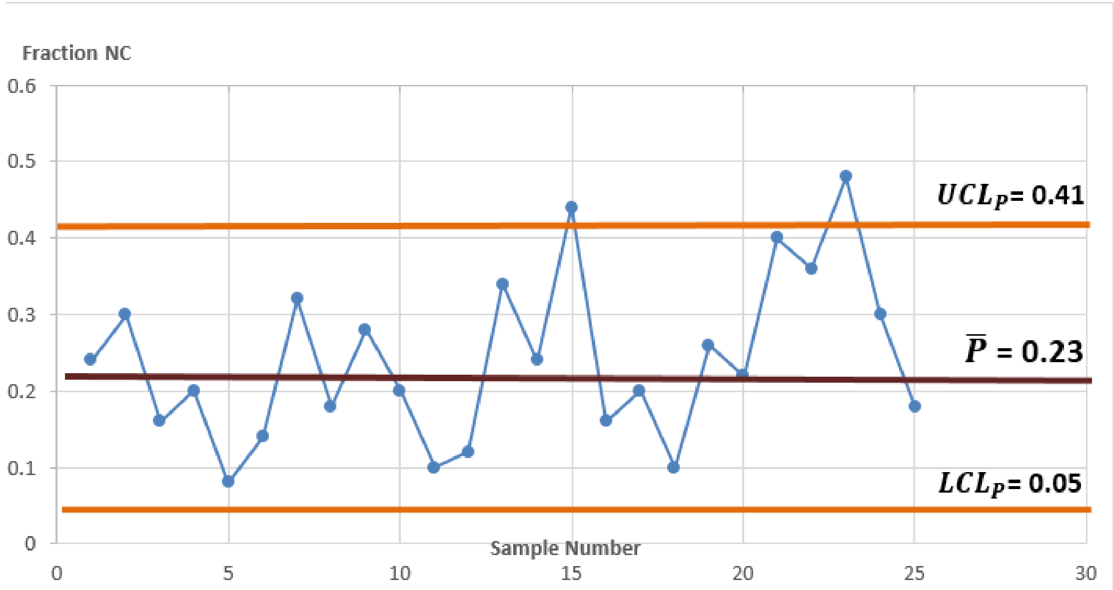

Figure 2: Example of a p-chart

The equations for constructing a p-chart:

$$\bar{p} = \frac{\sum_{i=1}^{m} p_i}{m} \tag{1.1}$$

$$UCL_p = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} \tag{1.2}$$

$$LCL_p = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} \tag{1.3}$$

where:

p_i ≡ fraction non-conforming at interval i

$\bar{p}$ ≡ average fraction of non-conforming based on m subgroups

n ≡ sample size within each subgroup

m ≡ number of subgroups

UCL_p ≡ Upper Control Limit for p chart

LCL_p ≡ Lower Control Limit for p chart

The two main decision variables in the design of p-chart are n and m. Ideally, more frequent samples with large sample size will detect assignable causes fast. However, such approach is expensive. The economic design of control chart aims at determining the optimum sample size and sampling interval that corresponds with the minimum quality and production cost. However, the statistical design of control charts aims at finding the optimum sample size and sampling interval that corresponds with the desired Average Run Length (ARL), which is defined as the average number of subgroups taken before the process goes out of control. ARL will be discussed in the next section.

On the other hand, the economic-statistical design gives the optimum sample size and sampling interval that corresponds with the minimum cost, while having a constraint for the ARL.

1.3. Average Run Length (ARL)

One way to measure the sensitivity of a control chart is by using the Average Run Length (ARL). It is the number of subgroups taken before a point plots out of control. In another words, it is the number of subgroups inspected before flagging an – out of control state. The ARL is calculated as the reciprocal of probability of detecting the assignable cause or shift in average fraction of nonconforming. In control chart decision making context, ARL is the reciprocal of the probability of having a sample point out of control. The probability of detection is estimated as the complement of probability of incorrectly accepting the hypothesis of statistical control (i.e., type II or β error). The probability of type II error or not detecting the shift β which can be estimated using binomial distribution as shown in equation 1.6. Alternatively, another metric called the Average Time to Signal (ATS) is also applied, which is the time units that pass before a point plots out of control.

$$ARL = \frac{1}{1-\beta} \tag{1.4}$$

$$ATS = ARL\, h \tag{1.5}$$

$$\beta = \Pr\{\hat{p} < UCL \mid p\} - \Pr\{\hat{p} \leq LCL \mid p\} =$$
$$\Pr\{D < nUCL \mid p\} - \Pr\{D \leq nLCL \mid p\} \tag{1.6}$$

where:

$\beta \equiv$ probability of not detecting a shift

$D \equiv$ number of defectives in a subgroup sample n

$h \equiv$ Sampling interval

In order to reduce ATS, either control limits are tightened or number of samples (n) are increased or sampling interval is reduced. Tightening control limits will increase false alarms, flagging a special cause when there is none while increasing n or reducing h will increase sampling cost.

It is worth noting here that one should differentiate between ARL interpretation during two states: state of in control and state of out of control. If the process is in control, ARL_0 is:

$$ARL_0 = \frac{1}{1-\beta_0} = \frac{1}{\alpha} \tag{1.7}$$

However, if the process is out of control, then:

$$ARL_1 = \frac{1}{1-\beta_1} \tag{1.8}$$

A typical Shewhart control chart would have a minimum ARL_0 of 270 and a low ARL_1. This means that the control chart will have a false alarm every 270 samples when process is in control and few samples when process is out of control.

Despite the wide spread of *p*-chart in service and manufacturing, few attempts were made to design and select the optimal settings of chart parameters. The objective of this research is to propose a method to select the optimum sample size *n* and sampling interval *h* for the *p*-chart using economical-statistical optimization design. The method considers both ability of chart to flag faulty states early on and minimizes total cost as well. The remainder of this report is as follows: Chapter two summarizes surveyed literature while chapter three outlines the problem on hand. Chapter four outlines the proposed method and model while a proof of concept and a case study are presented in chapters five and six respectively.

Chapter 2. Literature Review

Control chart are one of the main Statistical Quality Control (SPC) tools that is used to monitor and improve process stability and reduce variability. Several researchers proposed different ways to design control charts. Usually designs of charts addressing ARL criterion only are called statistical designs which mainly result in minimizing false alarm rate. Alternatively, economical design results in charts that minimize poor quality and sampling costs associated with the process. As a result, few researchers used a hybrid statistical economic design that accounted for both costs and ARL. Various assumptions have been used by different authors. One of the major assumptions is that process experiences a shift either due to single or multiple assignable causes. The multiple assignable causes assumption is more realistic than single.

2.1. Design of Variable Control Charts

Following Shewhart [2] introduction of the concept of control charts, many researchers built on his work and provided models for an efficient design of these charts. Most of these models focused on obtaining the optimal statistical properties of these charts, like the ARL and ATS without consideration of cost associated with these designs.

Since then, many researchers have addressed the design of control chart from an economic prospective, as the statistical design does not take into consideration the costs of sampling, treating the assignable causes or the cost of producing defective items [1]. Duncan [3] was the first to address the design of control charts from an economic prospective [1]. He proposed a model for the design of $\bar{X}$ charts that obtains the optimum design parameters by maximizing the average net income of the process.

Table 1 summaries surveyed literature focused on variable control charts based on chart type and design criteria.

Lorenzen and Vance [4] elaborated on Duncan's [3] work and derived the following function expected cost per time, that is applicable to different types of control charts.

Table 1: Summary of review papers on variable charts

No	Chart type	Authors	Statistical	Economic	Economic -statistical	single assignable cause	Multiple assignable causes	Method
1	EWMA	Borror et al. 1998	√			√		Markov Chain/Poisson distribution
2		Montgomery et al. 1995			√	√		Minimizing cost function
3		Amiri et al. 2015			√	√		robust design using genetic algorithm
4		Linderman and Love 2000			√	√		Multivariate model
5	X chart	Duncan 1956		√			√	Maximizing net income of the process.
6		Lee et al. 2016			√	√		Surrogate variable using GA/VSI model
7	X & R chart	Saniga 1989			√	√		Minimizing expected cost per hour
8		Bakir and Altunkaynak 2004			√		√	Genetic Algorithm/Multi-objective model
9	X-R & X-S charts	Sangia and Davis 2001			√	√		FORTRAN Program
10	CUSUM	Goel and Wu 1973		√		√		Long-run average cost/Pattern-search technique
11		Pan and Chen 2005		√		√		Revised Inverted Normal Loss Function (RINLF)/Estimate the expected cost of pollution.
12	ML Chart	Wu et al. 2004			√	√		Taguchi's Loss Function

Lorenzen and Vance [4] model considered the following types of costs:

$$E(C) = \frac{E(C_1) + E(C_2) + E(C_3)}{L} \quad (2.1a)$$

where:

$E(C_1) \equiv$ Expected cost per cycle due to nonconformities

$E(C_2) \equiv$ expected cost of false alarm and locating and repairing the true assignable cause

$E(C_3) \equiv$ the expected cost per cycle for sampling

$L \equiv$ the expected length of a production cycle

The costs identified in equation (*2.1a*) were estimated in [4] by the corresponding terms:

$$E(C) = \frac{\left\{\frac{C_0}{\lambda}+C_1(-\tau+nE+h(ARL2)+\delta_1 T_1+\delta_2 T_2)\right\}+\left\{\frac{sY}{ARL1}+W\right\}+\left\{\frac{(a+bn)\left(\frac{1}{\lambda}-\tau+nE+h(ARL2)+\delta_1 T_1+\delta_2 T_2\right)}{h}\right\}}{\frac{1}{\lambda}+\frac{(1-\delta_1)sT_0}{ARL1}-\tau+nE+h(ARL2)+T_1+T_2} \tag{2.1b}$$

where:

$C_0 \equiv$ Quality cost while producing in control

$C_1 \equiv$ Quality cost while producing out of control ($C_1 > C_0$)

$\lambda \equiv 1/$mean time process is in control

$\tau \equiv$ Expected time for the occurance of the assignable cause

$n \equiv$ Sample size

$E \equiv$ Time to sample and chart one item

$h \equiv$ Sampling interval

$ARL_1 \equiv$ average run length while in control $(1/\beta)$

$ARL_2 \equiv$ average run length while process out of control $(1/1-\beta)$

$\delta_1 = \begin{cases} 1 \text{ if production continues during searches} \\ 0 \text{ if production ceases during searches} \end{cases}$

$\delta_2 = \begin{cases} 1 \text{ if production continues during repair} \\ 0 \text{ if production ceases during repair} \end{cases}$

$T_1 \equiv$ Expected time to discover the assignable cause

$T_2 \equiv$ Expected time to repair the process

$s \equiv$ Expected number of samples taken while process in control

$Y \equiv$ Cost per false alarm

$W \equiv$ Cost to locate and repair the assignable cause

Lorenzen and Vance assumed that the process experiences only a single assignable cause. In reality processes usually undergoes multiple assignable causes with different times of occurrences.

Saniga [5] pioneered the economic statistical design by proposing a model for the design of $\bar{X}$ and R charts, in which he used constraints for type I error, power, and the ATS. Later on, Saniga and Davis [6] developed a comprehensive FORTRAN program for the economic statistical design of $\bar{X}$–R charts and $\bar{X}$–S charts, using Lorenzen and Vance's model. These two papers assumed a single assignable cause.

