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To cite this article: S Sakhi 2021 *J. Phys. Commun.* **5** 055011

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## PAPER

## OPEN ACCESS

RECEIVED  
22 March 2021REVISED  
28 April 2021ACCEPTED FOR PUBLICATION  
5 May 2021PUBLISHED  
14 May 2021

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# Renormalization functions of the tricritical $O(N)$ -symmetric $\Phi^6$ model beyond the next-to-leading order in $1/N$

S Sakhi<sup>1,2</sup>

Department of Physics, College of Arts and Sciences, American University of Sharjah, P O Box 26666 Sharjah, United Arab Emirates  
Materials Science and Engineering Research Institute, American University of Sharjah, United Arab Emirates

E-mail: [ssakhi@aus.edu](mailto:ssakhi@aus.edu)Keywords: renormalization, effective potential,  $1/N$  expansion

## Abstract

We investigate higher-order corrections to the effective potential of the tricritical  $O(N)$ -symmetric  $\Phi^6$  model in  $3-2\epsilon$  dimensions in its phase exhibiting spontaneous breaking of its scale symmetry. The renormalization group  $\beta$ -function and the anomalous dimension  $\gamma$  of this model are computed up to the next-to-next-to-leading order in the  $1/N$  expansion technique and using a dimensional regularization in a minimal subtraction scheme.

## 1. Introduction

The  $O(N)$ -symmetric model of  $N$  components scalar fields  $\Phi = (\phi_1, \phi_2, \dots, \phi_N)$  with a Landau-Ginzburg potential in three space-time dimensions given by  $V(\Phi) = r|\Phi|^2 + u|\Phi|^4 + \eta|\Phi|^6$  has an interesting phase diagram. When  $u > 0$ , there is a line of second order phase transition at  $r = 0$ . When  $u < 0$  there is a line of first order phase transition. These lines of transition terminate at a tricritical point where  $r = u = 0$  at which the potential has only a sextic interaction term  $\eta|\Phi|^6$ . At the classical level, this phase structure persists for all positive values of  $\eta$  and the sextic term is marginal. This model offers a unique platform to study several interesting aspects of quantum field theory [1], as well as critical and tricritical behavior in condensed matter systems observed, e.g., in liquid helium and in metamagnets [2].

The  $O(N)$  model has a soluble large  $N$  limit. This is facilitated by using auxiliary fields, which offer the advantage of containing no implicit  $N$  dependence. Consequently, the integration over the original field variables yields an effective theory in terms of the auxiliary fields in which all- $N$  dependence is fully explicit, enabling the analysis of higher order corrections in a systematic  $1/N$  expansion [3]. This technique leads to non-perturbative results, and has a long history in the study of critical phenomena, going back to the work of Stanley [4], who studied the  $N$  vector model using the saddle point technique and Ma [5], who calculated  $1/N$  corrections to the critical exponents and clarified in the same context the Wilson renormalization group ideas. Large- $N$  analysis was also applied successfully to model field theories such as the Gross-Neveu and the  $CP^{N-1}$  models [6], as well as to the topological Ginzburg-Landau theory of self-dual Josephson junction arrays [7].

In the strict  $N = \infty$  limit, the tricritical model has a nonperturbative UV fixed point at which a mass is dynamically generated, resulting in the spontaneous breaking of scale symmetry and the appearance of a massless dilaton, which is a Goldstone mode [8]. Several aspects of that phenomenon including  $1/N$  corrections were analyzed in [9, 10]. Other related investigations included models with abelian and non-Abelian Chern-Simons gauge fields [11] and the fate of light dilaton under  $1/N$  corrections [12]. We should also refer to recent work in [13] done within the functional renormalization group framework, which interprets the model's fixed point (BMB FP) in [8] as the intersection between a line of regular FPs and another line made of singular FPs. It delineates the dependence on  $N$  and on the space-time dimension  $d$  to preserve the critical behavior, and it shows the existence of non-perturbative fixed points with which the BMB FP can collide in the  $(N, d)$  phase space as  $N$  varies from infinity to finite values. In this work, we focus on higher-order corrections to the effective potential that were not considered before. We compute the renormalization group beta ( $\beta$ ) and anomalous dimension ( $\gamma$ ) functions up to the next-to-next-to-leading order in  $1/N$  expansion. These are important not

only for understanding the behavior of the running coupling constant of the model but also they can be used as an input to perform a renormalization group improvement to the effective potential, which should clarify the nature of the vacuum state of the model.

