Accepted Manuscript of: Karapetyants, A., Louhichi, I. (2020). Fractional integrodifferentiation and Toeplitz operators with vertical symbols. In: Bauer, W., Duduchava, R., Grudsky, S., Kaashoek, M. (Eds.), *Operator Algebras, Toeplitz Operators and Related Topics* (pp. 175-187). Birkhäuser. https://doi.org/10.1007/978-3-030-44651-2_13

FRACTIONAL INTEGRODIFFERENTIATION AND TOEPLITZ OPERATORS WITH VERTICAL SYMBOLS

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ABSTRACT. We consider the so-called vertical Toeplitz operators on the weighted Bergman space over the half plane. The terminology "vertical" is motivated by the fact that if a is a symbol of such Toeplitz operator, then $a(z)$ depends only on $y = \Im z$, where $z = x + iy$. The main question raised in this paper can be formulated as follows: given two bounded vertical Toeplitz operators T_a^λ and T_b^λ, under which conditions is there a symbol h such that $T_a^\lambda T_b^\lambda = T_h^\lambda$? It turns out that this problem has a very nice connection with fractional calculus! We shall formulate our main results using the well know theory of Riemmann-Liouville fractional integrodifferentiation.

Dedicated to the occasion of 70th birthday of Professor N.Vasilevski.

Keywords: Toepliz operators, Riemmann-Liouville fractional integrodifferentiation, Spaces of holomorphic functions

2010 MSC: 30H20, 47B35, 26A33,

1. Introduction

Let $\Pi = \{z = x+iy : x \in \mathbb{R},\ y > 0\}$ be the upper half plane in the complex plane $\mathbb{C}$, $dA(z) = dxdy$ is the Lebesgue area measure, and $dA_\lambda(z) = (\lambda+1)\,(2\Im z)^\lambda\, dA(z)$, $\lambda > -1$. We denote by $L^2_\lambda(\Pi) = L^2(\Pi; dA_\lambda)$ the space of square integrable functions on Π with respect to the measure dA_λ. Then the Bergman space $\mathcal{A}^2_\lambda(\Pi)$, known also as Bergman-Jerbashian space, (see [6, 20, 21, 7]), is the subspace of holomorphic functions in $L^2_\lambda(\Pi)$. Note that $\lambda = 0$ corresponds to the unweighted case. The corresponding orthogonal projection B^λ_Π, from $L^p_\lambda(\Pi)$ onto $\mathcal{A}^p_\lambda(\Pi)$, is given by the formula

$$B^\lambda_\Pi f(z) = \int_\Pi K_\lambda(z,w) f(w)\ dA_\lambda(w) = -\frac{1}{\pi}\int_\Pi \frac{f(w)}{(z-\overline{w})^{2+\lambda}}\ dA_\lambda(w),\ z \in \Pi,$$

and is bounded for $1 < p < \infty$.

In this study, we consider Toeplitz operators with symbols $g = g(2y)$, where $y = \Im z$, acting on $\mathcal{A}^2_\lambda(\Pi)$. These operators may be unbounded, but anyway, at least for $g \in L^1_\lambda(\Pi)$ they are densely defined by the rule

$$T^\lambda_g f = B^\lambda_\Pi g f. \tag{1}$$

We refer to the books [15, 10, 26, 25, 24] for a general modern theory of Bergman type spaces and operators on Bergman type spaces. More specifically, a very comprehensive study of special classes of Toeplitz operators, including the class in which we are interested in this paper, is presented in the N.Vasilevski monograph [24].

Using the well-known structural properties of $\mathcal{A}^2_\lambda(\Pi)$ in [11] (see also [12] and [24]), the Toeplitz operator T^λ_g with vertical symbol g is reduced to the operator of multiplication $\mathcal{M}^\lambda_g = \gamma^\lambda_g(x)I$ which acts on the space $L^2(\mathbb{R}_+)$. More details will be given by Theorem 2 below.

We shall exploit this idea to study the product (in a sense of composition) of two Toeplitz operators with vertical symbols. We recall our main problem which is: given two Toeplitz operators T^λ_a and T^λ_b is there a symbol h such that $T^\lambda_a T^\lambda_b = T^\lambda_h$? It appears that the solution to this problem is closely related to Laplace transform techniques and the theory of fractional integrodifferentiation in the general weighted cases.