Nenes *et al.* [7] proposed a general model for the design of fully adaptive $\bar{X}$ Shewhart control chart for processes subject to multiple assignable causes, using Markov chain model. They considered the following costs in their design:

$$E(C) = \frac{E(C_1) + E(C_2) + E(C_3) + E(C_4)}{L} \tag{2.2a}$$

where:

$E(C_1) \equiv$ the expected cost of sampling
$E(C_2) \equiv$ the expected cost of operating in the out of control states
$E(C_3) \equiv$ the expected cost of false alarms
$E(C_4) \equiv$ the cost of restoring the process to the in control state
$L \equiv$ the expected cycle time

The costs identified in equation (2.2a) were estimated in [7] by the corresponding terms:

$$E(C) = \frac{[b+cn_1\sum_i(\pi_{i0}+\pi_{i2})]+[cn_2\sum_i\pi_{i1}]+[\sum_i(\pi_{i0}K_i(h_1)+\pi_{i1}K_i(h_2)+\pi_{i2}K_0(h_1))]+[\sum_i\pi_{i2}L_i]}{h_1\sum_i\pi_{i0}+\sum_i(\pi_{i2}(h_1+T_i))+h_2\sum_i\pi_{i1}} \tag{2.2b}$$

Where:

$b \equiv$ fixed sampling cost
$c \equiv$ variable sampling cost per unit
$n_1 \equiv$ relaxed sampling size
$h_1 \equiv$ relaxed sampling interval
$n_2 \equiv$ tightened sampling size
$h_2 \equiv$ tightened sampling interval
$\pi_{ik} \equiv$ Steady state probabilities; $i \in \{0,1,\ldots,m\}$ represents the current state of the process and $k \in \{0,1,2\}$ represents the decision that should be taken.

$K_i \equiv$ Mean cost of operating under the effect of an assignable cause, given that the process is under the effect of assignable cause *i* at the beginning of a sampling interval.
$L_i \equiv$ cost of removing assignable cause i
$T_i \equiv$ time to search and remove assignable cause i

Unlike the previously discussed models, the above model considers the cost when it is possible for a process to shift from an already detected assignable cause to another assignable cause. Although, the fully adaptive design of control charts results in higher cost savings than the fixed parameter chart, it is often complex and difficult to administer in reality.

Bakir and Altunkaynak [8] proposed a multi-objective economic statistical design (MOESD) model for the $\bar{X}$ and R charts, using a genetic algorithm, assuming multiple assignable causes. Lee *et al.* [9] developed an economic statistical design of variable sample interval (VSI) $\bar{X}$ chart based on surrogate variable and using a genetic algorithm, assuming a single assignable cause.

A statistical design for the EWMA was proposed by Borror *et al.* [10] for processes that employ the Poisson distribution, using Markov chain approach. They concluded that the ARL for the Poisson EWMA chart is smaller than that of the *c*-chart.

Goel and Wu [11] provided a model for the design of CUSUM control charts, that gives the long-run average cost of the process, using the pattern-search technique. Pan and Chen [12] developed an economic design of the CUSUM chart to monitor environmental performance and estimate the expected cost of pollution, using the Revised Inverted Normal Loss Function (RINLF) for unilateral specification:

$$L(y) = \begin{cases} 0 & 0 \le y \le U \\ A\{1 - \exp\left(-\frac{(y-U)^2}{2\sigma_L^2}\right)\} & y > U \end{cases} \tag{2.3}$$

where:

$A \equiv$ the maximum loss if the process mean is deviated from the target
$U \equiv$ the upper bound under which no loss in incured
$\sigma_L \equiv$ the shape parameter of the loss function of the RINLF for unilateral specification

Both of these papers addressed the design of CUSUM charts assuming a single assignable cause.

Similarly, Montgomery *et al.* [13] presented a statistically constrained economic model for the design of EWMA control chart by minimizing the cost function proposed by Lorenzen and Vance [4]. Linderman and Love [14] have provided an economic statistical model for the multivariate EWMA chart, using also Lorenzen and Vance's [4] model. Amiri *et al.* [15] built on Montgomery's [13] work by developing a robust design for the statistical-economic model for EWMA control chart by using a genetic algorithm as an optimization method. All of these three papers assumed a single assignable cause.

Wu *et al.* [16] have proposed an economic statistical design by minimizing the overall mean of Taguchi's loss function for what they called an ML control chart, with constraints to the ATS_0 and the inspection rate, assuming a single assignable cause.

2.2. Design of Attribute Control Charts

The attribute control charts are very popular in practice, specifically the fraction non-conforming *p*-chart, since they could combine several failure modes, and they are easy to administer and implement. Hence, reduce the complexity and errors during inspection. Table 2 provides a summary of surveyed literature based on type of attribute charts and design criteria.

Ladany [17] pioneered the economic design of attribute control charts [18] by developing a model for the design of *p*-charts which minimizes the total cost within an interval:

$$TC = C_1 + C_2 + C_3 + C_4 \tag{2.4}$$

where:

$C_1 \equiv$ cost of sampling in the interval
$C_2 \equiv$ cost of not detecting a shift
$C_3 \equiv$ cost of false alarm
$C_4 \equiv$ cost of correction

Table 2: Summary of reviewed papers on attribute charts

No.	Chart type	Authors	Statistical	Economic	Economic -statistical	single assignable cause	Multiple assignable causes	Method
1	c-chart	Inghilleri and Lupo 2015		√				Double sampling
2		Lupo 2014			√	√		Multi-objective design/Taguchi's loss function
3	np-chart	Wu and Luo 2004	√					Based on ASI & ASS/Optimizing ATS
4		Jolayemi 2002	√					Multiple control regions
5		Kooli and Limam 2011		√		√		Based on VSS/Adaptive design
6		Kooli and Limam 2015		√		√		Based on VSI/Adaptive design
7		Kooli and Limam 2009		√		√		Based on VSS/Bayesian model
8		Gibra 1978		√		√		Minimizing expected cost per unit
9		Gibra 1981		√			√	Minimizing expected cost per unit
10		Chiu 1976		√			√	Grid search method
11		Chung 1995		√			√	Algorithm for solving Chiu's model
12		Williams et al. 1985		√				Curtailed sampling
13		Wang and Chen 1995			√			Fuzzy-set theory
14		Bashiri et al. 2013			√	√		Data Envelopment Analysis (DEA)/Multi-objective model
15		Collani 1989		√		√		Maximizing profit/Poisson distribution
16	p-chart	Ladnay 1973		√		√		Minimizing expected cost per time
17		Ladnay 1976		√				Optimal setup policy
18		Cozzucoli 2009	√					Two-sided multivariate p chart
19		Abooie and Nayeri 2009	√			√		Heuristic using Bayesian rule
20		Montgomery et al. 1975		√			√	Minimizing cost per unit
21		Duncan 1978		√		√		Minimizing expected cost
22		Kethley and Peters 2004		√		√		Genetic Algorithm
23		Calabrese 1995		√		√		Bayesian Rule
24		Sangia and Davis 1995			√			FORTRAN Language
25		Gunay and Kula 2016		√				Sample Average Approximation Algorithm (SAA)
26		Makis et al. 2016		√		√		An algorithm using Bayesian Rule, semi-Markov chain processes.
27		Namin and Hasanzada 2016			√	√		Cornish-Fisher Expansions and Non-linear Lexicography programming NLGP
28	General model	Lorenzen & Vance 1986		√		√		Minimizing expected cost function
29		Nenes et al. 2015			√		√	Markov Chain/Adaptive model

Similarly, Ladany and Bedi [19] presented a model for the selection of the optimal setup policy for a process that uses the *p*-chart, based on the minimum expected cost per unit of time.

Montgomery *et al.* [20] developed an economic model for the design of p-chart assuming a multiplicity of assignable causes, through the minimization of the following expected cost function per unit of product:

$$E(C) = E(C_1) + E(C_2) + E(C_3) \tag{2.5a}$$

where:

$E(C_1) \equiv$ the expected cost of sampling and testing per unit
$E(C_2) \equiv$ the unit cost of investigation and correction when process is out of control
$E(C_3) \equiv$ the expected cost per unit of producing defective product

The costs identified in equation (2.5a) were estimated in [20] by the corresponding terms:

$$E(C) = \frac{a_1 + a_2 n}{k} + \left(\frac{a_3}{k}\right)\sum_{i=0}^{s} q_i\, \alpha_i + a_4 \sum_{i=0}^{s} p_i \gamma_i \tag{2.5b}$$

where:

$a_1 \equiv$ fixed cost of sampling
$a_2 \equiv$ variable cost of sampling
$n \equiv$ sample size
$k \equiv$ number of units produced between the first items included in the successive samples.
$a_3 \equiv$ cost of investigating, correcting the process plus the cost of lost production
$q_i \equiv$ conditional probability that process is out of control when it is at state p_i
$\alpha_i \equiv$ probability the process is in state p_i when taking the sample
$a_4 \equiv$ the penalty cost of producing a defective unit
$p_i \equiv$ fraction defective corresponding with state i
$\gamma_i \equiv$ probability that the process in state p_i at any point in time

It is worth noting that Montgomery et al. model does not consider the cost of investigating false alarms, which for some processes might be significant. Moreover, they assumed that when the process is out of control due to an assignable cause, it remains free from the effect of other assignable causes, which is unrealistic. If the process remains out-of-control for a significant amount of time, other faults may arise and is often associated with costs. In addition, Montgomery *et al.* model used a state matrix describing the probability of transitioning from one state to another. Such matrix is complex and requires prior knowledge of processes under investigation. Moreover, their model is purely economical, and does not consider any constraint for important statistical parameters such as the ARL.

Chiu [21] developed an economic model for the design of *np*-charts, for a process that involved multiple assignable causes. The optimal design parameters are obtained through the maximization of the following function of the expected net profit per production cycle:

$$\frac{E(P_1) + E(P_2) - E(C_1) - E(C_2) - E(C_j)}{L} \tag{2.6a}$$

where:

$E(P_1) \equiv$ the expected profit when the process is in control
$E(P_2) \equiv$ the expected profit when the process is out of control
$E(C_1) \equiv$ the expected cost of investigating a false alarm
$E(C_j) \equiv$ the expected cost of investigating a true alarm assignable cause j ($j = 1,2,\dots,k$)
$E(C_3) \equiv$ the expected cost of sampling
$L \equiv$ the expected length of a production cycle

The costs identified in equation (2.6a) were estimated in [21] by the corresponding terms:

$$E(P) = \frac{V_0 + h\sum\lambda_j V_j B_j - hA_0B_0 - h\sum\lambda_j A_j - h(b+cn)(1+\sum\lambda_j B_j)}{1+\sum\lambda_j B_j + t_0B_0 + \sum\lambda_j t_j} \tag{2.6b}$$

where:

$V_0 \equiv$ Average profit from process when in control

$h \equiv$ Sampling interval

$\lambda_j \equiv$ Occurance rate of assignable cause j

$V_j \equiv$ Average profit from process when shifting to out of control state due to assignable cause j

$B_j \equiv$ Time before a true alarm is signalled due to assignable cause j

$A_0 \equiv$ the cost of investigating a false alarm

$B_0 \equiv$ Expected number of false alarms when process is in control

$A_j \equiv$ The cost of detecting and eliminating assignable cause j

$b \equiv$ Fixed cost of sampling

$c \equiv$ Variable cost of sampling

$n \equiv$ Sample size

$t_0 \equiv$ The average time needed for the search of a false alarm

$t_j \equiv$ The average time needed for detecting and eliminating assignable cause j