The paper is organized as follows: In section 2 we introduce our notation for the O(N)-invariant scalar field theory with sextic interaction in the Euclidean formulation and we derive the effective action in terms of auxiliary fields, including higher order interaction terms. In section 3, we analyze the effective potential and we evaluate the renormalization group flow functions including corrections up to  $1/N^2$ . In section 4, we give our conclusion and suggest further developments.

## 2. General formalism

We are interested in a theory with the following Hamiltonian density in Euclidean space

$$H = (\partial\Phi)^2/2 + NV(\Phi^2/N) \quad (1)$$

where  $\Phi$  is an N-component scalar field. We restrict ourselves to the model with renormalized  $u = r = 0$  and henceforward the potential is given by

$$V(\chi) = \frac{\eta}{6}\chi^3 \quad (2)$$

This model is renormalizable in  $(2 + 1)$ -dimensional space-time by the standard power counting procedure and possesses the usual UV divergences. It's worth noting that the  $|\Phi|^6$  term with a critical dimension  $d = 3$  is a special case of  $\Phi^{2n}$ -like interactions that can be renormalized perturbatively in  $d_c = 2n/(n - 1)$ , and its renormalization gives the critical properties in terms of an  $\varepsilon$ -expansion in  $d = d_c - \varepsilon$  dimensions. To derive an effective field theory in a form in which N appears explicitly as a parameter, we exploit the idea that for N large, O(N) invariant quantities like  $|\Phi|^2$  self-average and therefore have small fluctuations [3]. This suggests taking  $|\Phi|^2$  as a dynamical variable rather than  $\Phi$ . To implement this idea, we start from the generating functional of Euclidean correlation functions of Hamiltonian density (1), inside of which we insert  $1 = \int D\chi \delta(\chi - \Phi^2/N)$  and use a Fourier representation of the  $\delta$ -function by means of an integral over an auxiliary field  $\sigma$  along the imaginary axis as follows

$$\begin{aligned} Z[J] &= \int D\Phi \exp\left(-\int d^3x [(\partial\Phi_a)^2/2 - NV(\Phi^2/N) - J_a \cdot \Phi_a]\right) \\ &= \int D\Phi D\chi D\sigma \exp\left(-\int d^3x [(\partial\Phi)^2/2 + \sigma\Phi^2/2 - N\sigma\chi/2 + NV(\chi) - J_a \cdot \Phi_a]\right) \\ &= \int D\chi D\sigma \frac{1}{[\det(-\partial^2 + \sigma)]^{N/2}} \exp\left(-\int d^3x [-N\sigma\chi/2 + NV(\chi)]\right) \\ &\quad + \frac{1}{2} \int d^3x \int d^3y J_a(x) G(x, y) J_a(y) \\ &= \int D\chi D\sigma \exp\left(-S_{eff}(\chi, \sigma) + \frac{1}{2} \int d^3x \int d^3y J_a(x) G(x, y) J_a(y)\right) \end{aligned} \quad (3)$$

where

$$S_{eff} = \frac{N}{2} \text{Tr} \ln(-\partial^2 + \sigma) + N \int d^d x [-\sigma\chi/2 + V(\chi)] \quad (4a)$$

$G(x, y)$  is the Green function defined by

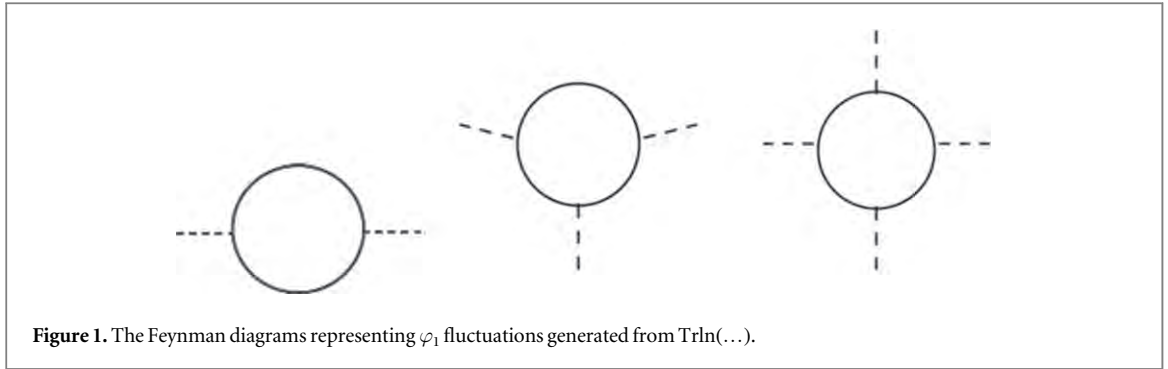
$$G^{-1}(x, y) = (-\partial_x^2 + \sigma(x))\delta(x - y) \quad (4b)$$

Such transformations were motivated in order to make the action quadratic in the  $\Phi$  scalar fields and facilitate the functional integration over them, resulting in a functional determinant, which was recast as the  $\text{Tr} \ln(\cdot)$  term in equation (4a).