It is worth mentioning here that products of Toeplitz operators in different holomorphic spaces have been studied extensively since the last two decades. Nevertheless and despite all the partial results obtained by different authors, we are still far from a complete answer to the question when the product of two Toeplitz operators is another Toeplitz operators. For more thorough treatments of this subject, the reader might refer to the following references [5, 1, 2, 4, 8, 16, 17, 18, 14].

Recently, a particular attention was paid to the so-called quasihomogeneous Toeplitz operators. A Toeplitz operator T_g is said to be quasihomogeneous if its symbol g can be written under the form $g(re^{i\theta}) = e^{ip\theta}\phi(r)$, where p is an integer and ϕ is a radial function. Quasihomogeneous symbols are considered to be a generalization of the class of radial symbols (i.e. when $p = 0$). Such operators act on the othogonal basis of the Bergman space of the unit disk as a shift operator with holomorphic weight. Various promising results were obtained for this class of operators. We refer the reader to [4, 16, 17, 19, 18, 14]. Furthermore, in [13] a partial answer to whether there exists or not a symbol h such that $T^\lambda_a T^\lambda_b = T^\lambda_h$ in the case of the unit ball of $\mathbb{C}$, by using Fourier analysis and the Wiener ring theory. Vertical Toeplitz operators certainly admit realization in the framework of the unit disk (see Section 2), but the corresponding symbols will not be radial (or quasihomogeneous) functions. At our best knowledge, we are not aware of any manuscript in the current literature dealing with the product of two vertical Toeplitz operators in the specific context of the mentioned above main problem.

The paper is organized as follows. In Section 2 we collect auxiliary facts. Section 3 is devoted to the specified above main problem. For the sake of clarity, we consider the unweighted and weighted cases separately, the former being a simplification of the latter. Finally, we state some open questions.

2. Auxiliary statements and definitions

2.1. **The class $AC^k(\alpha,\beta)$.** By $AC^1(\alpha,\beta) \equiv AC(\alpha,\beta)$ we denote the class of absolutely continuous functions on the interval (α,β). It is known that $f \in AC^1(\alpha,\beta)$ if and only if it is a primitive of a Lebesgues integrable function on (α,β). By $AC^k(\alpha,\beta)$, $k = 2, 3, \dots$, we denote the class of continuously differentiable up to the order $k-1$ functions f on (α,β) with $f^{(k-1)} \in AC^1(\alpha,\beta)$. The following known fact sheds a light on the functions from $AC^k(\alpha,\beta)$.

Lemma 1. *([23], Lemma 2.4). The class $AC^k(\alpha,\beta)$ consists only of functions f for which the following representation holds*

$$f(t)=\frac{1}{(k-1)!}\int_0^t (t-\tau)^{k-1}\varphi(\tau)d\tau+\sum_{j=0}^{k-1}C_j(t-\alpha)^j,$$

for some $\varphi\in L^1(\alpha,\beta)$, and some constants C_j.

2.2. **On spectral representation of Toeplitz operators with vertical symbols.** In order to simplify formulas here and in what follows we take $g=g(2y)$ for the (so-called vertical) symbol of Toeplitz operator T_g^λ acting on $\mathcal{A}_\lambda^2(\Pi)$. In [11] (see also [12] and [24]), it was shown that any such Toeplitz operator T_g^λ with vertical symbol g can be reduced to the operator of multiplication by the function

$$\gamma_g^\lambda(x)=\frac{x^{1+\lambda}}{\Gamma(1+\lambda)}\int_0^\infty g(t)\ t^\lambda\ e^{-xt}dt,\quad x\in\mathbb{R}_+, \tag{2}$$

which acts on the space $L^2(\mathbb{R}_+)$. The closure of the range of this function provides the spectrum for the corresponding Toeplitz operator. Precisely, the following theorem holds.

Theorem 2. *([11]) Toeplitz operator T_g^λ with the symbol $g=g(2y)$ acting on $\mathcal{A}_\lambda^2(\Pi)$, is unitary equivalent to the operator of multiplication $\mathcal{M}_g^\lambda=\gamma_g^\lambda I$, acting on $L^2(\mathbb{R}_+)$. Moreover,* $\mathrm{sp}T_g^\lambda=\overline{\{\gamma\in\mathbb{C}:\gamma=\gamma_g^\lambda(x),\ \text{all } x\in\mathbb{R}_+^1\}}$.