Chiu's model considers the cost of false alarms but fails to include the cost of producing defective items, which is often quite significant. Moreover, like Montgomery *et al.* the above model assumes that the process when out of control, remains under the effect of only one assignable cause, which is often unrealistic and associated with costs. Number of defective or non-conforming when the sample size is constant is known as np charts. Gibra [22] proposed two models for the economic design of *np*-charts for the case when process ceases during the search for a single assignable cause, and also for the case when the process continue to operate while the search for the assignable cause. Later, Gibra [23] developed a model for the design of *np*-charts by minimizing the following function of the expected total cost per unit time, when the process continues to operate during the search and assuming multiple assignable causes:

$$\frac{E(C_1) + E(C_2) + E(C_3) + E(C_4)}{L} \tag{2.7a}$$

where:

$E(C_1) \equiv$ the expected cost of inspection and charting

$E(C_2) \equiv$ the expected cost of searching due to false alarm

$E(C_3) \equiv$ the expected cost of search for true alarm, downtime and repair

$E(C_4) \equiv$ the cost incurred due to a higher rate of defectives when process is out of control

$L \equiv$ the expected cycle time

The costs identified in equation (2.7a) were estimated in [23] by the corresponding terms:

$$E(C) = \frac{(h+bn)\left[\frac{1+\sum \lambda_j \phi_j}{\lambda v}+\frac{\tau}{v}\right]+\left[\frac{s\alpha\tau}{[\exp(\lambda v)-1]}+st\right]+\sum\frac{\lambda_j t_j(d+c_j)}{\lambda}+\sum ur(p_j-p_0)(\phi_j+\tau)\lambda_j/\lambda}{\frac{1}{\lambda}+\sum(\lambda_j/\lambda)(\phi_j+t_j)+\tau} \quad (2.7b)$$

where:

$h \equiv$ Overhead cost of maintaining a np chart per inspected sample

$b \equiv$ Cost of inspection and charting per unit sampled

$n \equiv$ Sample size

$\lambda_j \equiv$ Occurance rate of assignable cause j

$\lambda \equiv \sum \lambda_j$

$\phi_j \equiv Expected\ time\ the\ process\ is\ out\ of\ control\ due\ to\ assignable\ cause\ \ j$

$v \equiv$ Sampling interval

$\tau \equiv$ Expected search time

$s \equiv$ Cost per unit time of searching for an assignable cause

$\alpha \equiv$ Probability of false alarm when process in control

$t \equiv$ Expected repair time of an assignable cause

$t_j \equiv$ Expected repair time due to the ocuurance of assignable cause j

$d \equiv$ cost of downtime per unit time

$c_j \equiv$ cost of repair per unit time due to the occurance of assignable cause j

$u \equiv$ penalty cost incured per defective item

$r \equiv$ production rate per unit time

$p_j \equiv$ Fraction defective when the process is out of control due to assignable cause j

$p_0 \equiv$ Fraction defective when the process is in control

Unlike the previously discussed models, Gibra's model considers the cost of operating an np-chart and also the cost of repair. However, it assumes that when the process is out of control, it remains under the effect of only one assignable cause and cannot shift further, which is often not realistic. In the same paper, Gibra [23] developed another model similar to the above-mentioned, where the process continues to operate during the search of assignable causes.

Chung [24] applied an algorithm for the economic design model of *np*-charts that was proposed by Chiu [21]. The algorithm overcomes some drawbacks in Chiu's model, makes it easy to solve the model and provide more accurate results. Similarly, Collani [25] developed a model for the economic design of both *c*- and *np*-charts, by maximizing the process profit and based on the Poisson distribution. Goh [26] developed an alternative charting technique for the *p*-chart, to be implemented in processes that have low-defective rate. Saniga and Davis [27] presented an economic-statistical model for the design of *p*-charts, that is optimized using FORTRAN language. Wang and Chen [28] have presented a statistical economic design of *np*-charts, using Lorenzen and Vance model and Chung results by using fuzzy-set theory as an optimization method.

Some researchers addressed the design of attribute charts using the Bayesian rule. For example, Kooli and Limam [29] economically optimized a Bayesian *np*-chart with a variable sample size for finite production runs, based on the probability that the process is operating under the out of control state. Similarly, Calabrese [30] developed a Bayesian model for the economic design of *p*-charts, that is based on the posterior probability of a process shift, which optimizes the control limit at any period during the process. Similarly, Abooie and Nayeri [31] proposed a heuristic for detecting small shifts in the *p*-chart, using the Bayesian rule. Makis *et al.* [32] developed an algorithm for the economic design of Bayesian *p*-charts, using semi-Markov chain processes. Both of these models use the prior probability of an out of control state, in order to update the posterior probability of a process being out of control.

Recently, some researchers proposed adaptive models for the design of attribute charts. For example, Wu and Luo [33] optimized the ATS for the *np*-chart using an Adaptive Sampling Interval (ASI) and an Adaptive Sample Size (ASS), which improve the effectiveness significantly in detecting small or moderate process shifts. Kooli and Limam [34] & [35] have developed an adaptive economic design of *np*-charts based on variable sampling size (VSS) and variable sampling interval (VSI), respectively. However, there are some practical difficulties when it comes to adaptive charts, as they are usually complex and hard to administer in reality.

Different approaches for the design of attribute charts have been proposed. For example, Jolayemi [36] developed a model for the statistical design *np*-chart that has multiple control regions. Cozzucoli [37] presented a two-sided multivariate *p*-chart to monitor the defined categories of non-conformities in the process.

Williams *et al.* [38] have presented an economic model for the design of *np*-charts based on curtailed sampling, which is based on the premise that sampling can be terminated once enough information has been acquired to render a decision. Their sampling plan is that the randomly selected items are inspected one at a time until either m defectives are observed or until n items are inspected. They showed that their model results in higher cost saving that the traditional models. However, curtailed sampling models are often complex and difficult to apply in practice.

Duncan [39] presented another model for the economic design of *p*-charts, similar to the model that he proposed earlier for the $\bar{X}$ chart. His model focused on minimizing the expected cost of the process, assuming a single assignable cause.

Recently new types charts were also proposed. Chan *et al.* [44] introduced the cumulative quantity control chart (*CQC*-Chart), and showed that it is more suitable than the *c*- or *u*-charts when process defect rate is low or moderate. Similarly, Bashiri *et al.* [40] developed an MOESD of *np*-charts by applying the Data Envelopment Analysis (DEA) as an optimization approach, and using Duncan's [39] model, which assumes a single assignable cause. Kethley and Peters [41] optimized the economic model of the loss-cost function presented by Duncan [39], for the design of *p*-chart using a genetic algorithm.

Namin and Hasanzadeh [42] have proposed an MOESD for the *p*-chart based on Cornish-Fisher expansions and by applying nonlinear lexicography goal programming (NLGP), assuming a single assignable cause.

Finally, Inghilleri *et al.* [45] proposed a double sampling scheme for the design of *c*-charts that resulted in a reduction for the number of observations required. Lupo

[46] has developed an economic statistical multi-objective model for the design of *c*-charts that incorporated the labor cost and included a constraint for the ATS_0:

$$\min C_T\,(n, h, k)$$
$$\min ARL_\delta\,(n, h, k)$$

Subject to:

$$ARL_0 \geq ARL_L$$
$$LR_{min} \leq LR \leq LR_{max}$$
$$h, k > 0$$
$$n\ integer \tag{2.8}$$

where:

$C_T \equiv$ Hourly total quality and labor costs
$ARL_0 \equiv$ In control average run length
$ARL_\delta \equiv$ Out of control average run length
$ARL_L \equiv$ Lower limit of the average run length
$LR \equiv$ Capacity of labor resource
$n \equiv$ Sample size
$h \equiv$ Sampling interval
$k \equiv$ Control limit width

He used the ε-constraint method and Taguchi's loss function to solve the model. However, he assumed a process with a single assignable cause, which is often not realistic.

Montgomery *et al.* [20] assumed that the values of the input parameters π (index for the probability distribution of shifting from 0 to state i) and λ (the average occurrence rate of the assignable causes) used for equation (*2.5b*) are given. They did not show how a method to be used to obtain these values.

In [20] they calculated the probability that the process remains in control, assuming a an exponential distribution:

$$P_{00} = e^{-\lambda k/R} \tag{2.9}$$

But, in equation (2.9) the distribution is memory-less, as it does not take into consideration the history of the process. Also, the same was used in calculating γ_i –the probability that the process is out of control at any point in time –without considering the previous information of the process being out of control.

Moreover, [20] did not include any statistical constraints for the ARL in their economic model. This would actually make the process subjected to frequent false alarms, which could increase the process variability. Woodall [43] pointed out some weaknesses of the economic design of control charts. He noted that the economically designed charts have considerably higher false alarm rates than those designed purely on statistical basis.

Calabrese [30] proposed a model to optimize the p-chart by minimizing the expected cost that is based on the posterior probability that the process is out of control. However, his model does not present a method to calculate the posterior probability that the process is out of control given the previous information of the non-conformities recorded. Moreover, like [20], it does not include statistical constraints for the ARL, and assumes a process with a single assignable cause.

2.4. Concluding Remarks

To our knowledge, very little research, if any, is done on economic-statistical design of *p*-charts for processes with multiple assignable causes. Moreover, few researches have considered taking into account the previous history of the process being out of control, in the design of control charts.

The multi-objective economic statistical design of control charts has been recently used by many researchers through the application of goal programming techniques, which offer flexibility in adjusting the ranges of the design parameters, as well as ranking the objectives as desired by the decision-maker.

Chapter 3. Problem Statement

The process of process p-chart design includes the determination of sampling size and sampling interval. The impact of this decision is critical since it impact the efficiency of the process, as the sampling process is both costly and sometimes requires production interruption. Hence, there is a need for design approaches that result in the optimum size of sample and interval length.

Many researches tried to optimize the design of p-charts by solely minimizing the quality and production costs. However, there have been a growing need to consider the statistical parameters like the ARL, as pure economic designs increased the false alarm rate. Moreover, the complexity of modern processes manifested in diverse assignable causes, which puts another reason for considering the design of p-charts considering multiple out-of-control states.

3.1. Research Objective

The objective of this research is to develop an economic-statistical Bayesian model for the design of p-charts for processes with multiple assignable causes. Moreover, there are a set of specific objectives that will be targeted in order to achieve the main goal. It starts by identifying an optimization algorithm to be used for solving our proposed model, followed by coding the model formulation using GAMS software. Also, a numerical study will be conducted to verify the model, and finally a sensitivity analysis will be performed to test how sensitive the output parameters are, to changes in the inputs.

3.2. Research Motivation and Contribution

It is usually the case that many processes undergo multiple out of control states. Moreover, economic models have resulted in less quality and production cost compared with the statistical models, but they usually make the processes subjected to increased rate of false alarms. So, to tackle this problem, we propose an economic-statistical model for the design of p-charts, for processes with multiple assignable causes. In addition, we will include the process history of the quantity non-conforming recorded at different intervals in order to determine the probability that the process is out of control.

Chapter 4. Proposed Methodology

In this research, we present a model that is used to optimally design a *p*-chart for processes with multiple assignable causes. Unlike the model shown in [20], the proposed model considers a finite production horizon, constraints for the ARL and a sampling frequency in units of time rather than in units produced. Moreover, our model incorporates the history of the process.

4.1 Assumptions

Several assumptions were made to simplify and manage the implementation of the proposed methodology:

1. The sample size used for inspecting the quality of the product or process n is constant.
2. The number of standard deviations of fraction nonconforming is fixed to 3 when developing the upper and lower control limits. This assumption will result in fixed control limits which is easier to understand and implement in the shop floor.
3. Process can shift from in control to out of control due to multiple assignable causes.