In the effective action (4a),  $N$  plays the role of  $1/\hbar$ ; hence, for large  $N$ , we can use the saddle-point method to expand the path integral around the configurations that make the effective action extremum. This yield the gap equations

$$\left. \frac{\delta S_{eff}}{\delta \sigma} \right|_{\chi=\bar{\chi}, \sigma=m^2} = 0 \quad (5a)$$

$$\left. \frac{\delta S_{eff}}{\delta \chi} \right|_{\chi=\bar{\chi}, \sigma=m^2} = 0 \quad (5b)$$



which give, after Fourier transform,

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 + m^2} = \chi \tag{6a}$$

$$m^2 = 2V'(\chi) \tag{6b}$$

The integral over momentum space on the left hand side of (6a) is ultraviolet divergent and needs to be regularized. Introducing a cut-off  $\Lambda$  followed by a non-multiplicative renormalization of the field,  $\chi_R = \chi - \Lambda/(2\pi^2)$ , leads to the dynamical generation of a mass for the  $\Phi$  fields given by  $m = -4\pi\chi$ , provided that the renormalized field  $\chi$  takes a negative vacuum expectation value. Furthermore, these steps necessitate fine-tuning the coupling constant  $\eta$  to the Bardeen-Moshe-Bander fixed-point value  $\eta = 16\pi^2$  [8].

### 3. The 1/N corrections

Having established a vacuum structure consisting of constant fields configurations for  $\chi$  and  $\sigma$ , which give the main non-perturbative properties of the theory, in order to go beyond the strict  $N = \infty$  limit, we expand the effective action around the vacuum and compute the radiative corrections to the effective potential and the scalar propagators in the shifted theory:  $\sigma \rightarrow m^2 + i\varphi_1/\sqrt{N}$ ,  $\chi \rightarrow \chi + \varphi_2/\sqrt{N}$ . At an intermediate step we find

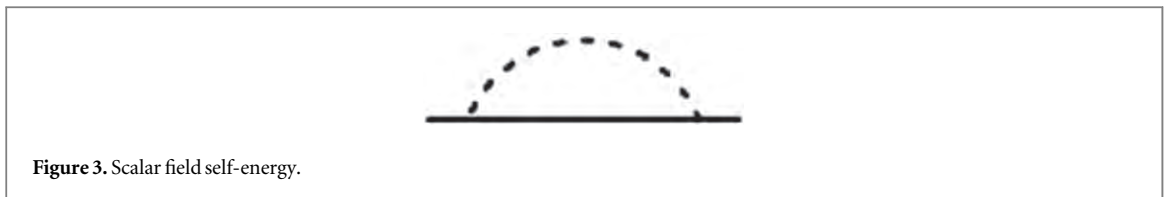
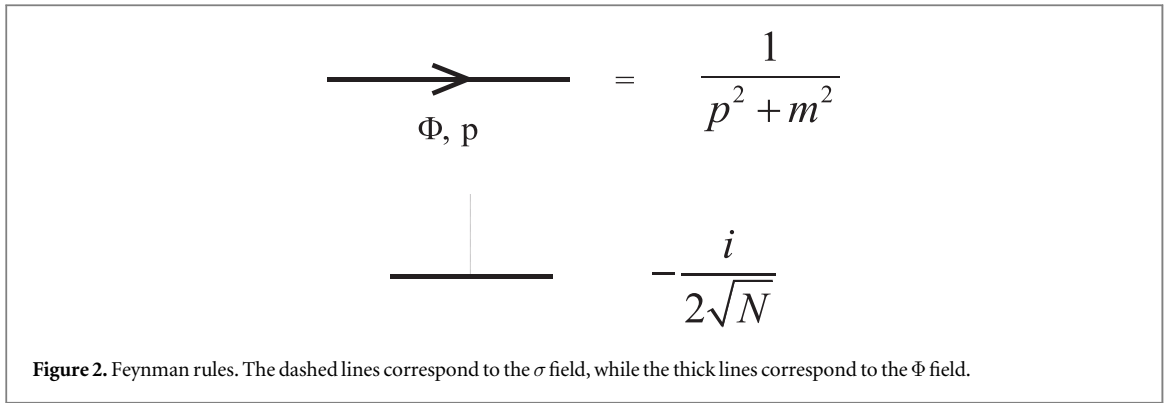
$$\begin{aligned} S = & \int -N \frac{m^3}{12\pi} - N \frac{m^2}{2} \chi + NV(\chi) \\ & + \frac{1}{2} \int (\varphi_1, \varphi_2) \begin{bmatrix} \Delta/2 & -i/2 \\ -i/2 & V'' \end{bmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} + \frac{\eta}{6\sqrt{N}} \int \varphi_2^3(x) \\ & - \frac{i}{6\sqrt{N}} \int S^{(3)}(x_1, x_2, x_3) \varphi_1(x_1) \varphi_1(x_2) \varphi_1(x_3) \\ & - \frac{1}{8N} \int S^{(4)}(x_1, x_2, x_3, x_4) \varphi_1(x_1) \varphi_1(x_2) \varphi_1(x_3) \varphi_1(x_4) + \dots \end{aligned} \tag{7}$$

