# Approval Signatures

We, the undersigned, approve the Master’s Thesis of Roozbeh Falah Ramezani.

Thesis Title: Non-linear modeling and control of unmanned air vehicle

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Abstract

The aim of this thesis is to design and simulate a Dynamic Inversion based autopilot for a fixed wing aircraft. The autopilot provides the aircraft motion stability by commanding the different aircraft control surfaces and achieve a given set of performance specifications. The Dynamic Inversion controller performance is evaluated and tested against classical autopilots. Issues related to stability and robustness are dealt with during the autopilot design. Almost all today’s autopilots are design based on linear aircraft models using PID techniques. To insure adequate performance of such simple PID autopilot, flight envelope is divided into many flight modes, each with different autopilot structure. Normally, a linear gain scheduler is used to transition these autopilots from one flight condition to the next. To achieve the performance required, a very complex code is designed and implemented. Such large software overhead makes the testing and certification very costly and very time consuming. In addition, under stringent flight conditions, it is very difficult to guarantee the performance and robustness of the system. Dynamic Inversion controllers are a natural solution to the performance and robustness problem. Dynamic Inversion reduces the code size significantly and simplifies the testing and validation process of the system. So the autopilot can be run on a smaller and lighter hardware. In addition, the stability and robustness of the Dynamic Inversion controller are taken into account during design using mu-synthesis and therefore less testing is required. In this thesis a 6 DOF, 12 states rigid body non-linear model of a small fixed-wing aircraft is developed. Gravity, aerodynamic, and propulsion forces and moments are considered. Linearized models are extracted for a set of given points by trimming the non-linear model about some specified flight conditions. A Dynamic Inversion control-law is derived and robustness and stability issues are addresses during design. Our simulated results show that the performance specification of the system is satisfied and that the tracking of the Dynamic Inversion controller is by far much better than that of a classical PID.

Search Terms: UAV, Dynamic Inversion, Autopilot, Non-linear controller, Aircraft
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Abbreviations

3D Three Dimensional
DOF Degree Of Freedom
DI Dynamic Inversion
PID Proportional Integral Derivative
PI Proportional Integral
PD Proportional Derivative
LFT Linear Fraction Transform
UAV Unmanned Air Vehicle
μ Singular structured value
m Mass
g Gravity acceleration
L Lift force
T Thrust force
D Drag force
α Angle of attack
β Side-slip attack
\(V_a\) Air speed
\(x_i\) Position x in inertial frame
\(y_i\) Position y in inertial frame
\(z_i\) Position z in inertial frame
\(x_b\) Position x in body frame
\(y_b\) Position y in body frame
\(z_b\) Position z in body frame
\(h\) Aircraft altitude in inertial frame
\(u\) Aircraft velocity in x direction in body frame
\(v\) Aircraft velocity in y direction in body frame
\(w\) Aircraft velocity in z direction in body frame
\(\varphi\) Roll angle
\(\theta\) Pitch angle
\(\psi\) Yaw angle
\(p\) Roll rate
\(q\) Pitch rate
\(r\) Yaw rate
\(\vec{F}\) Force vector
\(\vec{V}\) Velocity vector
\(\vec{V}_c\) Center of gravity velocity vector
\(\vec{r}\) Position vector
\(\vec{\omega}\) Angular velocity vector
\(\vec{M}\) Moment vector
\(\vec{H}\) Momentum vector
\(X\) Force in x direction
\(Y\) Force in y direction
\(Z\) Force in z direction
\(L\) Roll moment
\( M \) Pitch moment
\( N \) Yaw moment
\( I_x \) Moment of inertia about roll axis
\( I_y \) Moment of inertia about pitch axis
\( I_z \) Moment of inertia about yaw axis
\( I_{xy} \) Roll and pitch moment of inertia
\( I_{xz} \) Roll and yaw moment of inertia
\( I_{yz} \) Pitch and yaw moment of inertia
\( I \) Moment of inertia matrix
\( \text{det}(\ ) \) Determinant of matrix
\( \delta m \) Mass element
\( \delta F \) Force element
\( \delta M \) Moment element
\( \delta H \) Momentum element
\( b \) Wing span
\( S \) Effective wing area
\( \rho \) Air density
\( \bar{c} \) Mean aerodynamic chord
\( S_{\text{prop}} \) Propeller effective area
\( C_{\text{prop}} \) Propeller aerodynamic coefficient
\( k_{\text{motor}} \) Propeller DC motor constant
\( \delta_a \) Aileron deflection
\( \delta_e \) Elevator deflection
\( \delta_r \) Rudder deflection
\( \delta_t \) Thrust
\( R \) Flight path curve radius
\( \gamma \) Flight path angle
\( C_D \) Drag aerodynamic coefficient
\( C_L \) Lift aerodynamic coefficient
\( C_{l_{\alpha}} \) Lift curve slop
\( C_{D_{\alpha}} \) Drag curve slop
\( C_n \) Longitudinal static stability derivative
\( C_{l_{\beta}} \) Roll static stability derivative
\( C_{n_{\beta}} \) Yaw static stability derivative
\( C_{m_{\alpha}} \) Pitch damping derivative
\( C_{l_{\delta}} \) Roll damping derivative
\( C_{n_{\delta}} \) Yaw damping derivative
\( C_{m_{\delta}} \) Elevator control derivative
\( C_{l_{\kappa}} \) Aileron cross-control derivative
\( C_{n\varphi} \)  Rudder cross-control derivative
\( L_\beta \)  Dihedral effect
\( L_p \)  Roll rate damping
\( L_r \)  Rolling moment due to yaw rate
\( M_{\delta_e} \)  Change in the pitching moment due to elevator deflection
\( M_u \)  Stability derivative of the longitudinal motion in \( u \) direction
\( M_w \)  Stability derivative of the longitudinal motion in \( w \) direction
\( M_q \)  Stability derivative of the longitudinal motion in \( q \) direction
\( M_\theta \)  Stability derivative of the longitudinal motion in \( \theta \) direction
\( N_\beta \)  Directional stability
\( N_p \)  Yawing moment due to roll rate
\( N_r \)  Yaw rate damping
\( X_{\delta_e} \)  Stability derivative of axial force \( X \) due to elevator deflection
\( X_u \)  Stability derivative of the aerodynamic force \( X \) in \( u \) direction
\( X_w \)  Stability derivative of the aerodynamic force \( X \) in \( w \) direction
\( Y_\beta \)  Side force due to change in side-slip angle
\( Y_p \)  Side force due to yaw rate
\( Z_{\delta_e} \)  Stability derivative of the normal force due to elevator deflection
\( Z_u \)  Stability derivative of the normal force in \( u \) direction
\( Z_w \)  Stability derivative of the normal force in \( w \) direction
1 Introduction

A fixed wing aircraft is a complex non-linear system, as shown on Figure 1-1, four resultant forces are acting on the aircraft. Gravity (mg) due to weight of aircraft pulls it down toward earth. The Lift force (L) compensates for gravity forces and keep the aircraft airborne. Lift is generated by wings.

![Figure 1-1: Forces acts on aircraft](image)

Once the aircraft is moving with a velocity with respect to the air mass, the wings act as airfoils and lift, drag, side forces, and moments are generated. Figure 1-2, shows the air flow on top of the wing moving faster than that below the wing.

![Figure 1-2: Airflow over wing](image)

Faster airflow causes lower dynamic pressure, on the other hand slower airflow on bottom of wing makes higher dynamic pressure, such difference pressure multiply by effective area of the wing interpreted as lift force. Figure 1-3, shows the dimension of wing which is used to calculate effective area of the wing.

Whether aircraft is using propeller or jet engine, the role of them is generate thrust force (T), thrust force moves the aircraft toward surrounding air mass to have airspeed and consequently lift force to fly aircraft.

Drag forces (D) is the force apposing the movement of aircraft in the air, it is due to friction between air molecules and surface of airframe, by the way the air resistance aerodynamic behavior is complex and vary with the speed of the object moving in the air.
If airspeed considered as a velocity vector it is the most important parameter to control the aircraft.

In most of aeronautics texts, instead of considering airspeed as a velocity vector the magnitude of that are called airspeed and to have complete description, angle of attack ($\alpha$) and side-slip angle ($\beta$) is used as shown in Figure 1-4. Once aircraft is flying in the job of controller is keeping airspeed magnitude, angle of attack, side-slip angle and roll angle of aircraft. Airspeed magnitude is controlled by engine thrust (propeller or jet). For adjust the roll angle, aircraft is equipped with the control surfaces called aileron, as shown in Figure 1-5, ailerons are rotating in opposite direction and their deflection has major effect on the roll angle.

Angle of attack could be controlled by pitch angle of aircraft, in order to adjust pitch angle, elevator control surface is used, Figure 1-6.
Similar to the above control surfaces, rudder deflection is used to control yaw angle and side-slip angle, Figure 1-7.

In commercial aircrafts in addition to major control surfaces: aileron, elevator and rudder; there are other control surfaces such flaps, spoilers and so on. In this thesis we develop the controller using three major control surfaces only and in most of parts the thrust remains constant.

The thesis is divided into four sections, first we drive non-linear model of aircraft this section is according to procedures in [1], [2].

In next section conventional linear PID controller is developed for the Zagi aircraft model in [1].

After developing linear controller, using the method in [3], dynamic inversion controller is developed for the same aircraft model.

In the last section, dynamic inversion controller robustness and stability is tested using \( \mu \)-synthesis tool of MATLAB® under uncertain model parameter conditions, and finally comparison between the performance of PID and dynamic inversion controller with uncertain model parameters is evaluated.

In addition to the main procedure explained above, several other approaches are reviewed. In [7], a model predictive control strategy is used to regulate the aircraft motion about arbitrary time variant trajectories. The research resulted in a successful flight tests and the control system performed a range of aerobatic maneuvers autonomously.

In [9], two types of controller are developed to fly the FROG airframe model in simulation. Classical linear controllers are designed using Simulink Response Optimization (SRO) tool of MATLAB® / SIMULINK®. Optimal controller designed using linear quadratic (LQ) method.
In [10], a robust nonlinear controller is designed for a complete UAV mission, employing a combination of DI and H∞ control.

In [11], a continuous time model predictive controller is designed and simulated for “Ariel” airframe model using MATLAB® / SIMULINK®. The model is linearized at several different operating conditions.

In [12], the Generalized Dynamic Inversion method is developed for flight control, in this method a prescribed dynamic constraint will be inverted rather than inverting the system itself. As per [12], this method alleviates the need to make simplifying assumptions regarding the controlled plant that is often required in classical dynamic inversion to make deriving the inverse equation of motion feasible.

In [13], a global stability result is presented for nonlinear F-14 aircraft pitch-axis models with DI control laws. The stability region of the F-14 model is sketched in terms of the engine thrust and the selected pitch attitude.

In [14], modern control laws are developed for the AH-64D Longbow Apache helicopter to provide improved handling qualities for hover and low speed flight. The control laws use a model following approach to generate commands fed to the existing partial authority control system.

In [15], a new straightforward technique is proposed based on dynamic inversion which is applied for tracking the pilot commands using F-16 model.

In [16], the Yao’s adaptive robust control is implemented to an aircraft system. The control methodology is implemented as an outer loop controller to an aircraft under nonlinear dynamic inversion control. The control methodology is implemented on a full envelope, high fidelity simulation of the F-15 IFCS aircraft as well as on a lower fidelity full envelope F-5A simulation.

In [17], a control augmentation system is presented based on the dynamic model inversion architecture for a highly maneuverable aircraft. The pseudo-control hedging algorithm is applied to prevent or delay the instability of the control augmentation system due to a slow actuator or occurrence of actuator saturation.
2 Modeling

In this section the non-linear model of aircraft will be developed, in the modeling of aircraft we define two major axes system: Inertial or earth frame which is shown in Figure 2-1, the origin of the frame is some convenient point on surface, X axis points toward north, Y axis toward east and Z axis points toward center of earth.