During an interval – the process can shift from any assignable cause i to any assignable cause j, but it does not return to the in-control state without intervention. The probability of shifting from assignable cause i to assignable cause j is estimated using equation 4.1:

$$p_{ij} = p_i \quad \frac{p_j}{p_i + p_j} \tag{4.1}$$

where:

$p_i \equiv$ the probability of occurrence of assignable cause i

4. Using the same approach in [3], the expected time for the occurrence of an assignable cause τ within an interval h can be expressed by:

$$\tau = \frac{\int_{jh}^{(j+1)h} e^{-t}(t - jh)dt}{\int_{jh}^{(j+1)h} e^{-t}\, dt} = \frac{1 - (1 + h)e^{-h}}{(1 - e^{-h})} \tag{4.2}$$

The probability of false alarm when the system is in control (no shift) and the average $\bar{P} = P_0$ is estimated using the binomial distribution as shown in equation 4.3:

$$\alpha_i = 1 - \Pr\{[C_{nUCL}^{n} p_i^{nUCL}(1-p_i)^{(n-nUCL)} - C_{nLCL}^{n} p_i^{nLCL}(1-p_i)^{(n-nLCL)}] \mid \bar{P} = P_0\} \tag{4.3}$$

where:

$\alpha_i \equiv$ the probability of false alarm, i.e. plotting a sample outside the control limits while the process is in control ($\bar{P} = P_0$)

Alternatively, the p-chart may fail to detect the presence of cause i. The probability of failing to detect the cause when it happens is β_i:

$$\beta_i = \Pr\{[C_{nUCL}^{n} P^{nUCL}(1-P)^{(n-nUCL)} - C_{nLCL}^{n} P^{nLCL}(1-P)^{(n-nLCL)}] \mid P = P_i\} \tag{4.4}$$

Figure 3 shows the process stages of sampling and detection. The process starts from in-control state and decisions are made after sampling and inspection every h period. If production rate is R, then number of units produced between samples is hR. It is assumed that time process will take to switch to out of control is state follows an exponential distribution with an average τ. As a result number of defective units produced due to cause i is $R(h - \tau)$ units. Finally it is assumed that time it taken for sampling and investigation to detect cause is g.

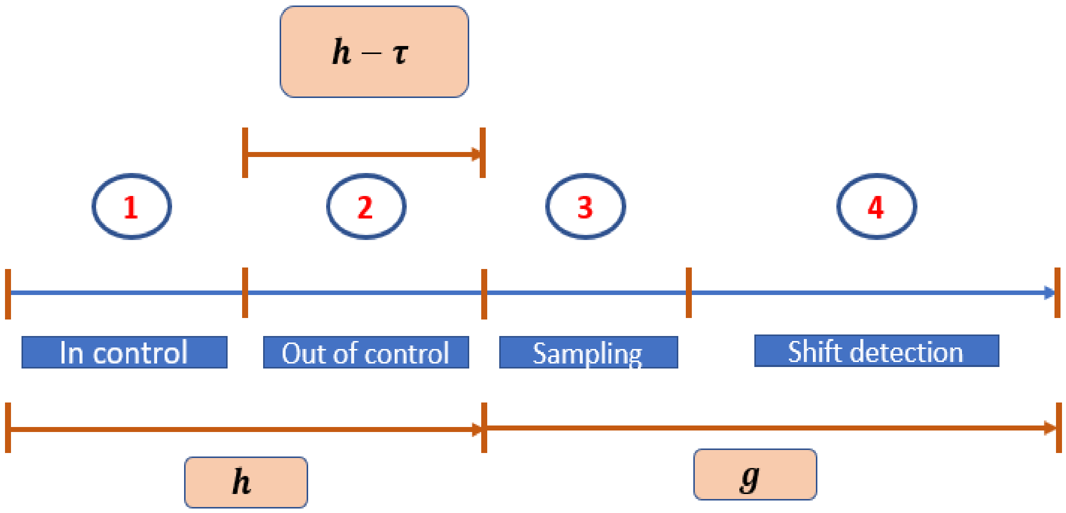

Figure 3: Process stages of sampling and detection of assignable causes

Keep in mind that the sample inspected may fail to detect the shift which in this case leads to an external cost of poor quality. In this case, the defective parts produced will be released to customer. Such cost is hard, i.e. monetary in addition to soft manifesting itself to reputation damage. Since soft cost may transfer into hard one in terms of liability or complaints or reduction of sales, the total external cost is higher than cost of detecting a defective product and fixing it. The latter is referred to as internal cost.

4.2 Objective Function

In this model, we are considering five types of costs, and the objective is to minimize the total cost per unit:

$$Min\ E(C_1) + E(C_2) + E(C_3) + E(C_4) + E(C_5) \tag{4.5}$$

where:

$E(C_1) \equiv$ the cost of sampling per unit produced

$E(C_2) \equiv$ expected cost of producing defective items when process is out of control

$E(C_3) \equiv$ the expected cost of investigating and correcting a true alarm.

$E(C_4) \equiv$ the expected cost of not detecting a shift of an occurrence of true failure, i.e. external cost

$E(C_5) \equiv$ the expected cost of false alarm

The sampling cost per unit produced is expressed by:

$$\frac{(a_1 + a_2 n)}{hR} \tag{4.6}$$

where:

$a_1 \equiv$ fixed cost of sampling

$a_2 \equiv$ variable cost of sampling

$n \equiv$ sample size

The cost of producing defective items per interval, when the process is out of control is expressed by:

$$\frac{\sum_{i}^{s}\sum_{j=1}^{s} b_{ij}\, p_{ij}\left(h-\tau_j\right)}{hR} \tag{4.7}$$

where:

$b_{ij} \equiv$ cost of producing a defective unit due to shifting from state i to state j

The expected cost of investigating and correcting a true alarm (cause is detected) per interval is expressed by:

$$\frac{\sum_{i=1}^{s} w_i(1-\beta_i)}{hR} \tag{4.8}$$

where:

$w_i \equiv$ cost of investigating and correcting a true alarm due to assignable cause i

$1-\beta_i \equiv$ the probability of detecting a shift of a true alarm due to assignable cause i

The expected cost of releasing a defective product to the customer. In this case, the system shifted due to cause i and the p-chart fail to detect this shift. This cost can be expressed:

$$\frac{\sum_{i=1}^{s} z_i\, \beta_i\, (h-\tau_i)}{hR} \tag{4.9}$$

where:

$z_i \equiv$ cost of releasing one defective product due to assignable cause i to customer

The expected cost of investigating a false alarm per interval is expressed by:

$$\frac{\sum_{i=1}^{s} c_i\, \alpha_i}{hR} \tag{4.10}$$

where:

$c_i \equiv$ cost of investigating and correcting a false alarm due to assignable cause i and

$\alpha_i \equiv$ false alarm rate due to cause i

4.3 Constraints

The model includes a constraint for ARL_i which is the Average Run Length when the process is out-of-control:

$$ARL_{1i} = \frac{1}{1-\beta_i} \geq u \quad \forall\, i = 1,2,\dots,s \tag{4.11}$$

where:

$u \equiv$ minmum number of points after which the chart plots a point out of control

Additionally, another constraint for the sample size taken in each interval, which should be at least one sample, and less than or equal to the quantity produced during the interval hR:

$$1 \leq n \leq hR \tag{4.12}$$

Moreover, adding a constraint for the sampling interval to be greater than time it takes to inspect samples

$$h \geq g \tag{4.13}$$

$g \equiv$ the time taken for sampling and inspection

As a result, the model can be summarized as follows:

$$\frac{(a_1 + a_2 n)}{hR} + \frac{\sum_i^s \sum_{j=1}^s b_{ij}\, p_{ij}\big(h - \tau_j\big)}{hR} + \frac{\sum_{i=1}^s w_i (1-\beta_i)}{hR} + \frac{\sum_{i=1}^s z_i\, \beta_i\, (h - \tau_i)}{hR} + \frac{\sum_{i=1}^s c_i\, \alpha_i}{hR} \tag{4.14}$$

Subject to:

$$ARL_{1i} = \frac{1}{1-\beta_i} \geq u \quad \forall\, i = 1,2,\dots,s$$

$$hR \geq n \geq 1$$

$$h \geq g$$

In the next chapter, the proposed model will be verified using a published case study. The results of the proposed model will be compared with the results of another model.

Chapter 5. Proof of Concept

In the paper by Montgomery *et al.* [20], their model is applied on a production process that is characterized by seven states, and described by the following parameters:

$p_i = [\ 0.01, 0.02, 0.04, 0.08, 0.16, 0.32, 0.64\ ]$

$a_1 = \$5$

$a_2 = \$0.1$

$a_3 = \$20$

$a_4 = \$10$

where:

$p_i \equiv$ fraction defective corresponding with state i

$a_1 \equiv$ fixed cost of sampling

$a_2 \equiv$ variable cost of sampling

$a_3 \equiv$ cost of investigating, correcting the process plus the cost of lost production

$a_4 \equiv$ the penalty cost of producing a defective unit

The resulting optimal solution for Montgomery's model, using a proposed search chart technique outlined in [20] is as follows:

$n = 14$

$k = 81$

$0.376 \leq L \leq 2.31$

$E(C) = \$0.35\ per\ unit\ produced$

where:

$L \equiv$ units of standard deviation

$k \equiv$ sampling interval in terms of units produced

As a proof of concept for the proposed model, we will apply it on the same production process that is used in Montgomery *et al.* [20]. It is worth noting the following differences between Montgomery's approach and our proposed model:

- Montgomery's model is purely economical, i.e. unconstrained cost objective. The proposed model is both statistical and economical since it includes constraints on the ARL.

- The decision variables in Montgomery's model includes k, n and control limits (L), while in our proposed model, the limits are fixed to ±3. Fixing the limits will make it more user-friendly for the inspector on the line.
- The proposed approach cost model is more comprehensive since it includes two additional cost terms. Namely, $E(C_4)$ and $E(C_5)$.
- The probability of transitioning from one assignable cause to another in Montgomery's approach requires the estimation of several terms x, y, z. The proposed model utilizes a simpler approach.

The formulation of the case is as follows:

$a_1 = \$5$

$a_2 = \$0.1$

$R = 81 \text{ units per hour}$

$K = 81 \text{ planned production}$

$\bar{p} = 0.1$

$g = 0$

$$b_{ij} = \begin{bmatrix} 10 & 20 & 20 & 20 & 20 & 20 & 20 \\ 20 & 10 & 20 & 20 & 20 & 20 & 20 \\ 20 & 20 & 10 & 20 & 20 & 20 & 20 \\ 20 & 20 & 20 & 10 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 & 10 & 20 & 20 \\ 20 & 20 & 20 & 20 & 20 & 10 & 20 \end{bmatrix}$$

$$p_{ij} = \begin{bmatrix} 0.005 & 0.007 & 0.008 & 0.009 & 0.009 & 0.010 & 0.010 \\ 0.007 & 0.010 & 0.013 & 0.016 & 0.018 & 0.019 & 0.019 \\ 0.008 & 0.013 & 0.020 & 0.027 & 0.032 & 0.036 & 0.038 \\ 0.009 & 0.016 & 0.027 & 0.040 & 0.053 & 0.064 & 0.071 \\ 0.009 & 0.018 & 0.032 & 0.053 & 0.080 & 0.107 & 0.128 \\ 0.010 & 0.019 & 0.036 & 0.064 & 0.107 & 0.160 & 0.213 \\ 0.010 & 0.019 & 0.038 & 0.071 & 0.128 & 0.213 & 0.320 \end{bmatrix}$$

$$w_i = \begin{bmatrix} 20 \\ 20 \\ 20 \\ 20 \\ 20 \\ 20 \end{bmatrix}$$

Since Montgomery's model does not include false alarm and external cost elements, the cost coefficients Z_i and c_i are assumed to be zero. The proposed model is re-written in 5.1:

$$\frac{(a_1 + a_2 n)}{hR} + \frac{\sum_i^s \sum_{j=1}^s b_{ij}\, p_{ij}\big(h - \tau_j\big)}{hR} + \frac{\sum_{i=1}^s w_i(1-\beta_i)}{hR} + \frac{\sum_{i=1}^s z_i\, \beta_i\,(h-\tau_i)}{hR} + \frac{\sum_{i=1}^s c_i\, \alpha_i}{hR} \tag{5.1}$$

subject to:

$$ARL_{1i} = \frac{1}{1-\beta_i} \geq u \quad \forall\, i = 1,2,\dots,s$$
$$hR \geq n \geq 1$$
$$h \geq g$$

The model was coded using GAMS and summarized in Appendix A. The resulting optimal solution is:

$n = 68\; units$
$h = 3\; hours$
$E(C) = \$0.32\; per\; unit\; produced$

From the above result, we notice that the total cost per unit produced from the proposed model, is less than Montgomery *et al.* [20] model by is \$0.03. This suggests that the proposed model brings about more savings per unit produced compared with Montgomery's model during a particular production cycle. This is because the proposed model includes a false alarm term in the objective function and a constraint for ARL_i. Such constraint results in false alarm rate reduction and hence saves on the production downtime and process investigation. The fifth cost term related to external failure did not make an impact since we assume $z_i = 0$.