Where  $\Delta$  represents a two-point function that arises from the quadratic term of the expansion of the  $\text{Tr} \ln(\dots)$  in (4a) and which can be visualized as a Feynman graph as shown in the first diagram in figure 1. The same expansion also generates higher order terms involving  $S^{(n)}$ , which represent the cubic, quadratic, etc interactions as shown in the second and third diagrams of figure 1.

The Feynman rules for constructing such diagrams are represented in figure 2.

The representation of  $\Delta(q, m)$  in  $d = 3 - 2\epsilon$  momentum space is

$$\begin{aligned} \Delta(q, m^2) = & \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2} \frac{1}{(p+q)^2 + m^2} \\ = & \frac{\Gamma(2 - d/2)}{(4\pi)^{d/2}} \left( m^2 + \frac{q^2}{4} \right)^{d/2-2} F \left[ 2 - \frac{d}{2}, \frac{1}{2}, \frac{3}{2}, \left( 1 + \frac{4m^2}{q^2} \right)^{-1} \right] \\ = & \frac{\Gamma(1/2 + \epsilon)}{(4\pi)^{3/2-\epsilon}} \frac{G(q, m, \epsilon)}{q^{2(1/2+\epsilon)}} \end{aligned} \tag{8a}$$



where the function  $G$  possesses the properties

$$G(q, m, \varepsilon) = \begin{cases} 2^{2\varepsilon} \sqrt{\pi} \frac{\Gamma(1/2 - \varepsilon)}{\Gamma(1 - \varepsilon)} & \text{for } q \rightarrow \infty \\ 2 \operatorname{Arctan}\left(\frac{q}{2m}\right) & \text{for } \varepsilon = 0 \end{cases} \quad (8b)$$

This form using dimensional regularization will be useful later when evaluating corrections of the effective potential beyond the next-to-leading order in the  $1/N$  expansion because calculation of graphs with a cutoff are extremely complicated and nearly impossible to perform when there are many loops.

### 3.1. Next-to-leading order corrections

Integrating out the quadratic small fluctuations of the  $\varphi$ -fields is straightforward and gives the next-to-leading order in the  $1/N$  expansion correction to the effective potential

$$V_{\text{eff}}^{(2)} = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \ln [1 + 2V''(\chi) \Delta(p)] \quad (9)$$

The integration in (9) presents UV divergence in the first three terms in the expansion of the logarithm in terms of  $V''(\chi)$  given in (2). To handle such divergences, we express the effective potential in terms of a renormalized mass  $M$  defined from the self-energy  $\Sigma$  shown in diagram of figure 3.

$$M^2 = m^2 + \frac{1}{N} \Sigma_1(0, m^2) - \frac{m^2}{N} \frac{\partial \Sigma_1(p^2, m^2)}{\partial p^2} \Bigg|_{p^2=0} \quad (10a)$$

with

$$\begin{aligned} \Sigma_1(p, m^2) &= \int \frac{d^3 q}{(2\pi)^3} \frac{D_{11}(q)}{(p+q)^2 + m^2} \\ &= \int \frac{d^3 q}{(2\pi)^3} \frac{4V''}{((p+q)^2 + m^2)(1 + 2V''\Delta(q))} \end{aligned} \quad (10b)$$

Where  $D_{11}(q)$  is the propagator of the  $\varphi_1$  field obtained by inverting the quadratic form in (7).