![Figure 2-1: Inertial Frame (Earth) [7]](image1)

Next frame is called body frame, origin is the center of gravity of aircraft, X axis points toward aircraft nose, Y axis points toward right wing and Z axis point toward down.

![Figure 2-2: Body Frame [7]](image2)

The aircraft assumed to be a rigid body in 3D space, the translation and rotation equation will be developed for the aircraft under influence of unknown forces and moments. Then forces and moments equation will be derived from aerodynamic coefficients. Once the non-linear model is developed, the linear model will be developed around a certain working condition for small change in the states, transfer functions and state space linear models will be developed in this section.
2.1 Rigid body dynamics

The states used here to develop the rigid body model are the same in [1], list of all twelve states as follows:

- $x_i$: North position of aircraft in inertial (earth) frame
- $y_i$: East position of aircraft in inertial (earth) frame
- $h = -z_i$: Altitude of aircraft in inertial (earth) frame
- $u$: Aircraft velocity along $x$ direction in body frame
- $v$: Aircraft velocity along $y$ direction in body frame
- $w$: Aircraft velocity along $z$ direction in body frame
- $\phi$: Roll angle of aircraft in body frame
- $\theta$: Pitch angle of aircraft in body frame
- $\psi$: Yaw angle of aircraft in body frame
- $p$: Roll rate in body frame
- $q$: Pitch rate in body frame
- $r$: Yaw rate in body frame

Rigid body equation of motion will be developed in four steps as follows:

1. Translation equation of motion
2. Rotation equation of motion
3. Navigation equation - Position
4. Navigation equation - Orientation

Translation equation of motion:

According to Newton laws we have:

$$\sum F = \frac{d}{dt}(mV) \quad (2-1)$$

Where, $V = [u \ v \ w]^T$

Let $\delta m$ be an element of mass then we have: $\delta F = \delta m \frac{d\bar{V}}{dt}$

Sum of all forces is: $\sum \delta F = F$

Velocity of $\delta m$ is: $\bar{V} = \bar{V}_c + \frac{d\bar{r}}{dt}$

$\bar{V}_c$ is the translation velocity of center of gravity and $\bar{r}$ is the position vector of $\delta m$ relative to center of gravity.

$$\sum \delta F = F = \frac{d}{dt} \left( \sum \bar{V}_c \delta m + \sum \frac{d\bar{r}}{dt} \delta m \right) = \frac{d}{dt} \bar{V}_c \sum \delta m + \frac{d}{dt} \sum \frac{d\bar{r}}{dt} \delta m$$

$\Rightarrow F = m \frac{d}{dt} \bar{V}_c + \frac{d^2}{dt^2} \sum \bar{r} \delta m$
Since \( \vec{r} \) is measured from center of gravity then \( \sum \vec{r}\delta m = 0 \), so we have
\[
\vec{F} = m \frac{d\vec{V}_C}{dt}
\] (2-2)

According to relative velocity equation we have:
\[
\frac{d\vec{V}_C}{dt}\bigg|_{LocalFrame} = \frac{d\vec{V}_C}{dt}\bigg|_{BodyFrame} + \vec{\omega} \times \vec{V}_C \Rightarrow \vec{F} = m \left( \frac{d\vec{V}_C}{dt}\bigg|_{BodyFrame} + \vec{\omega} \times \vec{V}_C \right)
\]

Where: \( \vec{V}_C = [u \ v \ w]^T \) and \( \vec{\omega} = [p \ q \ r] \)
\[
\vec{F} = m \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + [p \ q \ r] \times \begin{bmatrix} u \\ v \\ w \end{bmatrix} = m \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} i \\ j \\ k \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix} = m \begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + pw - ur \\ \dot{w} + pv - uq \end{bmatrix}
\]

\[
\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} qw - rv \\ pw - ur \\ pv - uq \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} m \frac{\dot{u}}{m} \\ m \frac{\dot{v}}{m} \\ m \frac{\dot{w}}{m} \end{bmatrix} = \begin{bmatrix} F_x - qw + rv \\ F_y - pw + ur \\ F_z - pv + uq \end{bmatrix}
\]

\[
\vec{\omega} \times \vec{V} = \frac{1}{m} \vec{F} - \vec{\omega} \times \vec{V}
\] (2-3)

Rotation equation of motion:

Newton momentum equation of \( \delta m \)
\[
\delta \vec{M} = \frac{d}{dt}\delta \vec{H} = \frac{d}{dt}(\vec{r} \times \vec{V}) \delta m
\]
\[
\vec{V} = \vec{V}_C + \frac{d\vec{r}}{dt} = \vec{V}_C + \vec{\omega} \times \vec{r}
\]

The total momentum is:
\[
\vec{H} = \sum \delta \vec{H} = \sum (\vec{r} \times \vec{V}_C) \delta m + \sum [\vec{r} \times (\vec{\omega} \times \vec{r})] \delta m = \sum \vec{r} \delta m \times V_C + \sum [\vec{r} \times (\vec{\omega} \times \vec{r})] \delta m
\]
Since \( \overline{r} \) is measured from center of gravity then \( \sum \overline{r} \delta m = 0 \)

\[ \Rightarrow \overline{H} = \sum \left[ \overline{r} \times (\overline{\omega} \times \overline{r}) \right] \delta m \]

\[ \overline{r} = [x \ y \ z] \]

\[ \overline{\omega} = [p \ q \ r] \]

\[ \overline{\omega} \times \overline{r} = \begin{bmatrix} i & j & k \\ p & q & r \\ x & y & z \end{bmatrix} \]

\[ \overline{r} \times (\overline{\omega} \times \overline{r}) = [x \ y \ z] \times \begin{bmatrix} qz - ry \\ -pz + rx \\ py - qx \end{bmatrix} = \begin{bmatrix} T & J & K \\ x & y & z \end{bmatrix} \]

\[ \overline{H} = \sum \left( \begin{bmatrix} py^2 - yqx + pz^2 - ryz \\ qx^2 - pyx + qx^3 - ryz \\ rx^2 - pxz + ry^2 - qyz \end{bmatrix} \right) \delta m \Rightarrow \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} p \sum (y^2 + z^2) \delta m - q \sum xy \delta m + r \sum z \delta m \\ q \sum (x^2 + z^2) \delta m - p \sum xy \delta m - r \sum yz \delta m \\ r \sum (x^2 + y^2) \delta m - p \sum z \delta m - q \sum yz \delta m \end{bmatrix} \]

We define moment of inertia as follows:

\[ I_x = \iiint (y^2 + z^2) \delta m \]
\[ I_y = \iiint (x^2 + z^2) \delta m \]
\[ I_z = \iiint (y^2 + x^2) \delta m \]
\[ I_{xy} = I_{yx} = \iiint xy \delta m \]
\[ I_{xz} = I_{zx} = \iiint xz \delta m \]
\[ I_{yz} = I_{zy} = \iiint yz \delta m \]

The momentum equations will be reduced to:

\[ \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} p I_x - q I_{xy} - r I_{xz} \\ -p I_{xy} + q I_y - r I_{yz} \\ -p I_{xz} - q I_{yz} + r I_z \end{bmatrix} = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \Rightarrow \overline{H} = I \overline{\omega} \]

Implementing the relative equation of motion we have:

\[ \sum \overline{\dot{M}} = \frac{d\overline{H}}{dt} \mid _{\text{BodyFrame}} + \overline{\omega} \times \overline{\dot{H}} \Rightarrow \sum \overline{\dot{M}} = \overline{\dot{H}} \mid _{\text{BodyFrame}} + \overline{\omega} \times \overline{\dot{H}} \Rightarrow \sum \overline{\dot{M}} = I \overline{\dot{\omega}} + \overline{\omega} \times I \overline{\dot{\omega}} \]

\[ \begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \]

(2-4)
\begin{equation}
\mathbf{\omega} \times \mathbf{I}\mathbf{\omega} = \begin{bmatrix}
\hat{i} & \hat{j} & \hat{k} \\
p & q & r \\
pl_{x} - ql_{xy} - rl_{xz} & - pl_{xy} + ql_{y} - rl_{yz} & - pl_{xz} - ql_{yz} + rl_{z}
\end{bmatrix}
\end{equation}

So the equation \((2-4)\) will be simplified to the following:

\begin{align*}
L &= I_{x}\dot{p} - I_{xy}\dot{q} - I_{xz}\dot{r} - qpl_{xz} - q^{2}I_{yz} + ql_{z} + qrI_{xz} + r^{2}I_{xz} \\
\Rightarrow L &= I_{x}\dot{p} - I_{xy}\dot{q} + qr \left(I_{z} - I_{y}\right) - pql_{xz} + l_{xy} \left(rp - \dot{q}\right) + l_{yz} \left(r^{2} - q^{2}\right) \\
M &= -I_{xy}\dot{p} + I_{y}\dot{q} - I_{yz}\dot{r} + p^{2}l_{xz} + pql_{yz} - prl_{z} - prl_{x} - qrI_{xy} - r^{2}I_{xz} \\
\Rightarrow M &= I_{y}\dot{q} + I_{x}\dot{p} - I_{yz}\dot{r} + pq \left(p^{2} - r^{2}\right) - I_{xy}\dot{p} - I_{xz}\dot{r} + pql_{yz} - qrI_{xy} \\
N &= -I_{xz}\dot{p} - I_{yz}\dot{q} + I_{xz}\dot{r} - p^{2}l_{xy} + pql_{xyz} - prl_{z} - pql_{xy} + q^{2}l_{xy} + qrI_{xz} \\
\Rightarrow N &= I_{z}\dot{r} - I_{xz}\dot{p} + qrI_{xz} + pq \left(l_{y} - l_{x}\right) - I_{xy}\dot{q} + I_{xy} \left(q^{2} - p^{2}\right) - prI_{yz}
\end{align*}

On the other hand, due to symmetry of the airframe we have: \(I_{yz} = I_{xy} = 0\), so the momentum equation will be simplified more as follows:

\begin{align*}
L &= I_{x}\dot{p} - I_{xz}\dot{r} + qr \left(I_{z} - I_{y}\right) - pql_{xz} \\
M &= I_{y}\dot{q} + I_{x}\dot{p} \left(p^{2} - r^{2}\right) \\
N &= I_{z}\dot{r} - I_{xz}\dot{p} + qrI_{xz} + pq \left(l_{y} - l_{x}\right)
\end{align*}

We call back the vector equation for rotation:

\begin{equation}
\sum M = I_{x}\mathbf{\omega} + \mathbf{\omega} \times I_{\mathbf{\omega}} \Rightarrow I_{\mathbf{\omega}} = \sum M - \mathbf{\omega} \times I_{\mathbf{\omega}} \Rightarrow \dot{\mathbf{\omega}} = I^{-1} \left(\sum M - \mathbf{\omega} \times I_{\mathbf{\omega}}\right)
\end{equation}

And recall the formula for calculation the inverse of 3x3 matrix:

\begin{equation}
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}^{-1} = \frac{1}{\Delta} \begin{bmatrix}
a_{33}a_{22} - a_{32}a_{23} & a_{32}a_{13} - a_{33}a_{12} & a_{33}a_{12} - a_{23}a_{13} \\
a_{33}a_{23} - a_{32}a_{21} & a_{32}a_{11} - a_{33}a_{11} & a_{33}a_{11} - a_{23}a_{11} \\
a_{33}a_{21} - a_{32}a_{21} & a_{32}a_{12} - a_{33}a_{11} & a_{33}a_{12} - a_{23}a_{11}
\end{bmatrix}
\end{equation}

\begin{equation}
\Delta = a_{11} \left(a_{33}a_{22} - a_{32}a_{23}\right) + a_{23} \left(a_{33}a_{11} - a_{33}a_{11}\right) + a_{31} \left(a_{23}a_{12} - a_{23}a_{12}\right)
\end{equation}