Moreover, the proposed model adds two costs that are not considered in Montgomery's model, which is the cost of releasing a defective product to the customer, and the cost of investigating a false alarm. Also, the cost of producing defective units is calculated considering the expected time for the occurrence of an assignable cause for the proposed model, and it include a constraint for the ARL, unlike Montgomery's.

In order to put things into prospective, the improvement of the proposed model in terms of reduced cost per unit produced (CPU) can be estimated by comparing CPU of proposed method CPU_{PP} with CPU provided by Montgomery model CPU_{Montg}:

$$\% \, improvement = \frac{CPU_{Montg} - CPU_{PP}}{CPU_{Montg}} X100\% = \frac{0.35 - 0.32}{0.35} X100\% = 0.86\%$$

The improvement percentage might seem small. However, considering mass production industries such as the water bottling case which will be discussed in the next chapter, the total cost reduction adds up. For example, for annual production of 1,000,000 units per year, the expected savings in quality inspection and improvement is $30,000.

In the next chapter, we apply the new proposed model on a production process of a local water bottling company (WBC) in UAE, with a daily production of 542,286 bottles.

Chapter 6. Case Study: Water Bottling Company (WBC)

The new proposed model is applied to a local water bottling company (WBC) in UAE. The average daily production of WBC is 542,286 bottles. Table 3 below shows historical data for WBC's inspection during the month of September 2017.

Table 3: Daily rejections and net production of WBC Sept 2017

	Bottles			
Date	**Low Fill rejections**	**Misapplied caps rejections**	**Label misalignment rejections**	**Net Production Volume**
1-Sep	786	2,144		590,400
2-Sep	946	3,785		572,688
3-Sep	1,582	6,329		316,248
4-Sep	2,552	10,190		347,208
5-Sep	1,004	4,014		331,032
6-Sep	1,103	4,414		434,592
7-Sep	1,104	4,414		490,176
8-Sep	1,498	5,985		449,208
9-Sep	3,033	12,134		628,776
10-Sep	2,983	11,934		658,080
11-Sep	3,257	13,032		653,496
12-Sep	2,753	11,013		559,176
13-Sep	1,058	4,231		547,656
14-Sep	1,788	7,152		664,848
15-Sep	1,877	7,508		759,984
16-Sep	1,538	6,153		634,224
17-Sep	931	3,722		399,384
18-Sep	2,124	8,495		563,328
19-Sep	2,053	8,214		637,512
20-Sep	1,102	4,407		609,240
21-Sep	2,205	8,821		720,360
22-Sep	1,622	6,489		568,848
23-Sep	2,383	9,534		515,304
24-Sep	1,255	5,022		409,632
25-Sep	2,253	9,011		537,264
26-Sep	2,501	10,005		615,312
27-Sep	2,056	8,223		469,776
28-Sep	1,722	6,886	70	691,200
29-Sep	1,027	4,109		487,800
30-Sep	1,780	7,199		405,840
Total	**53,876**	**214,569**	**70**	**16,268,592**

There are three main failure modes with bottling process: misapplied cap which require either high or low torque to open the cap (79.91% of the total rejections), low fill (20.06% of the total rejections) and label misalignment (0.03% of the total rejections). This data will be used to demonstrate the effectiveness of the proposed model.

The proposed mathematical model was used and coded using GAMS to find the optimal sample size and inspection interval. The code is summarized in Appendix A. The case parameters, objective function and constraints are as follows:

$a_1 = AED\ 2$

$a_2 = AED\ 0.1$

$R = 28{,}244\ units\ per\ hour$

$K = 677{,}858\ units\ of\ planned\ production$

$\bar{p} = 0.0165$

$v = 0.5$

$s = 3$

$p_1 = 0.0033$ where p_1 is the probability of ocurrance of low fill

$p_2 = 0.013$ where p_2 is the probability of ocurrance of misapplied cap

$p_3 = 0.000004$ is where p_2 the probability of occurrence of label misalignment

$$b_{ij} = \begin{bmatrix} 40\ 50\ 43 \\ 50\ 10\ 13 \\ 43\ 13\ 03 \end{bmatrix}$$

$$c_i = \begin{bmatrix} 15 \\ 60 \\ 08 \end{bmatrix}$$

$$w_i = \begin{bmatrix} 25 \\ 65 \\ 12 \end{bmatrix}$$

$$z_i = \begin{bmatrix} 50 \\ 70 \\ 15 \end{bmatrix}$$

Recalling from chapter 4, the transition probability matrix is obtained as follows:

$$p_{ij} = p_i \quad \frac{p_j}{p_i + p_j} \qquad where\ j\ \neq 0$$

$$p_{ij} = \begin{bmatrix} 0.00165 & 0.000668 & 0.003296 \\ 0.010368 & 0.0065 & 0.012996 \\ 0.00000 & 0.00000 & 0.000002 \end{bmatrix}$$

$u = 370$

Solving for the WBC case using GAMS, the resulting optimal solution is:

$n = 34\ units$

$h = 2\ hours$

$E(C) = AED\ 0.003/unit\ produced$

$u = 1$

In the following section, we perform sensitivity analysis to check how the sample size and sampling frequency change with respect to the total cost.

6.1. Sensitivity Analysis

For the new proposed model, we examine the sensitivity of the optimal sample size n and the sampling frequency h against the input cost coefficients to the model. Starting with making gradual increases to the variable cost of sampling a_2, we notice that the resulted optimal sample size decreases, as shown in Figure 4.

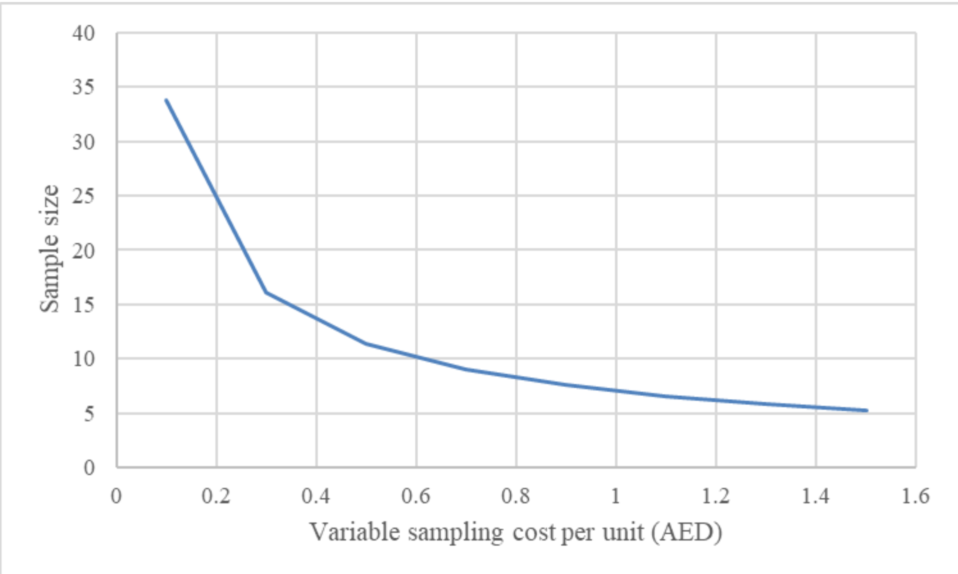

Figure 4: Variable sampling cost per unit vs sample size

Similarly, the resulting optimal sampling interval decreases when the unit cost of producing defectives increases. Relationship chart is shown is Figure 5.

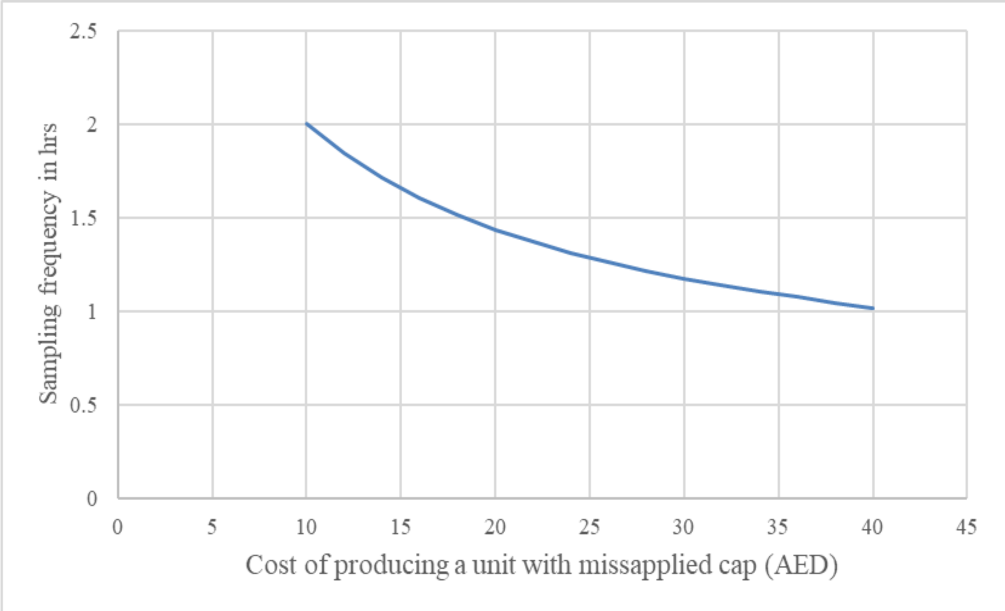

Figure 5: Cost of producing a unit with misapplied cap vs sampling frequency

Also, when checking the relationship between the cost of investigation and correction and the optimal sample size, we find that as the cost of investigation and correction increases, the resulting sample size increases as well.