At this order of the  $1/N$  expansion, the divergences in (10b) can be isolated either by a cutoff or by a dimensional regularization in the MS scheme. In what follows, we chose the latter technique because it is more practical later when calculating graphs at higher orders of the  $1/N$  expansion technique.

$$\begin{aligned}\Sigma_1(0, m^2) &\rightarrow \int \frac{d^d q}{(2\pi)^d} \frac{4V''}{q^2 + m^2} (1 - 2V''\Delta(q) + (2V'')^2\Delta^2(q) + \dots) \\ &\rightarrow -\frac{(V'')^2}{8\pi^2} \left( \frac{1}{\varepsilon} + 4 \ln\left(\frac{\mu}{m}\right) \right) + \text{finite terms}\end{aligned}\quad (10c)$$

We should note that  $\partial\Sigma/\partial p^2|_{p=0}$  has no divergences and hence there is no wave-function renormalization at this order. Using the renormalized mass, the leading terms in the effective potential (7) are transformed in this manner

$$-N\frac{m^3}{12\pi} - N\frac{m^2}{2}\chi \rightarrow -N\frac{M^3}{12\pi} - N\frac{M^2}{2}\chi - \frac{\eta^2\chi^2M}{2^6\pi^3\varepsilon} - \frac{\eta^2\chi^3}{2^4\pi^2\varepsilon}\quad (11)$$

Similarly, we extract the divergences in (9) as follows

$$\begin{aligned}\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \ln([1 + 2V''(\chi)\Delta(p)]) &\rightarrow \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} 2V''(\chi)\Delta(p) \\ &- [V'']^2 \int \frac{d^d p}{(2\pi)^d} \Delta^2(p) + \frac{4}{3} [V'']^3 \int \frac{d^d p}{(2\pi)^d} \Delta^3(p) + \dots \\ &\rightarrow \frac{\eta^2\chi^2M}{2^6\pi^3\varepsilon} + \frac{\eta^3\chi^3}{3 \cdot 2^{11}\pi^2} \left( \frac{1}{\varepsilon} + 8 \ln\left(\frac{\mu}{m}\right) \right) + \text{Finite terms}\end{aligned}\quad (12)$$

We notice the cancelation of the diverging terms  $\eta^2\chi^2M$  appearing in (11) and (12). Thus, the only remaining divergent terms involve only  $\chi^3$ , which are cancelled by adding counterterms to  $V(\chi)$  that in turn directly give rise to a relation between the bare coupling constant  $\eta$  and the renormalized one  $\eta_R$  at a scale  $\mu$  as follows

$$\eta = \mu^{4\varepsilon} \left( \eta_R + \frac{a(\eta_R)}{\varepsilon} \right)\quad (13a)$$

where the coefficient  $a(\eta)$  is chosen to cancel the poles in the  $\chi^3$  terms in (11) and (12), i.e.

$$a(\eta) = \frac{3\eta^2}{8\pi^2N} - \frac{\eta^3}{2^{10}\pi^2N}\quad (13b)$$

We note the arbitrary parameter  $\mu$ , which in dimensional regularization came from the fact that the coupling constant  $\eta$  outside the critical dimension  $d = 3$  has a mass dimension. That freedom in defining the coupling constant at some arbitrary scale  $\mu$  leads to the renormalization group functions. In particular, to obtain the beta function  $\beta(\eta_R, \varepsilon) = \mu d\eta_R/d\mu|_{\eta, \varepsilon}$ , we apply  $d/d\mu$  on both sides of (13a); we obtain (after dividing by  $\mu^{4\varepsilon}$ )

$$\beta(\eta_R, \varepsilon) = -4\varepsilon\eta_R + 4 \left( \eta_R \frac{d}{d\eta_R} - 1 \right) a(\eta_R)\quad (14a)$$