Now we apply the formula for \(I\):

\begin{equation}
I = \begin{bmatrix}
I_{x} & -I_{xy} & -I_{xz} \\
-I_{xy} & I_{y} & -I_{yz} \\
-I_{xz} & -I_{yz} & I_{z}
\end{bmatrix}
\end{equation}
\[ \text{det}(I) = I_y \left( I_y I_z - I_{yz}^2 \right) - I_{xy} \left( I_{xy} I_z - I_{xz}^2 \right) - I_{xu} \left( I_{xu} I_y - I_{xy}^2 \right) \]

\[ \Rightarrow \text{det}(I) = I_x I_z I_y - I_x I_y I_{yz} - I_{xy} I_z I_{xz} - I_{xz} I_{xy} I_{yz} - I_{yz} I_x I_{xy} - I_{xy} I_{xz} I_{yz} \]

\[ \Rightarrow \text{det}(I) = I_y \left( I_x I_z - I_{xz}^2 \right) - I_x I_y I_{xz} - I_{xy} I_z I_{xz} - I_{xz} I_{xy} I_{yz} - I_{yz} I_x I_{xy} - I_{xy} I_{xz} I_{yz} \]

\[ I^{-1} = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix}^{-1} = \frac{1}{\text{det}(I)} \begin{bmatrix} I_x I_y & I_x I_{xz} - I_{yz}^2 & I_y I_{xz} \\ 0 & I_x I_y - I_{xz}^2 & 0 \\ I_{xz} I_y & 0 & I_y I_x \end{bmatrix} \]

Since \( I_{yz} = I_{xy} = 0 \) due to symmetry the formula will be simplified as follows:

\[ \text{det}(I) = I_y \left( I_x I_z - I_{xz}^2 \right) \]

\[ I^{-1} = \frac{1}{\text{det}(I)} \begin{bmatrix} I_x I_y & 0 & I_y I_{xz} \\ 0 & I_x I_y - I_{xz}^2 & 0 \\ I_{xz} I_y & 0 & I_y I_x \end{bmatrix} \]

Continue to calculate the \( \hat{\omega} \), we have:

\[ \hat{\omega} = I^{-1} \left( \sum M - \hat{\omega} \times I \hat{\omega} \right) = \]

\[ = \frac{1}{\text{det}(I)} \begin{bmatrix} I_x I_y & 0 & I_y I_{xz} \\ 0 & I_x I_y - I_{xz}^2 & 0 \\ I_{xz} I_y & 0 & I_y I_x \end{bmatrix} \begin{bmatrix} M_x + qpl_{xz} + q^2 I_{yz} - qrI_{z} - rpl_{xy} + rql_{y} - r^2 I_{xz} \\ M_y - p^2 I_{xz} + prl_{x} - prI_{y} + qrI_{yz} + r^2 I_{xz} \\ M_z + p^2 I_{xz} - pqI_{y} + prl_{x} + pqI_{z} - q^2 I_{yz} - qrI_{xz} \end{bmatrix} \]

By define new variable \( \Delta \):

\[ \Delta = I_z I_x - I_{xz}^2 \Rightarrow \text{det}(I) = I_x \Delta \]

\[ \Rightarrow \frac{\hat{\omega}}{\Delta} = \frac{1}{\Delta} \begin{bmatrix} I_x I_y & 0 & I_y I_{xz} \\ 0 & I_x I_y - I_{xz}^2 & 0 \\ I_{xz} I_y & 0 & I_y I_x \end{bmatrix} \begin{bmatrix} M_x + qpl_{xz} - qrI_{z} + rql_{y} \\ M_y - p^2 I_{xz} + prl_{x} - prI_{y} + r^2 I_{xz} \\ M_z - pqI_{y} + pqI_{z} - q^2 I_{yz} - qrI_{xz} \end{bmatrix} \]

\[ I_x I_y M_x + I_x I_y qpl_{xz} - I_x qrl_{z} + I_y I_x^2 I_y^2 + I_y I_x^2 pq + I_y I_x^2 p q + I_y I_x^2 p q + I_y I_x^2 p q - I_y I_x^2 p q + I_y I_x^2 p q - I_y I_x^2 p q \]

\[ \begin{bmatrix} I_x I_y (I_x I_y - I_{xz}^2) (M_y - p^2 I_{xz} + prl_{x} - prI_{y} + r^2 I_{xz}) \\ I_x I_y M_z + I_x I_y qpl_{xz} - I_x qrl_{z} + I_y I_x^2 I_y^2 + I_y I_x^2 pq + I_y I_x^2 p q + I_y I_x^2 p q + I_y I_x^2 p q - I_y I_x^2 p q + I_y I_x^2 p q - I_y I_x^2 p q \end{bmatrix} \]
The summarized equation is shown in (2-5):

\[
\dot{\omega} = \begin{bmatrix}
    \dot{p} \\
    \dot{q} \\
    \dot{r}
\end{bmatrix} = \begin{bmatrix}
    \left( I_x M_x + I_{xx} M_x + I_x pq + I_2 qr \right) / \Delta \\
    \left( I_y M_y + (I_z - I_x) pr - \left( p^2 - r^2 \right) I_{xx} \right) / I_y \\
    \left( I_z M_z + I_{zz} M_z - I_z qr + I_3 pq \right) / \Delta
\end{bmatrix}
\]

(2-5)

Where,

\[\Delta = I_x I_y - I_{xx}^2\]
\[I_1 = I_{xx} \left( I_x - I_y + I_z \right)\]
\[I_2 = I_y I_z - I_x^2 - I_{zz}^2\]
\[I_3 = I_{zz}^2 - I_y I_z + I_x^2\]

**Navigation equation – Position**

The aircraft orientation is described by an ordered set of three Euler angles \(\phi, \theta, \psi\) which relates the orientation of the body axis relative to the local axis.

Assume that \([x, y, z]^T\) is the position of a point in the local frame. The position in the body frame will be calculated by implementing the following rotations:

First a rotation about \(\psi\) axis:

\[
\begin{bmatrix}
    x_2 \\
    y_2 \\
    z_2
\end{bmatrix} = \begin{bmatrix}
    \cos \psi & \sin \psi & 0 \\
    -\sin \psi & \cos \psi & 0 \\
    0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
    x_1 \\
    y_1 \\
    z_1
\end{bmatrix}
\]

Second a rotation about \(\theta\) axis:

\[
\begin{bmatrix}
    x_3 \\
    y_3 \\
    z_3
\end{bmatrix} = \begin{bmatrix}
    \cos \theta & 0 & -\sin \theta \\
    0 & 1 & 0 \\
    \sin \theta & 0 & \cos \theta
\end{bmatrix} \begin{bmatrix}
    x_2 \\
    y_2 \\
    z_2
\end{bmatrix}
\]
Third a rotation about $\varphi$ axis:

$$
\begin{bmatrix}
x_b \\
y_b \\
z_b
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \varphi & \sin \varphi \\
0 & -\sin \varphi & \cos \varphi
\end{bmatrix}
\begin{bmatrix}
x_3 \\
y_3 \\
z_3
\end{bmatrix}
$$

Combining these rotations yields:

$$
\begin{bmatrix}
x_b \\
y_b \\
z_b
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & 0 -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_i \\
y_i \\
z_i
\end{bmatrix}
$$

To convert a position from inertial (local or earth) frame to body frame the following matrix multiplication should be used:

$$
\begin{bmatrix}
x_b \\
y_b \\
z_b
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & -\sin \varphi & \sin \varphi \\
0 & -\cos \varphi & \cos \varphi
\end{bmatrix}
\begin{bmatrix}
\cos \theta & 0 & -\sin \theta \\
0 & \sin \theta & \cos \theta \\
\sin \theta & 0 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
x_i \\
y_i \\
z_i
\end{bmatrix}
$$

To generate the inverse transformation matrix i.e. once the position in body frame is known and position in inertial frame is desired, the inverse of the matrix in (2-6) is needed, since the transformation matrix in (2-6) is orthogonal, the inverse will be the transpose of that matrix as shown in equation (2-7):

$$
\begin{bmatrix}
x_i \\
y_i \\
z_i
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & \sin \varphi & 0 \\
-\sin \theta & \cos \varphi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos \varphi & \sin \varphi & 0 \\
0 & \cos \varphi & 0 \\
\sin \varphi & \cos \varphi & 0
\end{bmatrix}
\begin{bmatrix}
x_b \\
y_b \\
z_b
\end{bmatrix}
$$

Navigation equation – Orientation

The relationship between angular positions and angular rates are as follows:

$$
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix} =
\begin{bmatrix}
\dot{\phi} \\
0 \\
0
\end{bmatrix}
+ \begin{bmatrix}
0 & R_\phi & R_\psi \\
R_\phi & 0 & 0 \\
R_\psi & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix}
$$
The transformation matrix of each rotation specified in (2-8) are as follows:

\[
R_{\phi} \begin{bmatrix} 0 \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dot{\phi} \cos \phi \\ -\dot{\phi} \sin \phi \end{bmatrix} \quad (2-9)
\]

\[
R_{\psi}R_{\phi} \begin{bmatrix} 0 \\ \psi \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cos \phi \\ 0 & \cos \phi & \sin \phi \cos \theta \\ 0 & -\sin \phi \cos \phi & \sin \phi \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} 0 \\ \psi \end{bmatrix} \Rightarrow
\]

\[
R_{\psi}R_{\phi} \begin{bmatrix} 0 \\ \psi \end{bmatrix} = \begin{bmatrix} \cos \phi \sin \theta & \cos \phi & \sin \phi \cos \theta \\ \sin \phi \sin \theta & \sin \phi \cos \theta & \cos \phi \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} 0 \\ \psi \end{bmatrix} \quad (2-10)
\]

Combine the equations (2-8), (2-9) and (2-10):

\[
\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \sin \phi \cos \theta \\ 0 & -\sin \phi \cos \phi & \sin \phi \cos \theta \end{bmatrix} \begin{bmatrix} \phi \\ \dot{\phi} \end{bmatrix} \quad (2-11)
\]

In order to find Euler angles rates the inverse of (2-11) should be calculated as follows:

\[
\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \sin \phi \cos \theta \\ 0 & -\sin \phi \cos \phi & \sin \phi \cos \theta \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \\ r \end{bmatrix}
\]

\[
\begin{bmatrix} \cos \theta & \sin \theta & \cos \phi \sin \theta \\ \cos \phi \cos \theta & -\sin \phi \cos \theta & \cos \phi \sin \theta \\ 0 & \sin \phi & \cos \phi \end{bmatrix} = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{bmatrix}^{-1}
\]

\[
\begin{bmatrix} \phi \\ \dot{\phi} \\ \psi \end{bmatrix} = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad (2-12)
\]
Summary of kinematics and dynamics equation:

Force equations:

\[ X - mg \sin \theta = m(\ddot{u} + q\dot{w} - rv) \]
\[ Y + mg \cos \theta \sin \varphi = m(\ddot{v} + r u - p w) \]
\[ Z + mg \cos \theta \cos \varphi = m(\ddot{w} + p v - q u) \]

Moment equations:

\[ L = I_x \ddot{\phi} - I_{xz} \dot{q} + qr \left(I_y - I_z\right) - I_{xz} \rho q \]
\[ M = I_x \ddot{\rho} + rq \left(I_y - I_z\right) + I_{xz} \left(p^2 - r^2 \right) \]
\[ N = -I_{xz} \ddot{\rho} + I_z \ddot{\phi} + pq \left(I_y - I_z\right) + I_{xz} \rho r \]

Body angular velocities in terms of Euler angles and Euler rates:

\[ p = \dot{\phi} - \dot{\psi} \sin \theta \]
\[ q = \dot{\theta} \cos \phi + \dot{\psi} \cos \theta \sin \phi \]
\[ r = \dot{\psi} \cos \theta \cos \phi - \dot{\theta} \sin \phi \]

Euler rates in terms of Euler angles and body angular velocities:

\[ \dot{\theta} = q \cos \phi - r \sin \phi \]
\[ \dot{\phi} = p + q \sin \phi \tan \theta + r \cos \phi \tan \theta \]
\[ \dot{\psi} = q \sin \phi \sec \theta + r \cos \phi \sec \theta \]

Velocity of the aircraft in the inertial frame in terms of Euler angles and body velocity components:

\[
\begin{bmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt} \\
\frac{dz}{dt}
\end{bmatrix} =
\begin{bmatrix}
\cos \theta \cos \varphi & \sin \varphi \sin \theta \cos \varphi & -\cos \varphi \sin \theta \cos \varphi \\
\cos \theta \sin \varphi & \sin \varphi \sin \theta \sin \varphi + \cos \phi \cos \varphi & \cos \varphi \sin \theta \sin \varphi - \sin \phi \cos \varphi \\
-\sin \theta & \sin \varphi \cos \theta & \cos \varphi \cos \theta
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
\]
2.2 Aircraft stability

Before developing the formula for forces and moment, simple concept of aircraft static and dynamic stability from [2] and [4] is reviewed here. To define the stability assume that the aircraft is flying in certain flight course and equilibrium conditions i.e. lift is equal to weight and drag is equal to thrust, in addition moments (L, M, N) are zero with reference to center of gravity.