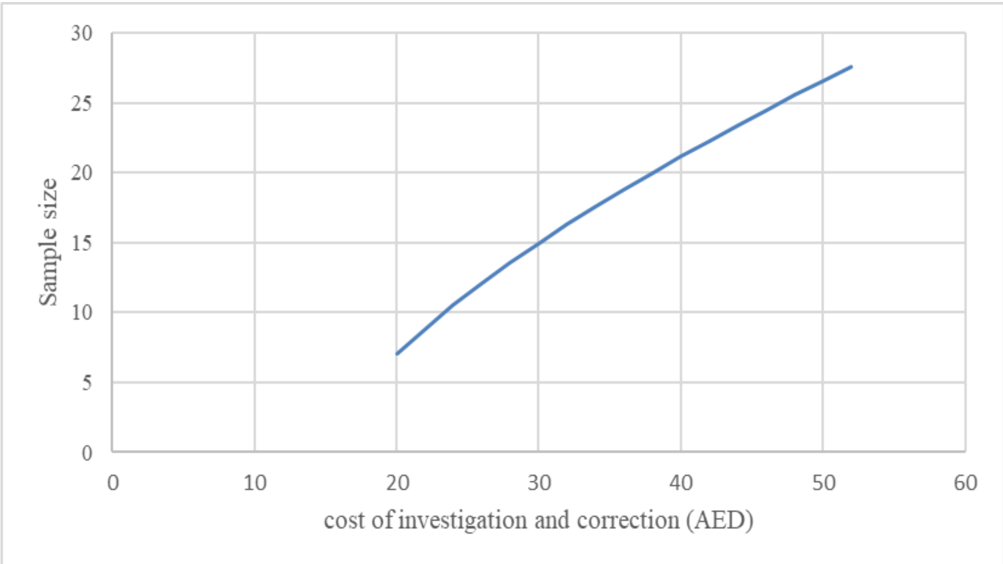

Figure 6: Cost of investigation and correction vs sample size

Also, it is shown in Figure 7 and Figure 8 that as the probability of failure due to the assignable cause of low fill and the misapplied caps becomes higher, the sampling interval *h* increases, but the resulting optimal sample size *n* decreases.

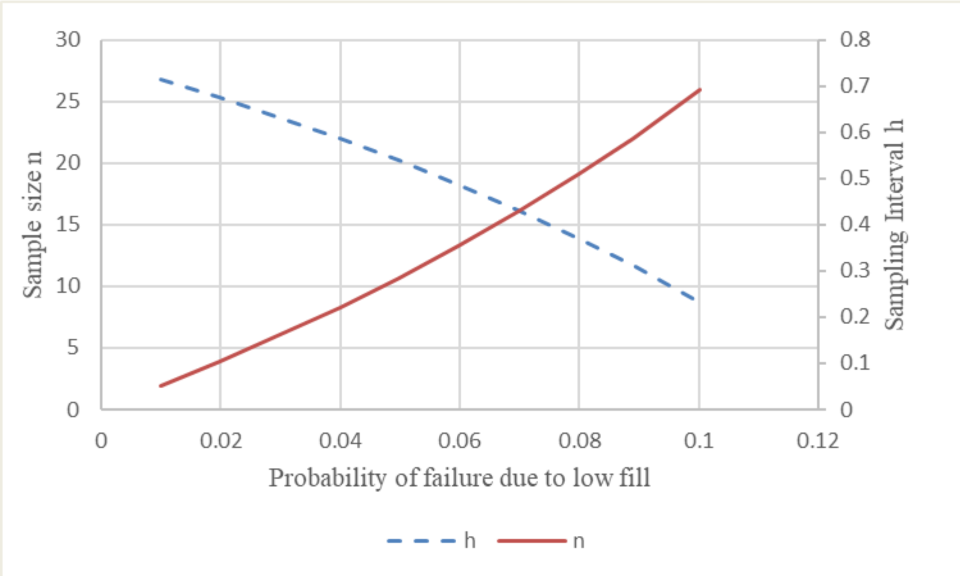

Figure 7: Probability of failure due to low fill vs n, h

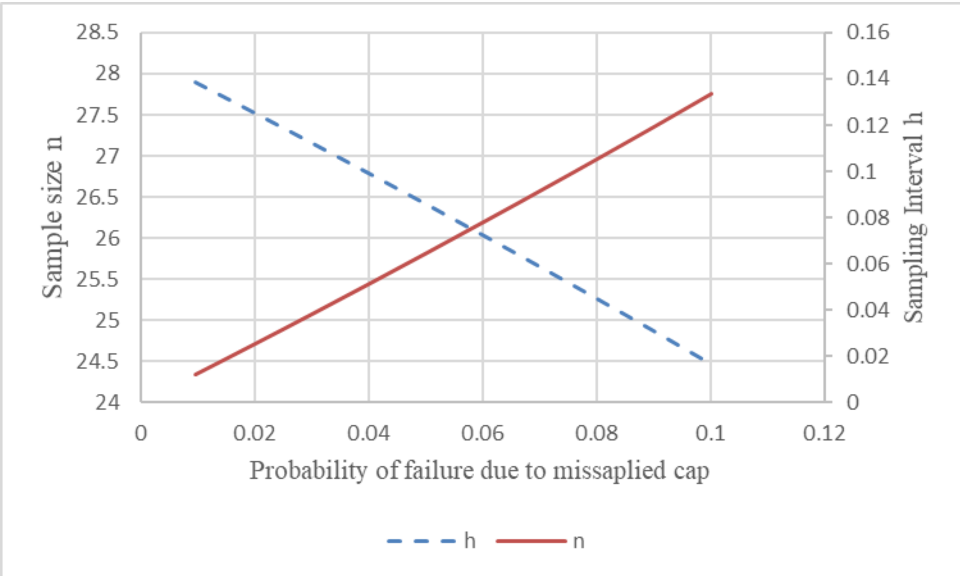

Figure 8: Probability of failure due to misapplied caps vs n, h

When comparing our model with the ones presented by Montgomery *et al.* [20] and Gibra [23], we notice that their model does not contain constraints for the ARL, and rather uses trial and error method to reach an optimal solution. In addition, the model is memoryless of the process history, and assumes that over time the process tends to move to an out-of-control state.

Chapter 7. Conclusion and Future Work

In this thesis, a statistical-economical model is proposed for the design of fraction of non-conforming control charts with multiple assignable causes. The effectiveness of the model is demonstrated by comparing it to a previous model and shows a reduction in the cost per unit produced by 0.86%. In practical applications, one of the most concerning issue -which is addressed by proposed model, is how to dynamically determine the optimal sample size and sampling frequency for a process with multiple assignable causes and random shift, that corresponds with the minimum expected cost. Previous modes have focused on p-chart optimization, considering only economical terms. However, the new proposed model considers both the economic and statistical parameters of control chart like the ARL, which allows for controlling the sensitivity of a *p*-chart by setting the probability of not detecting a shift to a specific desired value and getting the corresponding sample size and sampling frequency.

As future work, it would be more efficient to consider the dynamic sampling interval, proportional to the probability of failure. In the dynamic sampling, both sample size and sample interval are constantly changing based on inspection results. Another area that warrants further investigation is relaxing the constant control limits assumption. Although this assumption simplify the implementation of *p*-charts by operators on the shop floor, a comparison between constant and variable control limits is worth investigating in terms of cost and easiness of implementation. Finally, another limitation arises in processes where the production rate is not uniform across the production cycle, which makes it effective for the sampling interval to be in terms of unit produced rather than in units of time.

References

[1] D. C. Montgomery, *Introduction to statistical quality control*. Hoboken, NJ :: Wiley, 2013.

[2] W. A. Shewhart, *Economic control of quality of manufactured product*. London, ASQ Quality Press, 1931.

[3] A. J. Duncan, "The economic design of X charts used to maintain current control of a process," *Journal of the American Statistical Association,* vol. 51, no. 274, pp. 228-242, 1956.

[4] T. J. Lorenzen and L. C. Vance, "The Economic Design of Control Charts: A Unified Approach," *Technometrics,* vol. 28, no. 1, pp. 3-10, 1986.

[5] E. M. Saniga, "Economic Statistical Control-Chart Designs with an Application to $\bar{X}$ and R Charts," *Technometrics,* vol. 31, no. 3, pp. 313-320, 1989.

[6] E. Saniga and D. J. Davis, "Economic-statistical design of X and R or X and S charts," *J. Quality Technology,* vol. 33, pp. 234-241, 2001.

[7] G. Nenes, K. A. Tasias, and G. Celano, "A general model for the economic-statistical design of adaptive control charts for processes subject to multiple assignable causes," *International Journal of Production Research,* vol. 53, no. 7, pp. 2146-2164, 2015.

[8] M. Bakir and B. Altunkaynak, "The optimization with the genetic algorithm approach of the multi-objective, joint economical design of the X and R control charts," *Journal of Applied Statistics,* vol. 31, no. 7, pp. 753-772, 2004.

[9] T.-H. Lee, S.-H. Hong, H.-M. Kwon, and M. Lee, "Economic Statistical Design of Variable Sampling Interval X¯ $\overline X $ Control Chart Based on Surrogate Variable Using Genetic Algorithms," *Management and Production Engineering Review,* vol. 7, no. 4, pp. 54-64, 2016.

[10] C. M. Borror, C. W. Champ, and S. E. Rigdon, "Poisson EWMA control charts," *Journal of Quality Technology,* vol. 30, no. 4, p. 352, 1998.

[11] A. L. Goel and S. M. Wu, "Economically Optimum Design of Cusum Charts," *Management Science*, vol. 19, no. 11, pp. 1271-1282

[12] J.-N. Pan and S.-T. Chen, "The Economic Design of CUSUM Chart for Monitoring Environmental Performance," *Asia Pacific Management Review,* vol. 10, no. 2, p. n/a, 2005.

[13] D. C. Montgomery, J. C. Torng, J. K. Cochran, and F. P. Lawrence, "Statistically constrained economic design of the EWMA control chart," *Journal of Quality Technology,* vol. 27, no. 3, pp. 250-256, 1995.

[14] K. Linderman and T. E. Love, "Economic and economic statistical designs for MEWNA control charts," *Journal of Quality Technology,* vol. 32, no. 4, p. 410, 2000.

[15] A. Amiri, A. Moslemi, and M. H. Doroudyan, "Robust economic and economic-statistical design of EWMA control chart," *The International Journal of Advanced Manufacturing Technology,* vol. 78, no. 1-4, pp. 511-523, 2015.

[16] Z. Wu, M. Shamsuzzaman, and E. Pan, "Optimization design of control charts based on Taguchi's loss function and random process shifts," *International Journal of Production Research,* vol. 42, no. 2, pp. 379-390, 2004.

[17] S. P. Ladany, "Optimal use of control charts for controlling current production," *Management Science,* vol. 19, no. 7, pp. 763-772, 1973.

[18] W. K. Chiu, "Economic Design of Attribute Control Charts," *Technometrics,* vol. 17, no. 1, pp. 81-87, 1975.

[19] S. P. Ladany and D. N. Bedi, "Selection of the optimal setup policy," *Naval Research Logistics (NRL),* vol. 23, no. 2, pp. 219-233, 1976.

[20] D. C. Montgomery, R. G. Heikes, and J. F. Mance, "Economic Design of Fraction Defective Control Charts," *Management Science,* vol. 21, pp. 1272-1284, 1975.

[21] W. K. Chiu, "Economic Design of np Charts for Processes Subject to a Multiplicity of Assignable Causes," *Management Science (pre-1986),* vol. 23, no. 4, p. 404, 1976.

[22] I. N. Gibra, "Economically Optimal Determination of the Parameters of np-Control Charts," *Journal of Quality Technology* vol. 10, pp. 12-19, 1978.

[23] I. N. Gibra, "Economic Design of Attribute Control Charts for Multiple Assignable Causes," *Journal of Quality Technology* vol. 13, pp. 9-93, 1981.

[24] K.-J. Chung, "An algorithm for the economic design of np charts for a multiplicity of assignable causes," *Journal of the Operational Research Society,* vol. 46, no. 11, pp. 1374-1385, 1995.

[25] E. von Collani, "Economically optimalc-andnp-control charts," *Metrika,* vol. 36, no. 1, pp. 215-232, 1989.