Taking the limit  $\varepsilon \rightarrow 0$ , we obtain the beta function as

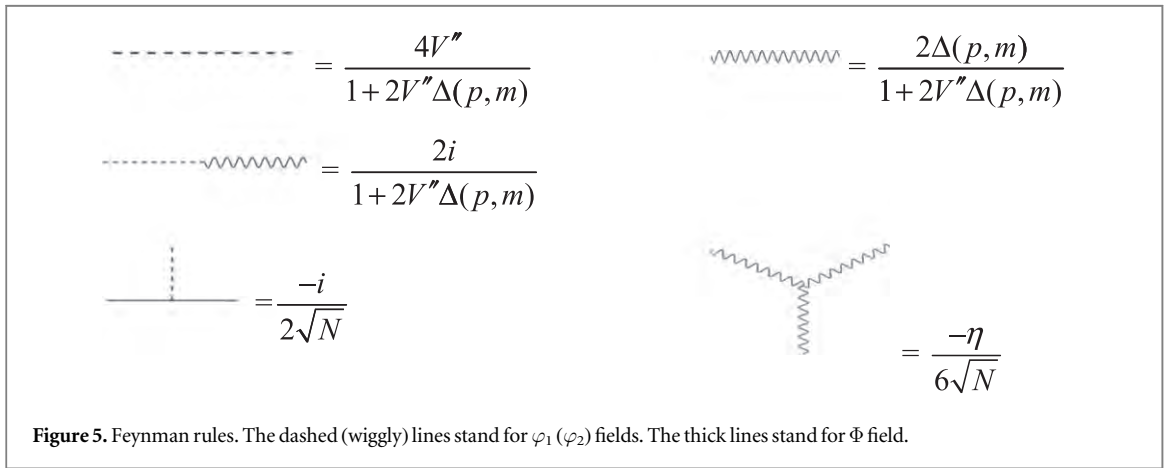
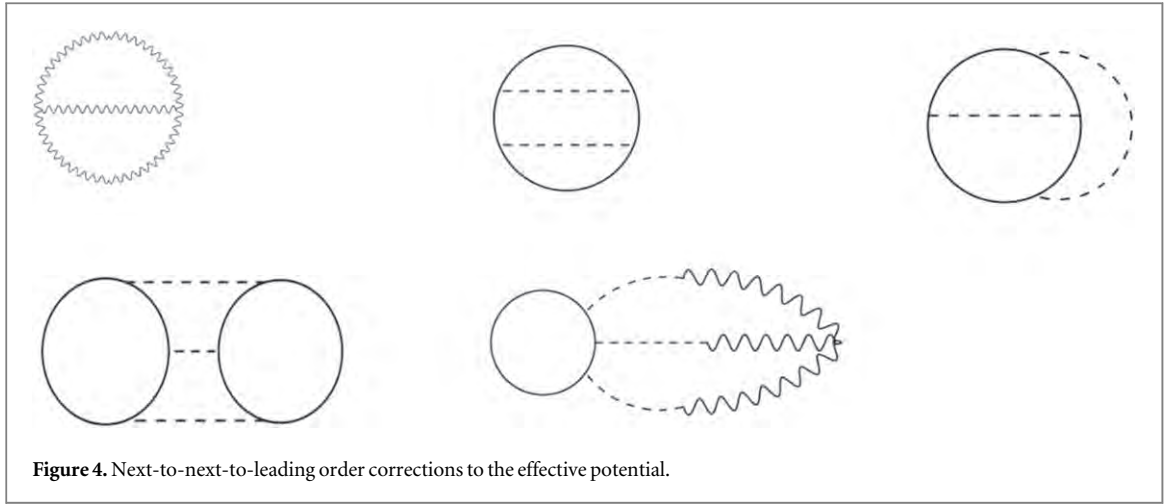
$$\beta(\eta_R) = \frac{3\eta_R^2}{2\pi^2N} - \frac{\eta_R^3}{2^7\pi^2N}\quad (14b)$$

This reproduces the large  $N$  limit of the known perturbative beta function [14, 15]. The resulting effective potential to the next-to-leading order expressed now in terms of the renormalized mass  $M$  and renormalized coupling is

$$\begin{aligned}V_{\text{eff}} &= N \int -\frac{M^3}{12\pi} - \frac{M^2}{2}\chi + \frac{\eta}{6}\chi^3 + \frac{1}{2N} \left( \chi + \frac{M}{4\pi} \right) \left( \Sigma_1^{\text{Finite}} - \frac{MV''}{2\pi} \right) \\ &+ \left[ \frac{1}{2N} \int \frac{d^3 p}{(2\pi)^3} \ln([1 + 2V''(\chi)\Delta(p)]) \right]_{\text{Finite part}} - \frac{1}{6} \chi^3 \beta(\eta) \ln\left(\frac{\mu}{M}\right)\end{aligned}\quad (15)$$

### 3.2. Next-to-next-to-leading order corrections

At the next-to-next-to-leading order in the  $1/N$  expansion (NNLO), we include in our computation the cubic and the quartic terms in (7). The diagrams shown in figure 4 give the new corrections to the effective potential and the new Feynman rules of the theory of three fields with nontrivial propagators derived from (7) are summarized in figure 5. It is clear one must consider additional functions besides the effective potential in order to determine the renormalization group parameters; we must take account of also self-energy corrections, which are shown in figure 6.