If for example a disturbance affects the aircraft flight path in such a way that generate a pitch in the nose up direction, as shown in Figure 2-3 (b).

If disturbed moments cause further nose up pitch angle then aircraft is statically unstable, if the aircraft hold the disturbed position as shown in Figure 2-3 (c) then aircraft has neutral static stability, if forces and moments generates in such a way to bring back the aircraft to equilibrium position then the aircraft is statically stable, as an example for the pitch disturbance tail stabilizer will generate opposing pitch moment to return back aircraft to equilibrium once it is disturbed for pitch angle.

In addition to static stability, the behavior of statically stable aircraft after disturbance will defines the dynamic stability of that, refer to Figure 2-4 (a), if the moments tend to return airplane to equilibrium and oscillation decays then aircraft is statically and dynamically stable, if the same condition as above but the oscillation does not decay then as shown in Figure 2-4 (b), aircraft is statically stable but has neutral dynamic stability. Finally, refer to Figure 2-4 (c) if moments tend to return aircraft to equilibrium but oscillation divergent then aircraft is statically stable and dynamically unstable.

Figure 2-3 Static Stability [4]
2.3 Aerodynamics forces and moments

To define the forces and moments based on the aerodynamic coefficients, the non-linear behavior of lift and drag coefficients is taking into consideration. The air speed, angle of attack and side-slip angle definition in the absence of external wind (respect to the inertial frame) are shown in Figure 2-5:

\[
V_a = \sqrt{u^2 + v^2 + w^2} \tag{2-13}
\]

\[
\alpha = \tan^{-1}\left(\frac{w}{u}\right) \tag{2-14}
\]

\[
\beta = \sin^{-1}\left(\frac{v}{V_a}\right) = \tan^{-1}\left(\frac{v}{\sqrt{u^2 + w^2}}\right) \tag{2-15}
\]
Lift and Drag coefficients are non-linear function of angle of attack as per details explained in [1] the non-linear behavior of lift coefficient is shown in Figure 2-6:

![Figure 2-6 Lift coefficient vs. Angle of attack [1]](image)

Drag coefficient non-linear behavior is shown in Figure 2-7:

![Figure 2-7 Drag coefficient vs. Angle of attack [1]](image)

Equations (2-16) and (2-17) are the mathematical expressions for lift and drag coefficients:

\[
C_L(\alpha) = \left[ 1 - \sigma(\alpha) \right] \left[ C_{L_0} + C_{L_\alpha} \alpha \right] + \sigma(\alpha) \left[ 2 \text{sign}(\alpha) \sin^2 \alpha \cos \alpha \right]
\]  \hspace{1cm} (2-16)

\[
C_D(\alpha) = C_{D_0} + \left[ 1 - \sigma(\alpha) \right] \left[ C_{D_\alpha} + C_{D_\alpha} \alpha^3 \right] + \sigma(\alpha) \left[ 2 \text{sign}(\alpha) \sin^3 \alpha \right]
\]  \hspace{1cm} (2-17)

Where: 
\[
\sigma(\alpha) = \frac{1 + e^{-M(\alpha-\alpha_0)}}{1 + e^{-M(\alpha+\alpha_0)}}
\]

M and \( \alpha_0 \) are positive constants.

As mentioned in [1] the forces and moments can be calculated according to the following equations:
Force X:
\[
F_x = -mg \sin \theta + \frac{1}{2} \rho V_a^2 \left[ C_D(\alpha) C_z(\alpha) \right] \left[ -\cos \alpha \sin \alpha \right] + \frac{1}{4} \rho V_a \overline{C_{D_0}} C_{L_0} \left[ -\cos \alpha \sin \alpha \right] q + \\
+ \frac{1}{2} \rho V_a^2 \left[ C_{D_{\alpha}} C_{L_{\alpha}} \right] \left[ -\cos \alpha \sin \alpha \right] \delta_{\alpha} + \frac{1}{2} \rho V_a \overline{C_{D_{\alpha}}} \overline{C_{L_{\alpha}}} \left[ \overline{(k_{\alpha} \overline{\delta_{\alpha}})^2} - V_a^2 \right]
\]
(2-18)

Force Y:
\[
F_y = mg \cos \theta \sin \phi + \frac{1}{2} \rho V_a^2 SC_{y_{\alpha}} C_{z_{\alpha}}(\alpha) + \frac{1}{2} \rho V_a^2 SC_{y_{\alpha}} C_{z_{\alpha}}(\alpha) \beta + \frac{1}{4} \rho V_a Sc_{y_{\alpha}} p + \frac{1}{4} \rho V_a Sc_{y_{\alpha}} \rho V_a \overline{Sc_{y_{\alpha}}} + \frac{1}{2} \rho V_a^2 SC_{y_{\alpha}} \delta_{\alpha} + \frac{1}{2} \rho V_a^2 SC_{y_{\alpha}} \delta_{\alpha}
\]
(2-19)

Force Z:
\[
F_z = mg \cos \theta \cos \phi + \frac{1}{2} \rho V_a^2 SC_{z_{\alpha}} C_{z_{\alpha}}(\alpha) + \frac{1}{2} \rho V_a^2 SC_{z_{\alpha}} C_{z_{\alpha}}(\alpha) \beta + \frac{1}{4} \rho V_a Sc_{z_{\alpha}} p + \frac{1}{4} \rho V_a Sc_{z_{\alpha}} \rho V_a \overline{Sc_{z_{\alpha}}} + \frac{1}{2} \rho V_a^2 SC_{z_{\alpha}} \delta_{\alpha} + \frac{1}{2} \rho V_a^2 SC_{z_{\alpha}} \delta_{\alpha}
\]
(2-20)

Moment L
\[
L = \frac{1}{2} \rho V_a^2 Sc_{I_{\alpha}} C_{L_{\alpha}}(\alpha) + \frac{1}{2} \rho V_a^2 Sc_{I_{\alpha}} C_{L_{\alpha}}(\alpha) \beta + \frac{1}{4} \rho V_a Sc_{I_{\alpha}} p + \frac{1}{4} \rho V_a Sc_{I_{\alpha}} \rho V_a \overline{Sc_{I_{\alpha}}} + \frac{1}{2} \rho V_a^2 Sc_{I_{\alpha}} \delta_{\alpha} + \frac{1}{2} \rho V_a^2 Sc_{I_{\alpha}} \delta_{\alpha}
\]
(2-21)

Moment M
\[
M = \frac{1}{2} \rho V_a^2 Sc_{C_{m_{\alpha}}} + \frac{1}{2} \rho V_a^2 Sc_{C_{m_{\alpha}}} \alpha + \frac{1}{4} \rho V_a Sc_{C_{m_{\alpha}}} q + \frac{1}{2} \rho V_a^2 Sc_{C_{m_{\alpha}}} \delta_{\alpha}
\]
(2-22)

Moment N
\[
N = \frac{1}{2} \rho V_a^2 Sc_{C_{n_{\alpha}}} + \frac{1}{2} \rho V_a^2 Sc_{C_{n_{\alpha}}} \beta + \frac{1}{4} \rho V_a Sc_{C_{n_{\alpha}}} p + \frac{1}{4} \rho V_a Sc_{C_{n_{\alpha}}} \rho V_a \overline{Sc_{C_{n_{\alpha}}} + \frac{1}{2} \rho V_a^2 Sc_{C_{n_{\alpha}}} \delta_{\alpha} + \frac{1}{2} \rho V_a^2 Sc_{C_{n_{\alpha}}} \delta_{\alpha}
\]
(2-23)

By implementing the equations (2-18) to (2-23) in the rigid body dynamic equations considering the non-linear behavior of lift and drag coefficients, the non-linear model of aircraft will be obtained.
2.4 Non-linear model

Summary of nonlinear equation of motion, (6 DOF, 12 States) which is the aircraft non-linear model as follows:

\[
\begin{align*}
\dot{x}_i &= (\cos \theta \cos \psi)u + (\sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi)v + (\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi)w \\
\dot{y}_i &= (\cos \theta \sin \psi)u + (\sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi)v + (\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi)w \\
\dot{h} &= -\dot{z}_i = u \sin \theta - v \sin \phi \cos \theta - w \cos \phi \cos \theta \\
u &= r v - q w - g \sin \theta + \frac{\rho V^2 S}{2m} \left[ C_x(\alpha) + C_{x_\psi}(\alpha) \frac{e_\psi}{V_a} + C_{x_\alpha}(\alpha) \delta_\alpha \right] + \frac{\rho S_{prop} \alpha_{prop}}{2m} \left( [k_{\alpha mod} \delta_\alpha]^2 - V_a^2 \right) \\
v &= p w - r u + g \cos \theta \sin \phi + \frac{\rho V^2 S}{2m} \left[ C_{y_\phi} + C_{y_\psi} \beta + C_{y_\alpha} \frac{e_\alpha}{2V_a} + C_{y_\delta} \frac{br}{2V_a} + C_{y_\delta_\alpha} \delta_\alpha + C_{y_\delta_\beta} \right] \\
w &= q u - p v + g \cos \phi \cos \theta + \frac{\rho V^2 S}{2m} \left[ C_{z_\phi} + C_{z_\psi} (\alpha) \frac{e_\psi}{2V_a} + C_{z_\alpha} (\alpha) \delta_\alpha \right] \\
\dot{\phi} &= p \cos \theta + q \sin \phi \tan \theta + r \cos \phi \tan \theta \\
\dot{\theta} &= q \cos \phi - r \sin \phi \\
\dot{\psi} &= q \sin \phi \sec \theta + r \cos \phi \sec \theta \\
\dot{p} &= \frac{I}{\Delta} p q + \frac{I}{\Delta} q r + \frac{1}{2} \frac{\rho V^2 S}{2m} \left[ C_{p_\phi} + C_{p_\psi} \beta + C_{p_\alpha} \frac{br}{2V_a} + C_{p_\delta} \frac{br}{2V_a} + C_{p_\delta_\alpha} \delta_\alpha + C_{p_\delta_\beta} \delta_\beta \right] \\
\dot{q} &= \frac{I_z}{\Delta} - \frac{L_z}{\Delta} \left( p^2 - r^2 \right) + \frac{\rho V^2 S \varepsilon}{2I_z} \left[ C_{m_\phi} + C_{m_\psi} \alpha + C_{m_\delta} \frac{e_\delta}{2V_a} + C_{m_\delta_\alpha} \delta_\alpha \right] \\
\dot{r} &= \frac{I_z}{\Delta} p q - \frac{L_z}{\Delta} q r + \frac{1}{2} \frac{\rho V^2 S}{2m} \left[ C_{r_\phi} + C_{r_\psi} \beta + C_{r_\alpha} \frac{br}{2V_a} + C_{r_\delta} \frac{br}{2V_a} + C_{r_\delta_\alpha} \delta_\alpha + C_{r_\delta_\beta} \delta_\beta \right] \\
\end{align*}
\]