Using Theorem 2, we can understand the action of the Toeplitz operator (1) on $L_\lambda^2(\Pi)$ considering it as being unitary equivalent to the operator of multiplication by $\gamma_g^\lambda I$, acting on $L^2(\mathbb{R}_+)$. Thus we may consider bounded and unbounded Toeplitz operators. However, we prefer to deal with operators which are initially bounded on $L_\lambda^2(\Pi)$. We shall specify the corresponding assumptions in the beginning of Section 3.

Recall that vertical Toeplitz operators possess the following realization in the framework of the unit disk (see [24]). Let $\mathbb{D}=\{z=x+iy:|z|<1\}$ stand for the unit disk, and $\partial\mathbb{D}$ stand for its boundary (unit circle). Consider the space

$$L_\lambda^2(\mathbb{D})=L^2(\mathbb{D},(\lambda+1)(1-|z|^2)^\lambda\frac{1}{\pi}dxdy),\quad \text{where}\quad z=x+iy,$$

and let $\mathcal{A}_\lambda^2(\mathbb{D})$ be the corresponding Bergman space. For a given a point $z_0\in\partial\mathbb{D}$, consider all Euclidean circles tangent to $\partial\mathbb{D}$ at z_0. Consider the class of all Toeplitz operators $T_{\widehat{g}}^\lambda$ acting on $L_\lambda^2(\mathbb{D})$ with the symbols $\widehat{g}$ which are constant on the above mentioned circles. All such classes are reduced (up to a rotation) to the class of operators corresponding to $z_0=i$. The conformal map $z=\Phi(w)=\frac{w-i}{1-iw}$, from Π onto $\mathbb{D}$, maps the lines $\{z\in\Pi:z=x+iy_0,\ x\in\mathbb{R},\ y_0>0 \text{ is fixed}\}$ into the circles tangent to $\partial\mathbb{D}$ at the point $z_0=i$. The unitary operator $U:L_\lambda^2(\mathbb{D})\to L_\lambda^2(\Pi)$

$$Uf(z)=\left(\frac{\sqrt{2}}{1-iw}\right)^{2+\lambda} f\circ\Phi(w), \tag{3}$$

provides the following relation

$$T_{g\circ\Phi}^\lambda=U^{-1}T_g^\lambda U \tag{4}$$

between the Toeplitz operator T_g^λ with vertical symbol acting on $L_\lambda^2(\Pi)$ and the Toeplitz operators $T_{\widehat{g}}^\lambda$ acting on $L_\lambda^2(\mathbb{D})$ with the symbol $\widehat{g}=g\circ\Phi$ which is constant on the mentioned above circles. We note that the inverse map is given by the relation:

$$U^{-1}\varphi(w) = \left(\frac{\sqrt{2}}{1+iw}\right)^{2+\lambda} \varphi \circ \Psi(w), \quad \Psi(w) = \frac{w+1}{1+iw}.$$

2.3. **On fractional integrodifferentiation and Laplace transform.** For fractional integrodifferentiation we refer to the books [23, 22]. We will use fractional integrals on the whole real axis which are defined by

$$I_+^\alpha \varphi(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x \frac{\varphi(t)}{(x-t)^{1-\alpha}} dt, \quad x \in \mathbb{R}, \tag{5}$$

$$I_-^\alpha \varphi(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{\varphi(t)}{(t-x)^{1-\alpha}} dt, \quad x \in \mathbb{R}, \tag{6}$$

or as for the convolution

$$I_\pm^\alpha \varphi(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \varphi(x \mp t) dt, \quad x \in \mathbb{R}. \tag{7}$$

Fractional integrals $I_\pm^\alpha$ are defined for functions $\varphi \in L^p(\mathbb{R})$ if $0 < \Re\alpha < 1$ and $1 < p < \frac{1}{\Re\alpha}$. It is known that

$$I_\pm^\alpha e^{\pm\theta x} = \theta^{-\alpha} e^{\pm\theta x}, \text{ for } \Re\theta > 0 \text{ and } \Re\alpha > 0. \tag{8}$$

We will also need the following fractional integration on half-axis

$$I_{0+}^\alpha \varphi(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}} dt, \quad x \in \mathbb{R}_+. \tag{9}$$