Divergences also arise at this next order in the  $1/N$  expansion with some diagrams having double poles in  $\varepsilon$ . Technically, we used the method of dimensional regularization in the  $\overline{\text{MS}}$  scheme to eliminate the logarithmic divergences at  $d = 3$  which show up as poles in  $\varepsilon$  ( $d = 3 - 2\varepsilon$ ) and by introducing wave function and coupling constant renormalization. The bare fields  $\Phi$  and couplings  $\eta$  are expressed via renormalized quantities as

$$\Phi = Z_\Phi^{1/2} \Phi_R \tag{16a}$$

$$\eta = Z_\eta \mu^{4\varepsilon} \eta_R = \frac{Z_3}{Z_\Phi^3} \mu^{4\varepsilon} \eta_R \tag{16b}$$

The  $Z$  factors are chosen to absorb all the poles at  $\varepsilon = 0$  in the expansion of the vertex functions. The freedom in defining coupling at some arbitrary scale  $\mu$  leads to the renormalization group functions (beta and anomalous dimension functions)

$$\beta(\eta) = \mu \frac{d\eta}{d\mu} = \frac{-4\varepsilon\eta}{1 + \eta \frac{d \ln(Z_\eta)}{d\eta}} \tag{17a}$$

$$\gamma_d = \beta(\eta) \frac{d \ln(\sqrt{Z_\Phi})}{d\eta} \tag{17b}$$

A detailed description of the renormalization scheme of minimal subtraction of  $\varepsilon$ -poles for dimensionally regularized vertex functions and the general rules used to derive the RG equations are given in [16]. We find

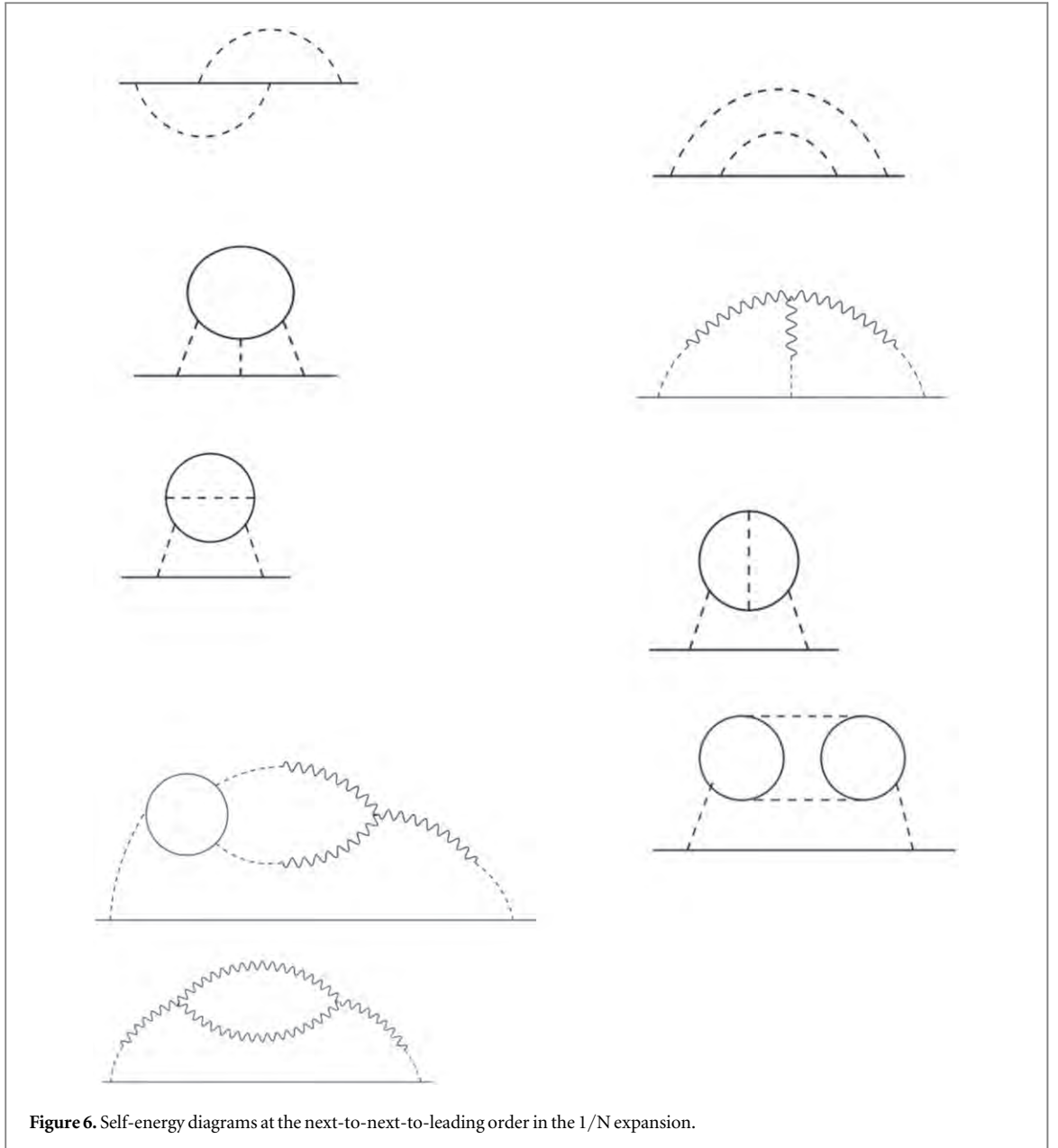


Figure 6. Self-energy diagrams at the next-to-next-to-leading order in the  $1/N$  expansion.