For ‘u’ and ‘w’ aerodynamic coefficients we have:

\[
\begin{align*}
C_x(\alpha) &= -C_{D_\alpha} \alpha \cos \alpha + C_{L_\alpha}(\alpha) \sin \alpha \\
C_{x_\phi}(\alpha) &= -C_{D_\alpha} \cos \alpha + C_{L_\alpha} \sin \alpha \\
C_{x_\psi}(\alpha) &= -C_{D_\alpha} \cos \alpha + C_{L_\alpha} \sin \alpha \\
C_{x_\psi}(\alpha) &= -C_{D_\alpha} \cos \alpha + C_{L_\alpha} \sin \alpha \\
C_x(\alpha) &= -C_{D_\alpha} \alpha \cos \alpha - C_{L_\alpha}(\alpha) \cos \alpha \\
C_{z_\phi}(\alpha) &= -C_{D_\alpha} \sin \alpha - C_{L_\alpha} \alpha \cos \alpha \\
C_{z_\alpha}(\alpha) &= -C_{D_\alpha} \sin \alpha - C_{L_\alpha} \alpha \cos \alpha \\
C_{z_\beta}(\alpha) &= -C_{D_\alpha} \sin \alpha - C_{L_\alpha} \alpha \cos \alpha \\
\end{align*}
\]

For ‘p’ and ‘q’ aerodynamic coefficients we have:

\[
\begin{align*}
C_{p_\phi} &= \frac{I}{\Delta} C_{L_\alpha} + \frac{I_z}{\Delta} \alpha, \text{ where '*' will be replaced by: } 0, \beta, p, r, \delta_\alpha, \delta_\beta \\
C_{p_\psi} &= \frac{I}{\Delta} C_{L_\alpha} + \frac{I_z}{\Delta} \alpha, \text{ where '*' will be replaced by: } 0, \beta, p, r, \delta_\alpha, \delta_\beta \\
\end{align*}
\]
During this thesis the model of Zagi airframe, Figure 2-8, is used to develop the controller.

Parameters for Zagi airframe and aerodynamics coefficients obtained from [1] shown in Figure 2-9:

```
%习近平总frame parameters
% Longitudinal Coef.
% Lateral Coef.

w=1.56;    % [Kg]   C_L=0.38;  % \( \frac{C}{h} \)
q=9.0;     % [m/s^2]  C_D=0.03;  % \( \frac{D}{h} \)
C_m=0;     % [Kg-m^3]  C_n=0;  % \( \frac{N}{h} \)

I_x=0.1147; % [Kg-m^2]  C_T=0.058;  % \( \frac{C_T}{h} \)
I_y=0.0876; % [Kg-m^2]  C_I=0.12;  % \( \frac{C_I}{h} \)
I_z=0.1712; % [Kg-m^2]  C_N=0.28;  % \( \frac{C_N}{h} \)
I_k=0.0015; % [Kg-m^2]

% Inertial Matrix
[\begin{bmatrix}
2 & -1 & 0 \\
-1 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix}]

S=0.2589;  % [m^2]  C_T=0.038;  % \( \frac{C_T}{h} \)
b=1.4234;  % [m]  C_I=0.17;  % \( \frac{C_I}{h} \)
c_bar=0.5302;  % [m]  C_N=0.158;  % \( \frac{C_N}{h} \)

\( \text{Sprop}=0.0314;  % [m^2] \)
\( \text{rho}=1.2022;  % [Kg/m^3] \)
\( \text{Kmotor}=2; \)
```

Figure 2-9 Zagi airframe parameters and aerodynamics coefficients

### 2.5 Linear models

In order to develop the linear model, as mentioned in [2], the aircraft is assumed to be flown in certain trim conditions. The effect of small disturbance in the most effective parameter will be derived, due to that assumption small values will be ignored and non-linear function will be linearized around the trim point. Equations for forces and moments based on small disturbance are as follows:
\[ u = u_0 + \Delta u \quad v = v_0 + \Delta v \quad w = w_0 + \Delta w \]

\[ p = p_0 + \Delta p \quad q = q_0 + \Delta q \quad r = r_0 + \Delta w \]

\[ X = X_0 + \Delta X \quad Y = Y_0 + \Delta Y \quad Z = Z_0 + \Delta Z \]

\[ M = M_0 + \Delta M \quad N = N_0 + \Delta N \quad L = L_0 + \Delta L \]

\[ \delta = \delta_0 + \Delta \delta \]

**X force equation:**

\[
X - mg \sin \theta = m(\ddot{u} + qw - rv)
\]

\[
X_0 + \Delta X - mg \sin(\theta_0 + \Delta \theta) = m\left[ \frac{d}{dt}(u_0 + \Delta u) + (q_0 + \Delta q)(w_0 + \Delta w) - (r_0 + \Delta r)(v_0 + \Delta v) \right]
\]

By ignoring the multiplication of small values, and assuming the following:

\[ v_0 = p_0 = q_0 = r_0 = \phi_0 = \psi_0 = 0 \]

The X-force equation can be simplified to:

\[
X_0 + \Delta X - mg \sin \left( \theta_0 + \Delta \theta \right) = m\ddot{u}
\]

\[
\sin \left( \theta_0 + \Delta \theta \right) = \sin \theta_0 \cos \Delta \theta + \cos \theta_0 \sin \Delta \theta = \sin \theta_0 + \Delta \theta \cos \theta_0
\]

\[
X_0 + \Delta X - mg \sin \left( \theta_0 + \Delta \theta \right) = m\ddot{u}
\]

\[
X_0 - mg \sin \theta_0 = 0
\]

\[
X_0 + \Delta X - mg (\sin \theta_0 + \Delta \theta \cos \theta_0) = m\ddot{u}
\]

\[
\Delta X - mg \Delta \theta \cos \theta_0 = m\Delta \ddot{u}
\]

\[
\Delta X = f \left( u, w, \delta_z, \delta \right) \Rightarrow \Delta X = \frac{\partial X}{\partial u} \Delta u + \frac{\partial X}{\partial w} \Delta w + \frac{\partial X}{\partial \delta_z} \Delta \delta_z + \frac{\partial X}{\partial \delta} \Delta \delta
\]

\[
\frac{\partial X}{\partial u} \Delta u + \frac{\partial X}{\partial w} \Delta w + \frac{\partial X}{\partial \delta_z} \Delta \delta_z + \frac{\partial X}{\partial \delta} \Delta \delta - mg \Delta \theta \cos \theta_0 = m\Delta \ddot{u}
\]

\[
\left( m \frac{d}{dt} - \frac{\partial X}{\partial u} \right) \Delta u - \frac{\partial X}{\partial w} \Delta w + mg \cos \theta_0 \Delta \theta = \frac{\partial X}{\partial \delta_z} \Delta \delta_z + \frac{\partial X}{\partial \delta} \Delta \delta
\]

\[ X_u = \frac{\partial X}{\partial u} / \ddot{u} / m \]

\[ X_w = \frac{\partial X}{\partial w} / m \]
\[
\left( \frac{d}{dt} - X_u \right) \Delta u - X_u \Delta w + g \cos \theta \Delta \theta = X_x \Delta \delta_y + X_x \Delta \delta_z \nabla \tag{2-24}
\]

Y force equation:
\[
Y + mg \cos \theta \sin \phi = m(\dot{v} + ru - pw)
\]
\[
Y_0 + \Delta Y + mg \cos(\theta_0 + \Delta \theta) \sin(\phi_0 + \Delta \phi) = m \left[ \frac{d}{dt} (v_0 + \Delta v) + \left( r_0 + \Delta r \right) (u_0 + \Delta u) - \left( p_0 + \Delta p \right) (w_0 + \Delta w) \right]
\]

By ignoring the multiplication of small values, and assuming the following:
\[
v_0 = w_0 = p_0 = q_0 = r_0 = \phi_0 = \psi_0 = 0
\]

The Y-force equation can be simplified to:
\[
Y_0 + \Delta Y + mg \cos(\theta_0 + \Delta \theta) \sin(\phi_0 + \Delta \phi) = \dot{m} + m \dot{u}_v \Delta r
\]

On the other hand:
\[
\sin(\phi_0 + \Delta \phi) = \sin \Delta \phi = \Delta \phi
\]
\[
\cos(\theta_0 + \Delta \theta) = \cos \theta_0 \cos \Delta \theta - \sin \theta_0 \sin \Delta \theta = \cos \theta_0 + \Delta \theta \sin \theta_0
\]
\[
Y_0 + \Delta Y + mg \Delta \phi \left( \cos \theta_0 + \Delta \theta \sin \theta_0 \right) = m \dot{v} + m \dot{u}_v \Delta r
\]
\[
Y_0 + mg \cos \theta_0 \sin \phi_0 = 0 \Rightarrow Y_0 = 0
\]
\[
\Delta Y + mg \Delta \phi \left( \cos \theta_0 + \Delta \theta \sin \theta_0 \right) = m \dot{v} + m \dot{u}_v \Delta r
\]
\[
\Delta Y + mg \Delta \phi \cos \theta_0 = m \dot{v} + m \dot{u}_v \Delta r
\]
\[
\Delta Y = f \left( v, p, r, \delta \right) \Rightarrow \Delta Y = \frac{\partial Y}{\partial v} \Delta v + \frac{\partial Y}{\partial p} \Delta p + \frac{\partial Y}{\partial r} \Delta r + \frac{\partial Y}{\partial \delta} \Delta \delta
\]
\[
\frac{\partial Y}{\partial v} \Delta v + \frac{\partial Y}{\partial p} \Delta p + \frac{\partial Y}{\partial r} \Delta r + \frac{\partial Y}{\partial \delta} \Delta \delta + mg \Delta \phi \cos \theta_0 = m \dot{v} + m \dot{u}_v \Delta r
\]
\[
\left( m \frac{d}{dt} - \frac{\partial Y}{\partial v} \right) \Delta v - \frac{\partial Y}{\partial p} \Delta p + \left( m \dot{u}_v - \frac{\partial Y}{\partial r} \right) \Delta r - mg \Delta \phi \cos \theta_0 = \frac{\partial Y}{\partial \delta} \Delta \delta
\]
\[
\left( \frac{d}{dt} - Y_r \right) \Delta v - Y_p \Delta p + \left( \dot{u}_v - \delta \right) \Delta r - g \cos \theta \Delta \phi = Y_x \Delta \delta_y \nabla \tag{2-25}
\]

Z force equation:
\[
Z + mg \cos \theta \cos \phi = m(\dot{w} + pv - qv)
\]
\[
Z_0 + \Delta Z + mg \cos(\theta_0 + \Delta \theta) \cos(\phi_0 + \Delta \phi) = m \left[ \frac{d}{dt} (w_0 + \Delta w) + (p_0 + \Delta p) (v_0 + \Delta v) - (q_0 + \Delta q) (u_0 + \Delta u) \right]
\]

By ignoring the multiplication of small values, and assuming the following:
\[
v_0 = w_0 = p_0 = q_0 = r_0 = \phi_0 = \psi_0 = 0
\]