The Laplace transform of a function φ defined for all real numbers $t \geqslant 0$, is the function $\mathcal{L}\varphi$, defined by:

$$\mathcal{L}\varphi(z) = \int_0^\infty \varphi(t)\, e^{-zt}\, dt, \quad z = x + iy. \tag{10}$$

The meaning of $\mathcal{L}\varphi$ depends on the class of functions of interest. A necessary condition for existence of the integral is that φ must be locally integrable on $\mathbb{R}_+^1$. For locally integrable φ that is of exponential type, the integral can be understood as a (proper) Lebesgue integral. For instance, if $|\varphi(t)| \leqslant \alpha e^{\beta t}$ for some nonnegative constants α, β and all $t > s_0$, then the Laplace transform $\mathcal{L}\varphi$ is correctly defined as a function in $\{z = x + iy \in \mathbb{C} : x > \beta\}$. The Laplace convolution product of two functions φ and ψ is defined by the integral

$$\varphi \circ \psi(x) = \int_0^x \varphi(t)\psi(x-t) dt, \quad x \in \mathbb{R}_+,$$

so that the Laplace transform of the convolution is given by

$$\mathcal{L}(\varphi \circ \psi)(z) = (\mathcal{L}\varphi)(z)\,(\mathcal{L}\psi)(z),$$

as long as the objects in the above formula exist.

3. Product of vertical Toeplitz operators

3.1. **Statement of the main problem.** We formulate the main problem as follows. Given two vertical Toeplitz operators T_a^λ and T_b^λ bounded on $\mathcal{A}_\lambda^2(\Pi)$ with the symbols a and b, find a function (symbol) h such that

$$T_a^\lambda T_b^\lambda = T_h^\lambda. \tag{11}$$

Here and everywhere below we consider operator relations on a dense set of polynomials in $\mathcal{A}_\lambda^2(\Pi)$. In view of the Theorem 2, the above problem is equivalent to the problem of finding a function h such that

$$\gamma_a^\lambda(x)\gamma_b^\lambda(x) = \gamma_h^\lambda(x), \quad x \in \mathbb{R}_+^1. \tag{12}$$

Here and in what follows, we impose the following admissibility conditions on a symbol g of vertical Toeplitz operator T_g^λ :

(i) $g(t)t^\lambda \in L^1(0,B)$, for all positive constants B;
(ii) for any $\varepsilon > 0$ there exist $A_\varepsilon \geqslant 0$, $t_\varepsilon > 0$, such that $|g(t)| \leqslant A_\varepsilon e^{\varepsilon t}$, all $t \geqslant t_\varepsilon$.
(iii) Toeplitz operator T_g^λ is bounded on $\mathcal{A}_\lambda^2(\Pi)$.

We say that a symbol g of a Toeplitz operator T_g^λ is admissible provided g satisfies the above stated conditions $(i) - (iii)$.

3.2. **Unweighted case ($\lambda = 0$).** We start with the unweighted case $\lambda = 0$. For the unweighted case we will avoid using the index "0" in the notation and simply write T_g, γ_g, $\mathcal{A}^2(\Pi)$, instead of T_g^0, γ_g^0, $\mathcal{A}_0^2(\Pi)$.

Theorem 3. *Let T_a and T_b be vertical operators on $\mathcal{A}^2(\Pi)$ with admissible symbols a and b. If there exists a function h on $\mathbb{R}_+^1$ satisfying $(i) - (ii)$ and such that*

$$\int_0^t [h(\tau) - a(\tau)b(t-\tau)]\, d\tau = 0, \quad \text{all} \quad t \in \mathbb{R}_+^1, \tag{13}$$

then

$$T_a T_b = T_h. \tag{14}$$

Proof. In view of Theorem 2, for a Toeplitz operator T_g, we consider the corresponding function γ_g given by

$$\begin{aligned}\gamma_g(x) &= x\int_0^\infty g\,(t)\, e^{-xt}dt = x^2\int_0^\infty g(t)\int_t^\infty e^{-x\tau}d\tau \\ &= x^2\int_0^\infty e^{-x\tau}d\tau\int_0^\tau g(t)dt = x^2\int_0^\infty \widetilde{g}(\tau)\; e^{-x\tau}d\tau,\end{aligned}$$

where we denoted

$$\widetilde{g}(t) = \int_0^t g(\tau)d\tau.$$

Hence, the relation

$$\gamma_a(x)\gamma_b(x) = \gamma_h(x), \quad x \in \mathbb{R}_+^1,$$

becomes

$$\left(\int_0^\infty a(t)e^{-xt}dt\right)\left(\int_0^\infty b(t)e^{-xt}dt\right) = \int_0^\infty \widetilde{h}(t)e^{-xt}dt, \quad x \in \mathbb{R}_+^1,$$

which, in terms of Laplace transform, reads as

$$\left(\mathcal{L}a\right)(x)\left(\mathcal{L}b\right)(x) = \left(\mathcal{L}\widetilde{h}\right)(x), \quad x \in \mathbb{R}_+^1.$$