$$\begin{aligned}
 Z_3 = 1 &+ \left[ \frac{3\eta}{2^3\pi^2\epsilon} - \frac{\eta^2}{2^{10}\pi^2\epsilon} \right] \frac{1}{N} \\
 &+ \left[ \frac{11\eta}{4\pi^2\epsilon} + \frac{9\eta^2}{2^6\pi^4\epsilon^2} - \frac{213\eta^2}{2^9\pi^4\epsilon} - \frac{17\eta^2}{2^9\pi^2\epsilon} \right. \\
 &+ \left. \left( -\frac{7}{2^{13}\pi^4\epsilon^2} + \frac{3}{2^9\pi^4\epsilon} - \frac{\ln(2)}{2^{10}\pi^4\epsilon} + \frac{5}{3 \cdot 2^{13}\pi^2\epsilon} \right) \eta^3 \right] \\
 &\times \frac{1}{N^2} + O\left(\frac{1}{N^3}\right)
 \end{aligned} \tag{18}$$

$$Z_\Phi = 1 - \frac{\eta^2}{3 \cdot 2^9\pi^4 N^2 \epsilon} + O\left(\frac{1}{N^3}\right) \tag{19}$$

The  $\beta$ -function and the anomalous dimension function  $\gamma_d$  are then obtained from (17a) and (17b) by repeating similar calculation as we did after equation (13a)



$$\beta(\eta) = \frac{3\eta^2}{2\pi^2 N} \left(1 - \frac{\eta}{3 \cdot 2^6}\right) + \frac{\eta^2}{\pi^2 N^2} \left[11 - \left(\frac{53}{2^4 \pi^2} + \frac{17}{2^6}\right)\eta\right] + \left(-\frac{3 \ln(2)}{2^8 \pi^2} + \frac{5}{2^{11}} + \frac{9}{2^7 \pi^2}\right)\eta^2 + \mathcal{O}\left(\frac{1}{N^3}\right) \quad (20)$$

$$\gamma_d = \frac{\eta^2}{3 \cdot 2^7 \pi^4 N^2} + \mathcal{O}\left(\frac{1}{N^3}\right) \quad (21)$$

## 4. Conclusion

In this paper we investigated higher-order corrections to the effective potential of the massless  $\Phi_3^6$  scalar model in its phase exhibiting spontaneous breaking of scale symmetry up to the next-to-next-to-leading order in the  $1/N$ -expansion. The use of the background field method and dimensional regularization in the minimal subtraction scheme simplified considerably the calculation of the renormalization group beta ( $\beta$ ) and anomalous dimension ( $\gamma$ ) functions. Higher-order corrections to the  $\beta$  and  $\gamma$  functions are useful not only in understanding the behavior of the running coupling constant of a theory but also they can be used as an input to RG-improve the effective potential. This is indeed the case since renormalization induces a scale parameter  $\mu$  that did not exist in the initial action, as a result the effective potential must satisfy a renormalization group partial differential equation involving the  $\beta$ -function of the renormalized coupling and the anomalous dimension  $\gamma$ . This means that the explicit dependence on RG scale  $\mu$  in the effective potential can be compensated by redefining the coupling constant, through its  $\beta$  function, and rescaling the field based on its anomalous dimension  $\gamma$ . From this perspective, the knowledge of  $\beta$  and  $\gamma$  at a higher order in the  $1/N$  expansion can be useful in the inverse problem of recovering the effective potential, through the method of characteristics as was done in the context of  $\Phi_4^4$  model in [17]. Namely, one can compute the potential to fixed order at a scale  $\mu_0$ , in terms of  $\eta(\mu_0)$  and  $\mu_0$  and then evolve to some other scale  $\mu$  by solving the RG equation simultaneously with the RG equation for the coupling constant. The solution of the RG equation would give an improved effective potential that sums the leading (and subleading...) logarithms, which would extend its domain of perturbative credibility and might shed more light on whether the RG improved potential has any bearing on the issue of stability of the scale symmetry-breaking phase. We hope to return to investigate this problem in the near future.

## Acknowledgments

This work was funded by a faculty research grant (FRG) from the American University of Sharjah.

## Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

## ORCID iDs

S Sakhi  <https://orcid.org/0000-0002-7651-4708>

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