The Z-force equation can be simplified to:
\[
Z_0 + \Delta Z + mg \cos(\theta_0 + \Delta \theta) \cos(\phi_0 + \Delta \phi) = m \dot{w} - m \dot{u}_v \Delta q
\]
On the other hand:

\[
\cos(\phi_0 + \Delta \phi) = \cos \Delta \phi = 1
\]

\[
\cos(\theta_0 + \Delta \theta) = \cos \theta_0 \cos \Delta \theta - \sin \theta_0 \sin \Delta \theta = \cos \theta_0 + \Delta \theta \sin \theta_0
\]

\[
Z_0 + \Delta Z + mg \cos(\theta_0 + \Delta \theta) \cos(\phi_0 + \Delta \phi) = mw - mu_0 \Delta q
\]

\[
Z_0 + mg \cos \theta_0 \cos \phi_0 = 0 \Rightarrow Z_0 + mg \cos \theta_0 = 0
\]

\[
Z_0 + \Delta Z + mg \cos \theta_0 + mg \Delta \theta \sin \theta_0 = mw - mu_0 \Delta q
\]

\[
\Delta Z + mg \Delta \theta \sin \theta_0 = mw - mu_0 \Delta q
\]

\[
\Delta Z = f(u, w, w, q, \delta_z, \delta_r) \Rightarrow \Delta Z = \frac{\partial Z}{\partial u} \Delta u + \frac{\partial Z}{\partial w} \Delta w + \frac{\partial Z}{\partial q} \Delta q + \frac{\partial Z}{\partial \delta_z} \Delta \delta_z + \frac{\partial Z}{\partial \delta_r} \Delta \delta_r
\]

\[
\Delta Z = \frac{\partial Z}{\partial u} \Delta u + \frac{\partial Z}{\partial w} \Delta w + \frac{\partial Z}{\partial q} \Delta q + \frac{\partial Z}{\partial \delta_z} \Delta \delta_z + \frac{\partial Z}{\partial \delta_r} \Delta \delta_r + mg \Delta \theta \sin \theta_0 = mw - mu_0 \Delta q
\]

\[
\Delta Z = \frac{\partial Z}{\partial w} \Delta w + \frac{\partial Z}{\partial q} \Delta q - mg \Delta \theta \sin \theta_0 - mu_0 \Delta q
\]

\[
Z = \nabla \cdot \left( \frac{Z}{\theta \cos \phi} \right) \Rightarrow \Delta Z = \frac{\partial Z}{\partial \delta_z} \Delta \delta_z + \frac{\partial Z}{\partial \delta_r} \Delta \delta_r
\]

\[
L \text{ moment equation:}
\]

\[
L = I_x \Delta p - I_{zx} \Delta r + qr(I_x - I_y) - I_{zx}pq
\]

\[
L_0 + \Delta L = I_x \frac{d}{dt} (p_0 + \Delta p) - I_{zx} \frac{d}{dt} (r_0 + \Delta r) + (q_0 + \Delta q)(r_0 + \Delta r)(I_x - I_y) - I_{zx}(p_0 + \Delta p)(q_0 + \Delta q)
\]

\[
v_0 = w_0 = p_0 = q_0 = r_0 = \phi_0 = \psi_0 = 0
\]

L moment equation can be simplified to:

\[
L_0 + \Delta L = I_x \frac{d}{dt} (\Delta p) - I_{zx} \frac{d}{dt} (\Delta r)
\]

\[
L_0 = 0
\]

\[
\Delta L = I_x \frac{d}{dt} \Delta p - I_{zx} \frac{d}{dt} \Delta r
\]

\[
\Delta L = f(v, p, r, \delta_z, \delta_r) \Rightarrow \Delta L = \frac{\partial L}{\partial v} \Delta v + \frac{\partial L}{\partial p} \Delta p + \frac{\partial L}{\partial r} \Delta r + \frac{\partial L}{\partial \delta_z} \Delta \delta_z + \frac{\partial L}{\partial \delta_r} \Delta \delta_r
\]
\[ \begin{align*}
\frac{\partial L}{\partial v} \Delta v + \frac{\partial L}{\partial p} \Delta p + \frac{\partial L}{\partial r} \Delta r + \frac{\partial L}{\partial \delta_r} \Delta \delta_r + \frac{\partial L}{\partial \delta_a} \Delta \delta_a &= I_s \frac{d}{dt} \Delta p - I_{xz} \frac{d}{dt} \Delta r \\
-\frac{\partial L}{\partial v} \Delta v - \frac{\partial L}{\partial p} \Delta p - \frac{\partial L}{\partial r} \Delta r + I_s \frac{d}{dt} \Delta p - I_{xz} \frac{d}{dt} \Delta r &= \frac{\partial L}{\partial \delta_r} \Delta \delta_r + \frac{\partial L}{\partial \delta_a} \Delta \delta_a \\
-\frac{\partial L}{\partial v} \Delta v + I_s \frac{d}{dt} \Delta p - \frac{\partial L}{\partial p} \Delta p - \frac{\partial L}{\partial r} \Delta r - I_{xz} \frac{d}{dt} \Delta r &= \frac{\partial L}{\partial \delta_r} \Delta \delta_r + \frac{\partial L}{\partial \delta_a} \Delta \delta_a \\
-\frac{\partial L}{\partial v} \Delta v + \left( I_s \frac{d}{dt} - L_p \right) \Delta p - \left( I_{xz} \frac{d}{dt} + L_r \right) \Delta r &= L_{\phi_r} \Delta \delta_r + L_{\phi_a} \Delta \delta_a \\
\end{align*} \]

\( (2-27) \)

M moment equation:

\[ M = I_s \dot{q} + rq(I_s - I_z) + I_{xz}(p^2 - r^2) \]

\[ M_0 + \Delta M = I_s \frac{d}{dt} \left( q_0 + \Delta q \right) + (r_0 + \Delta r)(q_0 + \Delta q)(I_s - I_z) + I_{xz} \left[ (p_0 + \Delta p)^2 - (r_0 + \Delta r)^2 \right] \]

\( \nu_0 = w_0 = p_0 = q_0 = r_0 = \phi_0 = \psi_0 = 0 \)

M moment equation can be simplified to:

\[ M_0 + \Delta M = I_s \frac{d}{dt} (\Delta q) \]

\[ M_0 = 0 \]

\[ \Delta M = I_s \frac{d}{dt} \Delta q \]

\[ \Delta M = f(u, w, \dot{w}, q, \delta_r, \delta_a) \Rightarrow \Delta M = \frac{\partial M}{\partial u} \Delta u + \frac{\partial M}{\partial w} \Delta w + \frac{\partial M}{\partial \dot{w}} \Delta \dot{w} + \frac{\partial M}{\partial q} \Delta q + \frac{\partial M}{\partial \delta_r} \Delta \delta_r + \frac{\partial M}{\partial \delta_a} \Delta \delta_a = I_s \frac{d}{dt} \Delta q \]

\[ \frac{\partial M}{\partial u} \Delta u = \frac{\partial M}{\partial w} \Delta w + \frac{\partial M}{\partial \dot{w}} \Delta \dot{w} + \frac{\partial M}{\partial q} \Delta q + \frac{\partial M}{\partial \delta_r} \Delta \delta_r + \frac{\partial M}{\partial \delta_a} \Delta \delta_a = I_s \frac{d}{dt} \Delta q \]

\[ -\frac{\partial M}{\partial u} \Delta u - \frac{\partial M}{\partial w} \Delta w - \frac{\partial M}{\partial \dot{w}} \Delta \dot{w} + I_s \frac{d}{dt} \Delta q = \frac{\partial M}{\partial \delta_r} \Delta \delta_r + \frac{\partial M}{\partial \delta_a} \Delta \delta_a \]

\[ -\frac{\partial M}{\partial u} \Delta u - \frac{\partial M}{\partial w} \Delta w - \frac{\partial M}{\partial \dot{w}} \Delta \dot{w} + I_s \frac{d}{dt} \Delta q - \frac{\partial M}{\partial q} \Delta q = \frac{\partial M}{\partial \delta_r} \Delta \delta_r + \frac{\partial M}{\partial \delta_a} \Delta \delta_a \]

\[ \Delta q = \Delta \theta \Rightarrow \frac{d}{dt} \Delta q = \frac{d^2}{dt^2} \Delta \theta \]

\[ -\frac{\partial M}{\partial u} \Delta u + \left( \frac{\partial M}{\partial w} \frac{d}{dt} + \frac{\partial M}{\partial \dot{w}} \right) \Delta w + \left( I_s \frac{d^2}{dt^2} - \frac{\partial M}{\partial q} \frac{d}{dt} \right) \Delta \theta = \frac{\partial M}{\partial \delta_r} \Delta \delta_r + \frac{\partial M}{\partial \delta_a} \Delta \delta_a \]

\[ -M_s \Delta u + \left( M_s \frac{d}{dt} + M_u \right) \Delta w + \left( \frac{d^2}{dt^2} - M_s \frac{d}{dt} \right) \Delta \theta = M_s \Delta \delta_r + M_u \Delta \delta_a \]

\( (2-28) \)
N moment equation:
\[ N = -I_{xz} p + I_{z} r + pq(I_{y} - I_{z}) + I_{w} qr \]
\[ N_{0} + \Delta N = -I_{xz} \frac{d}{dt}(p_{0} + \Delta p) + I_{z} \frac{d}{dt}(r_{0} + \Delta r) + (p_{0} + \Delta p)(q_{0} + \Delta q)(I_{y} - I_{z}) + I_{xz}(q_{0} + \Delta q)(r_{0} + \Delta r) \]
\[ v_{0} = w_{0} = q_{0} = r_{0} = \phi_{0} = \psi_{0} = 0 \]

N moment equation can be simplified to:
\[ N_{0} + \Delta N = -I_{xz} \frac{d}{dt}(\Delta p) + I_{z} \frac{d}{dt}(\Delta r) \]
\[ N_{0} = 0 \]
\[ \Delta N = -I_{xz} \frac{d}{dt}(\Delta p) + I_{z} \frac{d}{dt}(\Delta r) \]
\[ \Delta N = f(v, p, r, \delta_{r}, \delta_{a}) \Rightarrow \Delta N = \frac{\partial N}{\partial \nu} \Delta \nu + \frac{\partial N}{\partial p} \Delta p + \frac{\partial N}{\partial r} \Delta r + \frac{\partial N}{\partial \delta_{r}} \Delta \delta_{r} + \frac{\partial N}{\partial \delta_{a}} \Delta \delta_{a} = -I_{xz} \frac{d}{dt}(\Delta p) + I_{z} \frac{d}{dt}(\Delta r) \]
\[ \frac{\partial N}{\partial \nu} \Delta \nu + \frac{\partial N}{\partial p} \Delta p + \frac{\partial N}{\partial r} \Delta r - I_{xz} \frac{d}{dt} \Delta p + I_{z} \frac{d}{dt} \Delta r = \frac{\partial N}{\partial \delta_{r}} \Delta \delta_{r} + \frac{\partial N}{\partial \delta_{a}} \Delta \delta_{a} \]
\[ -\frac{\partial N}{\partial \nu} \Delta \nu - I_{xz} \frac{d}{dt} \Delta p - \frac{\partial N}{\partial r} \Delta r - I_{z} \frac{d}{dt} \Delta p + I_{z} \frac{d}{dt} \Delta r = \frac{\partial N}{\partial \delta_{r}} \Delta \delta_{r} + \frac{\partial N}{\partial \delta_{a}} \Delta \delta_{a} \]
\[ -\frac{\partial N}{\partial \nu} \Delta \nu \left( I_{xz} \frac{d}{dt} + N_{p} \right) \Delta p + \left( I_{z} \frac{d}{dt} - N_{r} \right) \Delta r = N_{\delta_{r}} \Delta \delta_{r} + N_{\delta_{a}} \Delta \delta_{a} \]

(2-29)

### 2.5.1 Transfer functions

In order to design a linear controller as detailed in [1] the approximate model of aircraft for each parameter will be used. For each state, the linearized transfer function is used while taking into consideration the most effective control surface. For example, roll angle of the aircraft takes more effect from aileron deflection compared to other control surfaces and on the other hand, the deflection of aileron has major effects on roll and roll rate compared to other states. To find the transfer function, the coupling effect of the systems on each other and non-linearity of the system are ignored. The coupling effect of the systems and higher order terms of non-linear model will be considered as disturbance on the linear controllers. The transfer functions for de-coupled systems are shown in Figure 2-10:
The parameters for the transfer functions are shown in Figure 2-11:

\[
\begin{align*}
  a_{phi1} &= -0.25 \rho c \thetaVa_{trim}S^*(b^2)c_p b; \\
  a_{phi2} &= 0.5 \rho c \theta(Va_{trim}b)^2S^*c_p \delta a; \\
  a_{beta1} &= -0.5 \rho c \theta Va_{trim}S^*C_{Ybeta}/m; \\
  a_{beta2} &= -0.5 \rho c \theta(Va_{trim}^2)S^*C_{Ydelta r}/m; \\
  a_{theta1} &= -0.25 \rho c \theta Va_{trim}^2c_{bar}^2S^*_C m q/Iy; \\
  a_{theta2} &= -0.5 \rho c \theta(Va_{trim}^2)c_{bar}^2S^*_C m \alpha/Iy; \\
  a_{theta3} &= 0.5 \rho c \theta(Va_{trim}^2)c_{bar}^2S^*_C m \delta e/Iy; \\
  a_{Y1} &= (\rho c \theta Va_{trim}/m)(S^*(C_D trim+C_D delta e*delta e trim)+Sprop*Cprop); \\
  a_{V2} &= \rho c \theta Cprop*Cprop(S^*(\text{motor}^2)*delta_b trim/m); \\
\end{align*}
\]

Transfer functions will be used to design PID controllers in section 3.2.