It suffices to check the above equality for $\widetilde{h}(t) = \int_0^t h(\tau)d\tau$. In virtue of (13) we have for such $\widetilde{h}$ and for $x \in \mathbb{R}^1_+$,

$$\begin{aligned}\left(\mathcal{L}\widetilde{h}\right)(x) &= \int_0^\infty \widetilde{h}(t)e^{-xt}dt = \int_0^\infty e^{-xt}dt \int_0^t h(\tau)d\tau \\ &= \int_0^\infty e^{-xt}dt \int_0^t a(\tau)b(t-\tau)d\tau = \int_0^\infty a(\tau)d\tau \int_\tau^\infty b(t-\tau)e^{-xt}dt \\ &= \int_0^\infty a(\tau)e^{-x\tau}d\tau \int_0^\infty b(t)e^{-xt}dt = (\mathcal{L}a)(x)\,(\mathcal{L}b)(x), \quad x \in \mathbb{R}^1_+,\end{aligned}$$

where the change of order of integration is justified by Foubini's theorem. This formula, in view of Theorem 2, proves the statement of the theorem. □

Theorem 4. *Let T_a and T_b be vertical Toeplitz operators on $\mathcal{A}^2(\Pi)$ with admissible symbols a and b. Assume either a or b is differentiable and has finite limit value at origin (for instance, let it be the function b). Then the function h defined by*

$$h(t) = a(t)b(0) + \int_0^t a(\tau)\ b'(t-\tau)\ d\tau$$

is such that

$$T_aT_b = T_h.$$

Proof. By differentiating equation (13) we obtain

$$h(t) = \widetilde{h}'(t) = a(t)b(0) + \int_0^t a(\tau)\ b'(t-\tau)\ d\tau.$$

This formula, in view of Theorem 2, proves the statement of the theorem. □

3.3. Weighted case $\lambda > -1$.

Lemma 5. *If a function g satisfies $(i)-(ii)$, then the following relation holds*

$$\gamma_g^\lambda(x) = \left(\frac{x^{1+\lambda}}{\Gamma(1+\lambda)}\right)^2 \int_0^\infty \widetilde{g}_\lambda(\xi)e^{-x\xi}d\xi, \quad x \in \mathbb{R}^1_+,$$

where

$$\widetilde{g}_\lambda(t) \equiv \int_0^t \frac{g(\xi)\xi^\lambda}{(t-\xi)^{-\lambda}}d\xi = \Gamma(1+\lambda)\left(I_{0+}^{1+\lambda}g(\xi)\xi^\lambda\right)(t). \tag{15}$$

Proof. We start with the integral on the right side of equation (2). Taking into account equation (8), we have

$$\begin{aligned}\int_0^\infty g(t)t^\lambda e^{-xt}dt &= x^{1+\lambda}\int_0^\infty g(t)t^\lambda \frac{1}{x^{1+\lambda}}e^{-xt}dt \\ &= x^{1+\lambda}\int_0^\infty g(t)t^\lambda \left(I_-^{1+\lambda}e^{-x\xi}\right)(t)dt \\ &= x^{1+\lambda}\int_0^\infty g(t)t^\lambda \left(\frac{1}{\Gamma(1+\lambda)}\int_t^\infty \frac{e^{-x\xi}}{(\xi-t)^{-\lambda}}d\xi\right)dt \\ &= \frac{x^{1+\lambda}}{\Gamma(1+\lambda)}\int_0^\infty e^{-x\xi}d\xi \int_0^\xi \frac{g(t)t^\lambda}{(\xi-t)^{-\lambda}}dt \\ &= \frac{x^{1+\lambda}}{\Gamma(1+\lambda)}\int_0^\infty \widetilde{g}_\lambda(\xi)e^{-x\xi}d\xi.\end{aligned}$$

The change of order of integration is justified by Fubini's theorem. Comparing the obtained formula with equation (2) completes the proof.