[26] T. Goh, "A charting technique for control of low-defective production," *International Journal of Quality & Reliability Management,* vol. 4, no. 1, pp. 53-62, 1987.

[27] E. M. Saniga, D. J. Davis, and T. P. McWilliams, "Economic, statistical, and economic-statistical design of attribute charts," *Journal of Quality Technology,* vol. 27, no. 1, pp. 56-73, 1995.

[28] R.-C. Wang and C.-H. Chen, "Economic statistical np-control chart designs based on fuzzy optimization," *The International Journal of Quality & Reliability Management,* vol. 12, no. 1, p. 82, 1995.

[29] I. Kooli and M. Limam, "Bayesian np control charts with adaptive sample size for finite production runs," *Quality and Reliability Engineering International,* vol. 25, no. 4, pp. 439-448, 2009.

[30] J. M. Calabrese, "Bayesian process control for attributes," *Management science,* vol. 41, no. 4, pp. 637-645, 1995.

[31] M. Abooie and M. A. Nayeri, "A heuristic fixed limit Bayesian p chart," *South African Journal of Industrial Engineering,* vol. 20, no. 2, pp. 117-125, 2009.

[32] V. Makis, F. Naderkhani, and L. Jafari, "Optimal Bayesian Control of the Proportion of Defectives in a Manufacturing Process," *World Academy of Science, Engineering and Technology, International Journal of Computer, Electrical, Automation, Control and Information Engineering,* vol. 10, no. 9, pp. 1658-1663, 2016.

[33] Z. Wu and H. Luo, "Optimal design of the adaptive sample size and sampling interval np control chart," *Quality and Reliability Engineering International,* vol. 20, no. 6, pp. 553-570, 2004.

[34] I. Kooli and M. Limam, "Economic design of an attribute np control chart using a variable sample size," *Sequential Analysis,* vol. 30, no. 2, pp. 145-159, 2011.

[35] I. Kooli and M. Limam, "Economic design of attribute <i>np</i> control charts using a variable sampling policy," *Applied Stochastic Models in Business and Industry,* vol. 31, no. 4, pp. 483-494, 2015.

[36] J. Jolayemi, "Statistical design of np-charts with multiple control regions," *International Journal of Production Research,* vol. 40, no. 11, pp. 2591-2608, 2002.
[37] P. C. Cozzucoli, "Process Monitoring with Multivariate " *International Journal of Quality, Statistics, and Reliability,* vol. 2009, 2009.
[38] W. W. Williams, S. W. Looney, and M. H. Peters, "Use of curtailed sampling plans in the economic design of np-control charts," *Technometrics,* vol. 27, no. 1, pp. 57-63, 1985.
[39] A. J. Duncan, "The economic design of p-charts to maintain current control of o process: Some numerical results," *Technometrics,* vol. 20, no. 3, pp. 235-243, 1978.
[40] M. Bashiri, A. Amiri, A. Asgari, and M. Doroudyan, "Multi-objective efficient design of np control chart using data envelopment analysis," *International Journal of Engineering,* vol. 26, no. 6, pp. 621-630, 2013.
[41] R. B. Kethley and M. H. Peters, "Extending economic design of p charts to handle user specified constraints using a genetic algorithm," *IIE Transactions,* vol. 36, no. 9, pp. 855-864, 2004.
[42] S. Jafarian-Namin and H. Hasanzadeh, "Multi-objective economic-statistical design of fraction defective control chart for high quality processes using a heuristic algorithm," in *Proceedings of the 12rd International Conference on Industrial Engineering, Tehran, Iran*, 2016, pp. 6404-6406.
[43] W. H. Woodall, "Weaknesses of the economic design of control charts," *Technometrics,* vol. 28, no. 4, pp. 408-409, 1986.
[44] L. Chan, M. Xie, and T. Goh, "Cumulative quantity control charts for monitoring production processes," *International Journal of Production Research,* vol. 38, no. 2, pp. 397-408, 2000.
[45] R. Inghilleri, T. Lupo, and G. Passannanti, "An effective double sampling scheme for the c control chart," *Quality and Reliability Engineering International,* vol. 31, no. 2, pp. 205-216, 2015.
[46] T. Lupo, "Economic design approach for an SPC inspection procedure implementing the adaptive C chart," *Quality and Reliability Engineering International,* vol. 30, no. 4, pp. 545-558, 2014.

Appendix A

Proposed model coded in GAMS:

```
$funclibin stolib stodclib
Functions cdfnorm / stolib.cdfnormal /;

Sets i assignable cause state i /state1, state2, state3, state4, state5, state6, state7/
j assignable cause state j /state1, state2, state3, state4, state5, state6, state7/;

Scalar a1 fixed cost of sampling /5/;
Scalar a2 variable cost of sampling /0.1/;
Scalar R rate of production per hour /81/;

Variables
n sample size
h sampling frquency in hours
s units of standard deviation
y total cost per unit produced
c1 sampling cost
c2 cost of producing defective items
c3 cost of investigating and correcting true alarm
c4 cost of releasing defective product to customer
c5 cost of false alarm;

Positive Variables n, h, s;

Scalar pbar overall mean of rejections in the last production cycle /0.1/ ;

Table b(i,j) cost of producing a defective unit when shifting from assignable cause i to
assignable cause j
state1  state2  state3  state4  state5  state6 state7
state1      10      10      10      10      10      10   10
state2      10      10      10      10      10      10   10
state3      10      10      10      10      10      10   10
state4      10      10      10      10      10      10   10
state5      10      10      10      10      10      10   10
state6      10      10      10      10      10      10   10
state7      10      10      10      10      10      10   10;

Table p(i,j) probability of shifting from assignable cause i to assignable cause j
state1      state2      state3      state4      state5      state6      state7
state1      0.005       0.007       0.008       0.009       0.009       0.010       0.010
state2      0.007       0.010       0.013       0.016       0.018       0.019       0.019
```

state3	0.008	0.013	0.020	0.027	0.032	0.036	0.038
state4	0.009	0.016	0.027	0.040	0.053	0.064	0.071
state5	0.009	0.018	0.032	0.053	0.080	0.107	0.128
state6	0.010	0.019	0.036	0.064	0.107	0.160	0.213
state7	0.010	0.019	0.038	0.071	0.128	0.213	0.320;

Parameters d(i) probability of occurance of assignable cause i
/state1 0.01
state2 0.02
state3 0.04
state4 0.08
state5 0.16
state6 0.32
state7 0.64/

w(i) cost of investigating and correcting a true alarm due to assigable cause i
/state1 20
state2 20
state3 20
state4 20
state5 20
state6 20
state7 20/

z(i) cost of releasing one defective product due to assigable cause i to customer
/state1 0
state2 0
state3 0
state4 0
state5 0
state6 0
state7 0/

c(i) cost of false alarm due to assigable cause i
/state1 0
state2 0
state3 0
state4 0
state5 0
state6 0
state7 0/;

Equations
cost objective function
sample_lower sampling lower limit

```
sample_upper sampling upper limit
frequency_lower frequency lower limit
frequency_upper frequency upper limit
sd_upper units of standard deviation upper limit
sd_lower units of standard deviation lower limit
cost_of_sampling cost of sampling
cost_of_producing_defective_items cost of producing defective items
cost_of_investigating_and_correcting_true_alarm cost of investigating and correcting
true alarm
cost_of_releasing_defective_product cost of releasing defective product to customer
cost_of_false_alarm cost of false alarm
ARL1 average run length state1
ARL2 average run length state2
ARL3 average run length state3
ARL4 average run length state4
ARL5 average run length state5
ARL6 average run length state6
ARL7 average run length state7;

cost.. y  =e=  (a1 + a2*n)/(h*R + 0.0001) + sum((i,j), b(i,j) *  p(i,j) * (h - (1-
(1+h)*exp(-h))/(1-exp(-h)+0.001)))/(h*R + 0.0001) + sum(i,  w(i) * (  (cdfnorm(  d(i)
+ s* sqrt(d(i)*(1-d(i))/(n+0.001)  ),  d(i),  sqrt(d(i)*(1-d(i))  ))) - (cdfnorm(  d(i) - s*
sqrt(d(i)*(1-d(i))/(n+0.001)  ),  d(i),  sqrt(d(i)*(1-d(i))  ))))   )/(h*R + 0.0001) + sum(i,
z(i) * (1 -  ((cdfnorm(  d(i) + s* sqrt(d(i)*(1-d(i))/(n+0.001)  ),  d(i),  sqrt(d(i)*(1-d(i))
))) - (cdfnorm(  d(i) - s* sqrt(d(i)*(1-d(i))/(n+0.001)  ),  d(i),  sqrt(d(i)*(1-d(i))  )))))
)/(h*R + 0.0001) + sum(i,  c(i) * (  (cdfnorm(  pbar + s* sqrt(pbar*(1-pbar)/(n+0.001)
),  pbar,  sqrt(pbar*(1-pbar)  ))) - (cdfnorm(  pbar - s* sqrt(pbar*(1-pbar)/(n+0.001)  ),
pbar,  sqrt(pbar*(1-pbar)  ))))   )/(h*R + 0.0001); ;
sample_lower.. n =g= 1;
sample_upper.. n =l= h*R;
frequency_lower.. h =g= 0;
frequency_upper.. h =l= 3;
sd_upper.. s =g= 1;
sd_lower.. s =l= 3;
cost_of_sampling.. c1 =e= (a1 + a2*n)/(h*R + 0.0001);
cost_of_producing_defective_items.. c2 =e= sum((i,j), b(i,j) *  p(i,j) * (h - (1-
(1+h)*exp(-h))/(1-exp(-h)+0.001)))/(h*R + 0.0001);
cost_of_investigating_and_correcting_true_alarm.. c3 =e= sum(i,  w(i) * (1- (
(cdfnorm(  d(i) + s* sqrt(d(i)*(1-d(i))/(n+0.001)  ),  d(i),  sqrt(d(i)*(1-d(i))  ))) -
(cdfnorm(  d(i) - s* sqrt(d(i)*(1-d(i))/(n+0.001)  ),  d(i),  sqrt(d(i)*(1-d(i))  ))))
))/(h*R + 0.0001);
cost_of_releasing_defective_product.. c4 =e= sum(i,  z(i) * ( ( (cdfnorm(  d(i) + s*
sqrt(d(i)*(1-d(i))/(n+0.001)  ),  d(i),  sqrt(d(i)*(1-d(i))  ))) - (cdfnorm(  d(i) - s*
sqrt(d(i)*(1-d(i))/(n+0.001)  ),  d(i),  sqrt(d(i)*(1-d(i))  )))))   ) * ((h - (1-(1+h)*exp(-
h))/(1-exp(-h)+0.001)))/(h*R + 0.0001);
```

```
cost_of_false_alarm.. c5 =e=  sum(i,  c(i) * (  (cdfnorm(  pbar + s* sqrt(pbar*(1-pbar)/(n+0.001)  ),  pbar,  sqrt(pbar*(1-pbar)  ))) - (cdfnorm(  pbar - s* sqrt(pbar*(1-pbar)/(n+0.001)  ),  pbar,  sqrt(pbar*(1-pbar)  ))))   )/(h*R + 0.0001);
ARL1.. 1/(1-  ((cdfnorm(  d('state1') + s* sqrt(d('state1')*(1-d('state1'))/(n+0.001)  ), d('state1'),  sqrt(d('state1')*(1-d('state1'))  ))) - (cdfnorm(  d('state1') - s* sqrt(d('state1')*(1-d('state1'))/(n+0.001)  ),  d('state1'),  sqrt(d('state1')*(1-d('state1')) ))) + 0.0001 )) =g= 1;
ARL2.. 1/(1-  ((cdfnorm(  d('state2') + s* sqrt(d('state2')*(1-d('state2'))/(n+0.001)  ), d('state2'),  sqrt(d('state2')*(1-d('state2'))  ))) - (cdfnorm(  d('state2') - s* sqrt(d('state2')*(1-d('state2'))/(n+0.001)  ),  d('state2'),  sqrt(d('state2')*(1-d('state2')) ))) + 0.0001 )) =g= 1;
ARL3.. 1/(1-  ((cdfnorm(  d('state3') + s* sqrt(d('state3')*(1-d('state3'))/(n+0.001)  ), d('state3'),  sqrt(d('state3')*(1-d('state3'))  ))) - (cdfnorm(  d('state3') - s* sqrt(d('state3')*(1-d('state3'))/(n+0.001)  ),  d('state3'),  sqrt(d('state3')*(1-d('state3')) ))) + 0.0001 )) =g= 1;
ARL4.. 1/(1-  ((cdfnorm(  d('state4') + s* sqrt(d('state4')*(1-d('state4'))/(n+0.001)  ), d('state4'),  sqrt(d('state4')*(1-d('state4'))  ))) - (cdfnorm(  d('state4') - s* sqrt(d('state4')*(1-d('state4'))/(n+0.001)  ),  d('state4'),  sqrt(d('state4')*(1-d('state4')) ))) + 0.0001 )) =g= 1;
ARL5.. 1/(1-  ((cdfnorm(  d('state5') + s* sqrt(d('state5')*(1-d('state5'))/(n+0.001)  ), d('state5'),  sqrt(d('state5')*(1-d('state5'))  ))) - (cdfnorm(  d('state5') - s* sqrt(d('state5')*(1-d('state5'))/(n+0.001)  ),  d('state5'),  sqrt(d('state5')*(1-d('state5')) ))) + 0.0001 )) =g= 1;
ARL6.. 1/(1-  ((cdfnorm(  d('state6') + s* sqrt(d('state6')*(1-d('state6'))/(n+0.001)  ), d('state6'),  sqrt(d('state6')*(1-d('state6'))  ))) - (cdfnorm(  d('state6') - s* sqrt(d('state6')*(1-d('state6'))/(n+0.001)  ),  d('state6'),  sqrt(d('state6')*(1-d('state6')) ))) + 0.0001 )) =g= 1;
ARL7.. 1/(1-  ((cdfnorm(  d('state7') + s* sqrt(d('state7')*(1-d('state7'))/(n+0.001)  ), d('state7'),  sqrt(d('state7')*(1-d('state7'))  ))) - (cdfnorm(  d('state7') - s* sqrt(d('state7')*(1-d('state7'))/(n+0.001)  ),  d('state7'),  sqrt(d('state7')*(1-d('state7')) ))) + 0.0001 )) =g= 1;

Model sampling /all/;
Solve sampling using NLP minimizing y;
Display y.l, n.l, h.l, c1.l, c2.l, c3.l, c4.l, c5.l, ARL1.l, ARL2.l, ARL3.l, ARL4.l, ARL5.l, ARL6.l, ARL7.l, s.l;
```