2.5.2 Lateral state-space equation

Following the method explained in [1], the UAV dynamics can be represented by the general nonlinear system of equation:

\[ \dot{x} = f(x, u) \]

Where \( x \in R^{12} \) is the state, \( u \in R^4 \) is the input (control vector), and \( f \) is given by above equations. The system is said to be in equilibrium at the state \( x^* \) and input \( u^* \) if

\[ f(x^*, u^*) = 0 \]

A trim input \( u^* \) and state \( x^* \) are found such that

\[ \dot{x}^* = f(x^*, u^*) \]
Letting $\bar{x} = x - x^*$ and $\bar{u} = u - u^*$:

$$
\begin{align*}
\dot{x} &= \dot{x} - \dot{x}^* = f(x,u) - f(x^*,u^*) = f(x + x^* - x^*, u + u^* - u^*) - f(x^*,u^*) = \\
&= f(x^* + \bar{x}, u^* + \bar{u}) - f(x^*,u^*)
\end{align*}
$$

Taking the Taylor series expansion of the first term about the trim state we get

$$
\dot{x} = f(x^*, u^*) + \frac{\partial f(x^*, u^*)}{\partial x} \bar{x} + \frac{\partial f(x^*, u^*)}{\partial u} \bar{u} + H.O.T. - f(x^*, u^*)
$$

Therefore, the linearized dynamics are determined by finding $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial u}$, evaluated at the trim conditions.

Since these trim conditions are independent of inertial position, $x_i$ and $y_i$ do not factor into trim calculations. In addition the dynamics for fixed wing aircraft can be approximately decomposed into longitudinal and lateral motion, for most airframes the coupling is small and can be handled by control algorithms designed for disturbance rejection.

States and inputs vector are defined as follows:

For lateral model:

$$
x_{lat} = \begin{bmatrix} v \\ \rho \\ p \\ r \\ \phi \\ \theta \\ \psi \end{bmatrix}, \quad u_{lat} = \begin{bmatrix} \delta_a \\ \delta_e \end{bmatrix}
$$

and for longitudinal:

$$
x_{lon} = \begin{bmatrix} u \\ w \\ q \\ \theta \\ h \end{bmatrix}, \quad u_{lon} = \begin{bmatrix} \delta_e \end{bmatrix}
$$

$$
\dot{v} = pw - ru + g \cos \phi \sin \phi + \rho \sqrt{u^2 + \dot{v}^2 + w^2} \frac{b}{2} \left[ C_{\rho, p} + C_{\phi, r} \right] +
\rho \frac{u^2 + v^2 + w^2}{2m} \left[ C_{\phi, p} + C_{\phi, p} \tan^{-1} \left( \frac{v}{\sqrt{u^2 + w^2}} \right) + C_{\phi, \phi} \delta_a + C_{\phi, \phi} \delta_e \right]
$$

$$
\dot{p} = \frac{l_1}{\Delta} pq - \frac{l_1}{\Delta} qr + \rho \frac{u^2 + v^2 + w^2}{2} \frac{b^2}{2} \left[ C_{\rho, p} + C_{\rho, p} \right] +
\frac{1}{2} \rho \frac{u^2 + v^2 + w^2}{2} \left[ C_{\rho, \phi} + C_{\rho, \phi} \tan^{-1} \left( \frac{v}{\sqrt{u^2 + w^2}} \right) + C_{\rho, \rho} \delta_a + C_{\rho, \rho} \delta_e \right]
$$

$$
\dot{r} = \frac{l_1}{\Delta} pq - \frac{l_1}{\Delta} qr + \rho \frac{u^2 + v^2 + w^2}{2} \frac{b^2}{2} \left[ C_{\phi, p} + C_{\phi, r} \right] +
\frac{1}{2} \rho \frac{u^2 + v^2 + w^2}{2} \left[ C_{\phi, \phi} + C_{\phi, \phi} \tan^{-1} \left( \frac{v}{\sqrt{u^2 + w^2}} \right) + C_{\phi, \rho} \delta_a + C_{\phi, \rho} \delta_e \right]
$$

$$
\dot{\phi} = p + q \sin \phi \tan \theta + r \cos \phi \tan \theta
$$

$$
\dot{\psi} = q \sin \phi \sec \theta + r \cos \phi \sec \theta
$$
\[
\begin{bmatrix}
\ddot{\nu} \\
\dot{p} \\
\tilde{r} \\
\gamma\
\end{bmatrix} =
\begin{bmatrix}
Y_{\nu} & Y_{p} & Y_{r} & g \cos \theta' \cos \phi' \\
L_{p} & L_{p} & L_{r} & 0 \\
N_{\nu} & N_{p} & N_{r} & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\nu \\
p \\
r \\
\gamma\
\end{bmatrix} +
\begin{bmatrix}
Y_{\nu} & Y_{p} \\
L_{p} & L_{p} \\
N_{\nu} & N_{p} \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\ddot{\nu} \\
\ddot{p} \\
\ddot{r} \\
\ddot{\gamma}\
\end{bmatrix}
\] (2-30)

\[
Y_{\nu} = \frac{\rho S b v^*}{4 m V_{v}} \left[ C_{v} p^* + C_{p} r^* \right] + \frac{\rho Sv^*}{m} \left[ C_{v} + C_{p} \beta^* + C_{p} \delta^* + C_{p} \beta^* \right] + \frac{\rho S C_{v} C_{p}}{2 m} \sqrt{u^2 + w^2}
\]

\[
Y_{p} = w^* + \frac{\rho V_{v}^2 S b}{4 m} C_{p}
\]

\[
Y_{r} = -u^* + \frac{\rho V_{v}^2 S b}{4 m} C_{p}
\]

\[
Y_{\nu} = \frac{\rho V_{v}^2 S}{2 m} C_{p}
\]

\[
Y_{r} = \frac{\rho V_{v}^2 S}{2 m} C_{p}
\]

\[
L_{u} = \frac{\rho S b v^*}{4 V_{v}^2} \left[ C_{p} p^* + C_{p} r^* \right] + \frac{\rho S b v^*}{m} \left[ C_{v} + C_{p} \beta^* + C_{p} \delta^* + C_{p} \beta^* \right] + \frac{\rho S C_{v} C_{p}}{2} \sqrt{u^2 + w^2}
\]

\[
L_{p} = \frac{l_{1}}{\Delta} q^* + \frac{\rho V_{v}^2 S b^2}{4} C_{p}
\]

\[
L_{r} = \frac{l_{2}}{\Delta} q^* + \frac{\rho V_{v}^2 S b^2}{4} C_{p}
\]

\[
L_{\nu} = \frac{\rho V_{v}^2 S b}{2} C_{p}
\]

\[
L_{\nu} = \frac{\rho V_{v}^2 S b}{2} C_{p}
\]

\[
N_{v} = \frac{\rho S b v^*}{4 V_{v}^2} \left[ C_{v} p^* + C_{v} r^* \right] + \frac{\rho S b v^*}{m} \left[ C_{v} + C_{v} \beta^* + C_{v} \delta^* + C_{v} \beta^* \right] + \frac{\rho S C_{v} C_{p}}{2} \sqrt{u^2 + w^2}
\]

\[
N_{p} = \frac{l_{1}}{\Delta} q^* + \frac{\rho V_{v}^2 S b^2}{4} C_{v}
\]

\[
N_{r} = \frac{l_{2}}{\Delta} q^* + \frac{\rho V_{v}^2 S b^2}{4} C_{v}
\]

\[
N_{\nu} = \frac{\rho V_{v}^2 S b}{2} C_{v}
\]

\[
N_{\nu} = \frac{\rho V_{v}^2 S b}{2} C_{v}
\]

The lateral equation are often given in terms of $\ddot{\beta}$ instead of $\ddot{\nu}$, we have

$\nu = V_{v} \sin \beta \rightarrow$ Linearizing around $\beta = \beta^*$, we get $\ddot{\nu} = V_{v} \cos \beta^* \dddot{\beta}$ which implies that:

38
2.5.3 Longitudinal state-space equation

\[ u = rv - qw - g \sin \theta + \frac{\rho V_s^2 S}{2m} \left[ C_{x_0} + C_{x_v} \alpha + C_{x_v} \cos \theta + C_{x_v} \delta \right] + \frac{\rho S_{prop} C_{prop}}{2m} \left[ (k_{motor}) \right] - V_a^2 \]

\[ \dot{w} = qu - pv + g \cos \theta \cos \phi + \frac{\rho V_s^2 S}{2m} \left[ C_{x_0} + C_{x_v} \alpha + C_{x_v} \cos \theta + C_{x_v} \delta \right] \]

\[ q = \frac{l_z - l_x}{\Delta} \left( pr - \frac{I_v}{\Delta} (p^2 - r^2) \right) + \frac{\rho V_s^2 S}{2m} \left[ C_{n_0} + C_{n_v} \alpha + C_{n_v} \cos \theta + C_{n_v} \delta \right] \]

\[ \dot{\theta} = q \cos \phi - r \sin \phi \]

\[ \dot{h} = -z_i = u \sin \theta - v \sin \phi \cos \theta - w \cos \phi \cos \theta \]

Assuming that the lateral states are zero, i.e., set \( \phi = p = r = \beta = v = 0 \) and substituting

\[ \alpha = \tan^{-1} \left( \frac{w}{u} \right) \]

\[ V_a = \sqrt{u^2 + w^2} \]