□

Theorem 6. *Let $\lambda > -1$ and let T_a^λ and T_b^λ be vertical Toeplitz operators on $\mathcal{A}_\lambda^2(\Pi)$ with admissible symbols a and b. If there exists a function h on $\mathbb{R}_+^1$ satisfying $(i)-(ii)$ and such that*

$$\int_0^t [h(\tau) - a(\tau)b(t-\tau)]\,\tau^\lambda (t-\tau)^\lambda d\tau = 0, \quad t \in \mathbb{R}_+^1, \tag{16}$$

then

$$T_a^\lambda T_b^\lambda = T_h^\lambda. \tag{17}$$

Proof. In view of Lemma 5, equation (12), i.e.,

$$\gamma_a^\lambda(x)\gamma_b^\lambda(x) = \gamma_h^\lambda(x), \quad x \in \mathbb{R}_+^1,$$

becomes

$$\left(\mathcal{L}a(t)t^\lambda\right)(x)\left(\mathcal{L}b(t)t^\lambda\right)(x) = \Gamma(1+\lambda)\left(\mathcal{L}\left(I_{0+}^{1+\lambda}h(\xi)\xi^\lambda\right)(t)\right)(x).$$

It suffices to check the above equality for h satisfying (16). We have for $x \in \mathbb{R}_+^1$,

$$\begin{aligned}
\Gamma(1+\lambda)\left(\mathcal{L}\left(I_{0+}^{1+\lambda}h(\xi)\xi^\lambda\right)(t)\right)(x) &= \int_0^\infty \left(I_{0+}^{1+\lambda}h(\xi)\xi^\lambda\right)(t)e^{-xt}dt \\
&= \int_0^\infty \left(I_{0+}^{1+\lambda}a(\xi)b(t-\xi)\xi^\lambda\right)(t)e^{-xt}dt \\
&= \int_0^\infty e^{-xt}dt \int_0^t a(\tau)\tau^\lambda\, b(t-\tau)(t-\tau)^\lambda\, d\tau \\
&= \left(\mathcal{L}a\right)(x)\left(\mathcal{L}b\right)(x), \quad x \in \mathbb{R}_+^1,
\end{aligned}$$

where the change of order of integration is justified by Foubini's theorem.This formula, in view of Theorem 2, proves the statement of the theorem. □

In what follows, we shall give necessary and sufficient conditions for the product of two vertical Toeplitz operators to be again a vertical Toeplitz operator using Riemann-Liouville operators. For simplicity, we introduce the following notations:

$$\begin{aligned}
\varphi_\lambda(t) &= h(t)t^\lambda, \\
f_\lambda(t) &= \frac{1}{\Gamma(1+\lambda)}\int_0^t a(\tau)\tau^\lambda\, b(t-\tau)(t-\tau)^\lambda\, d\tau.
\end{aligned}$$

First, we consider the case $-1 < \lambda < 0$.

Theorem 7. *Let $-1 < \lambda < 0$ and let T_a^λ and T_b^λ be vertical Toeplitz operators on $\mathcal{A}_\lambda^2(\Pi)$ with admissible symbols a and b. There exists admissible symbol h such that*

$$T_a^\lambda T_b^\lambda = T_h^\lambda$$

if and only if

$$I_{0+}^{-\lambda}f_\lambda(t) = \frac{1}{\Gamma(-\lambda)}\int_0^t \frac{f_\lambda(\tau)}{(t-\tau)^{1+\lambda}}d\tau \in AC(0,B), \text{ for any } B > 0, \text{and} \tag{18}$$

$$I_{0+}^{-\lambda}f_\lambda(0) = \frac{1}{\Gamma(-\lambda)}\int_0^t \frac{f_\lambda(\tau)}{(t-\tau)^{1+\lambda}}d\tau\bigg|_{t=0} = 0. \tag{19}$$

Proof. As in Theorem 6 we see that our problem reduces to the well-known Abel equation

$$I_{0+}^{1+\lambda}\varphi(t) = f(t),$$

considered on an arbitrary interval $(0, B)$, $B > 0$, and where we should replace φ with φ_λ and f with f_λ :

$$I_{0+}^{1+\lambda}\varphi_\lambda(t) = f_\lambda(t), \quad t \in (0, B).$$

Certainly, if we solve the Abel equation with $\varphi = \varphi_\lambda$ and $f = f_\lambda$ for an arbitrary $B > 0$, then we recover the function h. It is known that if the solution φ of the Abel equation exists, then it must be of the form

$$\varphi(t) = \frac{1}{\Gamma(-\lambda)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{1+\lambda}} d\tau.$$