GAMS code of the new proposed model applied on Montgomery's case:

```
$funclibin stolib stodclib
Functions cdfnorm / stolib.cdfnormal /;

Sets i assignable cause state i /low_fill, misapplied_cap, label_misalignment/
    j assignable cause state j /low_fill, misapplied_cap, label_misalignment/;
```

```
Scalar a1 fixed cost of sampling /2/;
Scalar a2 variable cost of sampling /0.1/;
Scalar R rate of production per hour /28244/;

Variables
        n sample size
        h sampling frquency in hours
        s units of standard deviation
        y total cost per unit produced
        c1 sampling cost
        c2 cost of producing defective items
        c3 cost of investigating and correcting true alarm
        c4 cost of releasing defective product to customer
        c5 cost of false alarm;

Positive Variables n, h, s;

Scalar pbar overall mean of rejections in the last production cycle /0.1/ ;

Table b(i,j) cost of producing a defective unit when shifting from assignable cause i to
assignable cause j
                    low_fill   misapplied_cap  label_misalignment
        low_fill            40        50             43
        misapplied_cap       50        10             13
        label_misalignment    43        13              03;

Table p(i,j) probability of shifting from assignable cause i to assignable cause j
                    low_fill   misapplied_cap  label_misalignment
        low_fill            0.002        0.003          0.000004
        misapplied_cap       0.003        0.007          0.000004
        label_misalignment    0.000004     0.000004       0.000002;

Parameters  d(i) probability of occurance of assignable cause i
        /low_fill 0.0033
         misapplied_cap 0.013
         label_misalignment 0.000004/

        w(i) cost of investigating and correcting a true alarm due to assigable cause i
        /low_fill 25
         misapplied_cap 65
         label_misalignment 12/

        z(i) cost of releasing one defective product due to assigable cause i to customer
        /low_fill 50
         misapplied_cap 70
```

label_misalignment 15/

c(i) cost of false alarm due to assigable cause i
/low_fill 15
misapplied_cap 60
label_misalignment 08/;

Equations
cost objective function
sample_lower sampling lower limit
sample_upper sampling upper limit
frequency_lower frequency lower limit
frequency_upper frequency upper limit
sd_upper units of standard deviation upper limit
sd_lower units of standard deviation lower limit
cost_of_sampling cost of sampling
cost_of_producing_defective_items cost of producing defective items
cost_of_investigating_and_correcting_true_alarm cost of investigating and correcting true alarm
cost_of_releasing_defective_product cost of releasing defective product to customer
cost_of_false_alarm cost of false alarm
ARL1 average run length state1
ARL2 average run length state2
ARL3 average run length state3;

cost.. y =e= (a1 + a2*n)/(h*R + 0.0001) + sum((i,j), b(i,j) * p(i,j) * (h - (1-(1+h)*exp(-h))/(1-exp(-h)+0.001)))/(h*R + 0.0001) + sum(i, w(i) * ((cdfnorm(d(i) + s* sqrt(d(i)*(1-d(i))/(n+0.001)), d(i), sqrt(d(i)*(1-d(i))))) - (cdfnorm(d(i) - s* sqrt(d(i)*(1-d(i))/(n+0.001)), d(i), sqrt(d(i)*(1-d(i)))))))/(h*R + 0.0001) + sum(i, z(i) * (1 - ((cdfnorm(d(i) + s* sqrt(d(i)*(1-d(i))/(n+0.001)), d(i), sqrt(d(i)*(1-d(i))))) - (cdfnorm(d(i) - s* sqrt(d(i)*(1-d(i))/(n+0.001)), d(i), sqrt(d(i)*(1-d(i))))))))/(h*R + 0.0001) + sum(i, c(i) * ((cdfnorm(pbar + s* sqrt(pbar*(1-pbar)/(n+0.001)), pbar, sqrt(pbar*(1-pbar)))) - (cdfnorm(pbar - s* sqrt(pbar*(1-pbar)/(n+0.001)), pbar, sqrt(pbar*(1-pbar))))))/(h*R + 0.0001); ;
sample_lower.. n =g= 1;
sample_upper.. n =l= h*R;
frequency_lower.. h =g= 0;
frequency_upper.. h =l= 2;
sd_upper.. s =e= 3;
sd_lower.. s =e= 3;
cost_of_sampling.. c1 =e= (a1 + a2*n)/(h*R + 0.0001);
cost_of_producing_defective_items.. c2 =e= sum((i,j), b(i,j) * p(i,j) * (h - (1-(1+h)*exp(-h))/(1-exp(-h)+0.001)))/(h*R + 0.0001);

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cost_of_investigating_and_correcting_true_alarm.. c3 =e= sum(i, w(i) * (1- ( (cdfnorm( d(i) + s* sqrt(d(i)*(1-d(i))/(n+0.001) ), d(i), sqrt(d(i)*(1-d(i)) ))) - (cdfnorm( d(i) - s* sqrt(d(i)*(1-d(i))/(n+0.001) ), d(i), sqrt(d(i)*(1-d(i)) )))) ))/(h*R + 0.0001);
cost_of_releasing_defective_product.. c4 =e= sum(i, z(i) * ( ( (cdfnorm( d(i) + s* sqrt(d(i)*(1-d(i))/(n+0.001) ), d(i), sqrt(d(i)*(1-d(i)) ))) - (cdfnorm( d(i) - s* sqrt(d(i)*(1-d(i))/(n+0.001) ), d(i), sqrt(d(i)*(1-d(i)) ))))) )/(h*R + 0.0001);
cost_of_false_alarm.. c5 =e= sum(i, c(i) * ( (cdfnorm( pbar + s* sqrt(pbar*(1-pbar)/(n+0.001) ), pbar, sqrt(pbar*(1-pbar) ))) - (cdfnorm( pbar - s* sqrt(pbar*(1-pbar)/(n+0.001) ), pbar, sqrt(pbar*(1-pbar) )))) ) * ((h - (1-(1+h)*exp(-h))/(1-exp(-h)+0.001))))/(h*R + 0.0001);
ARL1.. 1/(1- ((cdfnorm( d('low_fill') + s* sqrt(d('low_fill')*(1-d('low_fill'))/(n+0.001) ), d('low_fill'), sqrt(d('low_fill')*(1-d('low_fill')) ))) - (cdfnorm( d('low_fill') - s* sqrt(d('low_fill')*(1-d('low_fill'))/(n+0.001) ), d('low_fill'), sqrt(d('low_fill')*(1-d('low_fill')) ))) + 0.0001 )) =g= 1;
ARL2.. 1/(1- ((cdfnorm( d('misapplied_cap') + s* sqrt(d('misapplied_cap')*(1-d('misapplied_cap'))/(n+0.001) ), d('misapplied_cap'), sqrt(d('misapplied_cap')*(1-d('misapplied_cap')) ))) - (cdfnorm( d('misapplied_cap') - s* sqrt(d('misapplied_cap')*(1-d('misapplied_cap'))/(n+0.001) ), d('misapplied_cap'), sqrt(d('misapplied_cap')*(1-d('misapplied_cap')) ))) + 0.0001 )) =g= 1;
ARL3.. 1/(1- ((cdfnorm( d('label_misalignment') + s* sqrt(d('label_misalignment')*(1-d('label_misalignment'))/(n+0.001) ), d('label_misalignment'), sqrt(d('label_misalignment')*(1-d('label_misalignment')) ))) - (cdfnorm( d('label_misalignment') - s* sqrt(d('label_misalignment')*(1-d('label_misalignment'))/(n+0.001) ), d('label_misalignment'), sqrt(d('label_misalignment')*(1-d('label_misalignment')) ))) + 0.0001 )) =g= 1;

Model sampling /all/;
Solve sampling using NLP minimizing y;
Display y.l, n.l, h.l, c1.l, c2.l, c3.l, c4.l, c5.l, ARL1.l, ARL2.l, ARL3.l, s.l;
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Vita

Emad Aldin Mohammed Abdelkreem Mohammed was born in 1992 in Libya. He received his B.Sc. degree in Chemical Engineering from the University of Khartoum in 2013. From 2013 to 2014, he worked as an Occupational Health, Safety and Environmental Engineer in DAL Group – Sayga company in Sudan.

In August 2016, he joined the Engineering Systems Management master's program in the American University of Sharjah as a graduate teaching assistant. During his master's study, he co-authored a paper which was presented in the UAE Graduate Students Research Conference (GSRC) in April 2018. His research interests are in the statistical quality control field, and optimization modelling for industrial processes.