Thus the solution φ is unique for any arbitrary B, and hence it is uniquely defined on $\mathbb{R}^1_+$. Moreover, it is well-known that the Abel equation has a solution $\varphi \in L^1(0, B)$ (for any arbitrary $B > 0$) if and only if the conditions (18) and (19) are satisfied. Therefore, $\varphi_\lambda(t) = h(t)t^\lambda \in L^1(0, B)$ for arbitrary $B > 0$, and also satisfies (ii), and the operator identity $T_a^\lambda T_b^\lambda = T_h^\lambda$, considered on a dense set of polynomials in $\mathcal{A}^2_\lambda(\Pi)$, is valid by construction. Hence, h generates bounded operator which means that the symbol h is admissible. This finishes the proof. □

Remark 1. *If $f \in AC(\alpha, \beta)$, then $I_{0+}^\theta f \in AC(\alpha, \beta)$, for any $\theta \in (0, 1)$. Therefore, the condition $f_\lambda \in AC(0, B)$ is sufficient for the validity of the condition (18) (i.e. for $I_{0+}^{-\lambda} f_\lambda \in AC(0, B)$).*

Since the unweighted case $\lambda = 0$ was considered separately at the beginning of this section, we therefore turn our attention to the case $\lambda > 0$. We shall denote by $[\lambda]$ the entire part of λ.

Theorem 8. *Let $\lambda > 0$ and let T_a^λ and T_b^λ be vertical Toeplitz operators on $\mathcal{A}^2_\lambda(\Pi)$ with admissible symbols a and b. Then there exists admissible symbol h such that*

$$T_a^\lambda T_b^\lambda = T_h^\lambda$$

if and only if

$$(20) \quad I_{0+}^{1+[\lambda]-\lambda} f_\lambda(t) = \frac{1}{\Gamma(1+[\lambda]-\lambda)} \int_0^t \frac{f_\lambda(\tau)}{(t-\tau)^{-[\lambda]+\lambda}} d\tau \in AC^{1+[\lambda]}(0, B),$$

for any $B > 0$, and

$$\left(\frac{d}{dt}\right)^k I_{0+}^{1+[\lambda]-\lambda} f_\lambda(0) = \frac{1}{\Gamma(1+[\lambda]-\lambda)} \left(\left(\frac{d}{dt}\right)^k \int_0^t \frac{f_\lambda(\tau)}{(t-\tau)^{-[\lambda]+\lambda}} d\tau \right) \Bigg|_{t=0} = 0,$$

for $k = 0, 1, \ldots, [\lambda]$.

Proof. The proof follows the lines of the proof of Theorem 7 with the use of Theorem 2.3 from [23]. We leave it to the reader. □

Remark 2. *The condition $f_\lambda \in AC^{1+[\lambda]}(0, B)$ is sufficient for the validity of the condition (20) (i.e. for $I_{0+}^{1+[\lambda]-\lambda} f_\lambda \in AC^{1+[\lambda]}(0, B)$).*

4. Conclusion

We strongly believe that the technique developed here will help in studying particular problems and questions related to the product of vertical Toeplitz operators, such as the existence of Brown-Halmos type theorem i.e., a description of symbols a and b such that $T_a^\lambda T_b^\lambda = T_{ab}^\lambda$ or the nonzero zero divisor i.e., are there nonzero symbols a and b such that $T_a^\lambda T_b^\lambda = 0$? Partial result to these two problems were obtained for some specific class of Toeplitz operators (e.g. quasihomogeneous Toeplitz operators) in the unit disk, the ball of $\mathbb{C}^n$ and other domains (see the references provided below). However, we believe that for the case of vertical Toeplitz operators more challenging computations may occur.

Acknowledgements Alexey Karapetyants is partially supported by the Russian Foundation for Fundamental Research, projects 18-01-00094 and 18-51-05009. Part of this research was conducted during his Fulbright Research Scholarship research stay at SUNY-Albany and under support of the Fulbright Outreach Lecturing Fund.

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