Then we have:

\[ \dot{u} = -qw - g \sin \theta + \frac{\rho u^2 + w^2}{2m} \left[ C_{x_0} + C_{x_v} \tan^{-1} \left( \frac{w}{u} \right) + C_{x_v} \delta \right] + \frac{\rho \sqrt{u^2 + w^2}}{4m} \left[ C_{x_v} \cos \theta \right] \]

\[ \dot{w} = qu + g \cos \theta \cos \phi + \frac{\rho u^2 + w^2}{2m} \left[ C_{n_0} + C_{n_v} \tan^{-1} \left( \frac{w}{u} \right) + C_{n_v} \delta \right] + \frac{\rho \sqrt{u^2 + w^2}}{4m} \left[ C_{n_v} \cos \theta \right] \]

\[ \dot{q} = \frac{1}{2l_y} \rho (u^2 + w^2) \left[ C_{n_0} + C_{n_v} \tan^{-1} \left( \frac{w}{u} \right) + C_{n_v} \delta \right] + \frac{1}{4l_y} \rho \sqrt{u^2 + w^2} \left[ C_{n_v} \cos \theta \right] \]

\[ \dot{\theta} = q \]

\[ \dot{h} = -z_i = u \sin \theta - w \cos \theta \]

\[
\begin{bmatrix}
\ddot{u} \\
\ddot{w} \\
\ddot{q} \\
\ddot{h}
\end{bmatrix}
= \begin{bmatrix}
x_x & x_w & x_q & -g \cos \theta \cos \phi & 0 \\
Z_u & Z_w & Z_q & -g \sin \theta & 0 \\
M_u & M_w & M_q & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\dot{u} \\
\dot{w} \\
\dot{q} \\
\dot{h}
\end{bmatrix}
+ \begin{bmatrix}
X_\delta & X_\delta \vspace{10pt}
\end{bmatrix}
\begin{bmatrix}
\ddot{u} \\
\ddot{w} \\
\ddot{q} \\
\ddot{h}
\end{bmatrix}
\]

(2-32)
\[
X_u = \frac{u^* \rho S}{m} \left[ C_{x_u} + C_{x_u}^* + C_{x_u} \alpha^* + C_{x_u} \alpha^* \right] - \frac{\rho S w^* C_{x_u}}{2m} + \frac{\rho S \tau C_{x_u} u^* q^*}{4mV_a^*} - \frac{\rho S_{prop} C_{prop} u^*}{m}
\]

\[
X_w = -q^* + \frac{w^* \rho S}{m} \left[ C_{x_w} + C_{x_w}^* + C_{x_w} \alpha^* + C_{x_w} \alpha^* \right] + \frac{\rho S \tau C_{x_w} w^* q^*}{4mV_a^*} + \frac{\rho S C_{x_w} u^*}{2m} - \frac{\rho S_{prop} C_{prop} w^*}{m}
\]

\[
X_q = -w^* + \frac{\rho S V_a^* \tau C_{x_q}}{4m}
\]

\[
X_{q_v} = \frac{\rho S V_a^* \tau C_{x_{q_v}}}{2m}
\]

\[
X_{q_i} = \frac{\rho S_{prop} C_{prop} k^2 \tau C_{q_i}}{m}
\]

\[
Z_u = q^* + \frac{u^* \rho S}{m} \left[ C_{z_u} + C_{z_u}^* + C_{z_u} \alpha^* + C_{z_u} \alpha^* \right] + \frac{\rho S \tau C_{z_u} u^* q^*}{4mV_a^*} - \frac{\rho S_{z_u} w^*}{2m}
\]

\[
Z_w = \frac{w^* \rho S}{m} \left[ C_{z_w} + C_{z_w}^* + C_{z_w} \alpha^* + C_{z_w} \alpha^* \right] + \frac{\rho S \tau C_{z_w} w^* q^*}{4mV_a^*} + \frac{\rho S_{z_w} u^*}{2m}
\]

\[
Z_q = u^* + \frac{\rho S V_a^* \tau C_{z_q}}{4m}
\]

\[
Z_{q_v} = \frac{\rho S V_a^* \tau C_{z_{q_v}}}{2m}
\]

\[
M_u = \frac{u^* \rho S}{I_y} \left[ C_{m_u} + C_{m_u}^* + C_{m_u} \alpha^* + C_{m_u} \alpha^* \right] - \frac{\rho S \tau C_{m_u} w^*}{2I_y} + \frac{\rho S \tau^2 C_{m_u} q^* u^*}{4I_y V_a^*}
\]

\[
M_w = \frac{w^* \rho S}{I_y} \left[ C_{m_w} + C_{m_w}^* + C_{m_w} \alpha^* + C_{m_w} \alpha^* \right] + \frac{\rho S \tau C_{m_w} u^*}{2I_y} + \frac{\rho S \tau^2 C_{m_w} q^* w^*}{4I_y V_a^*}
\]

\[
M_q = \frac{\rho V_a^* \tau^2 C_{m_q}}{4I_y}
\]

\[
M_{q_v} = \frac{\rho V_a^* \tau^2 C_{m_{q_v}}}{2I_y}
\]
The longitudinal equations are often given in terms of $\bar{\alpha}$ instead of $\bar{w}$, we have:
\[ w = V_a \sin \alpha \cos \beta = V_a \sin \alpha \]

Where we have set $\beta = 0$, linearizing around $\alpha = \alpha^*$ we get:
\[ \bar{w} = V_a^* \cos \alpha^* \bar{\alpha} \Rightarrow \dot{\bar{\alpha}} = \frac{1}{V_a^* \cos \alpha^*} \dot{\bar{w}} \]

\[
\begin{bmatrix}
\pi \\
\sigma \\
\eta \\
\end{bmatrix} = \begin{bmatrix} X_a & X_a V_a' \cos \alpha^* & X_a & -g \cos \theta^* & 0 \\
Z_a & Z_a & Z_a & -g \sin \theta^* & 0 \\
V_a' \cos \alpha^* & V_a' \cos \alpha^* & V_a' \cos \alpha^* & 0 & 0 \\
M_a & M_a V_a' \cos \alpha^* & M_a & 0 & 0 \\
0 & 0 & 0 & u' \cos \theta^* + w' \sin \theta^* & 0 \\
\end{bmatrix} \begin{bmatrix} \pi \\
\sigma \\
\eta \\
\end{bmatrix} + \begin{bmatrix} X_\delta & X_\delta \\
Z_\delta & 0 \\
V_\delta & 0 \\
M_\delta & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix} \begin{bmatrix} \pi \\
\sigma \\
\eta \\
\end{bmatrix}
\]

(2-33)
3 Linear Controller

3.1 Trim

This section is based on the trim algorithm for aircraft turn with constant climb rate explained in [1]. However, for a straight line flight, the radius of turn will be set to infinity and flight path angle will be set to zero. Since the trim conditions are independent of the aircraft’s x and y positions in the inertial frame, they will be ignored.

Let \( x = [h \ u \ v \ w \ \phi \ \psi \ p \ q \ r]^T \) and \( u = [\delta_e \ \delta_i \ \delta_a \ \delta_r]^T \).

The trim states \( x^* \) and \( u^* \) should satisfy equations in section 2.4, consider the desired flight condition with a constant airspeed \( V_a \), a flight path angle of \( \gamma \) and a turning radius of \( R \). All the states will be constant except \( h \) and \( \psi \), therefore

\[
\dot{x}^* = \begin{bmatrix}
\dot{h}^* = V_a \sin \gamma \\
\dot{u}^* = 0 \\
\dot{v}^* = 0 \\
\dot{w}^* = 0 \\
\dot{\phi}^* = 0 \\
\dot{\theta}^* = 0 \\
\dot{\psi}^* = \frac{V_a}{R} \\
p^* = 0 \\
q^* = 0 \\
r^* = 0
\end{bmatrix} \tag{3-1}
\]

Objective of trim algorithms are to make function \( J \) in become minimum or ideally zero.

\[
J = \|\dot{x}^* - f(x^*, u^*)\|^2 \tag{3-2}
\]

In equation (3-2), \( \dot{x}^* \) comes from (3-1), while \( f(x^*, u^*) \) comes from equations in section 2.4. Since \( h \) and \( \psi \) do not appear in the right side of any remaining equations in section 2.4, they are not entered in the trim calculations. The equations for the rest of the states will be as follows:

\[
\begin{align*}
\dot{u}^* &= V_a \cos \alpha^* \cos \beta^* \\
\dot{v}^* &= V_a \sin \beta^* \\
\dot{w}^* &= V_a \sin \alpha^* \cos \beta^* \\
\dot{\theta}^* &= \alpha^* + \gamma
\end{align*} \tag{3-3-6}
\]
\[ p^* = \frac{V_u}{R} \sin \theta^* \]  \hspace{1cm} (3-7) \\
\[ q^* = \frac{V_u}{R} \sin \phi^* \cos \theta^* \]  \hspace{1cm} (3-8) \\
\[ r^* = \frac{V_u}{R} \cos \phi^* \cos \theta^* \]  \hspace{1cm} (3-9) 

To calculate trim control inputs equations in Figure 3-1 will be used:

Following algorithm shows the steps to calculate J function:

1. Input \( \alpha, \beta, \phi, V_u, R, \gamma \).
2. Compute \( x^* \) from equation (3-1).
3. Compute trim states from equations (3-3) to (3-9).
4. Compute trim inputs from equations in Figure 3-1.
5. Compute \( f(x^*, u^*) \) from equations of section 2.4.
6. Compute J from equation (3-2).

Algorithm in Figure 3-2 shows the steps for minimizing J function, with all these algorithms and equations now we are able to find trim states as per following steps:

1. Input: desired airspeed \( V_u \), flight path angle \( \gamma \), and turn radius \( R \).
2. Compute: \( (\alpha^*, \beta^*, \phi^*) \) by minimizing J function, Figure 3-2.
3. Compute trim states from equations (3-3) to (3-9).
4. Compute trim inputs from equations in Figure 3-1.
1: Input: $\alpha^{(0)}, \beta^{(0)}, \phi^{(0)}, V_a, R, \gamma.$
2: for $k = 1$ to $N$ do
3: $\alpha^+ = \alpha^{(k-1)} + \epsilon$
4: $\beta^+ = \beta^{(k-1)} + \epsilon$
5: $\phi^+ = \phi^{(k-1)} + \epsilon$
6: $\frac{\partial J}{\partial \alpha} = \frac{J(\alpha^+, \beta^{(k-1)}, \phi^{(k-1)}) - J(\alpha^{(k-1)}, \beta^{(k-1)}, \phi^{(k-1)})}{\epsilon}$
7: $\frac{\partial J}{\partial \beta} = \frac{J(\alpha^{(k-1)}, \beta^+, \phi^{(k-1)}) - J(\alpha^{(k-1)}, \beta^{(k-1)}, \phi^{(k-1)})}{\epsilon}$
8: $\frac{\partial J}{\partial \phi} = \frac{J(\alpha^{(k-1)}, \beta^{(k-1)}, \phi^+) - J(\alpha^{(k-1)}, \beta^{(k-1)}, \phi^{(k-1)})}{\epsilon}$
9: $\alpha^{(k)} = \alpha^{(k-1)} - \kappa \frac{\partial J}{\partial \alpha}$
10: $\beta^{(k)} = \beta^{(k-1)} - \kappa \frac{\partial J}{\partial \beta}$
11: $\phi^{(k)} = \phi^{(k-1)} - \kappa \frac{\partial J}{\partial \phi}$
12: end for

Figure 3-2 Algorithm for minimize $J$ [1]
3.2 Transfer functions controllers

Different linear controller is designed for control of each parameter based on the transfer functions developed in section 2.5.1 each controller step response to the input command or disturbance is evaluated.

3.2.1 Roll PID

![Roll PID Controller Diagram]

**Figure 3-3 Roll PID Controller**

![Roll PID Controller Step Response]

**Figure 3-4 Roll PID Controller Step Response**
3.2.2 Side-slip PI

Side_slip PI Controller

Figure 3-5 Side-Slip PI Controller

Figure 3-6 Side-Slip PI Controller Step Response
3.2.3 Yaw PI

Yaw Controller Using Successive Loop (Roll PID and Yaw PI)

Figure 3-7 Yaw PI Controller

Figure 3-8 Yaw PI Controller Step Response
3.2.4 Pitch PD

Figure 3-9 Pitch PD controller

Figure 3-10 Pitch PD controller Step Response
3.2.5 Altitude PI

Altitude Controller Using Successive Loop (Pitch PD and Altitude PI)

\[ Va_{trim} = a_{\theta 3.s} + a_{\theta 1.s} + a_{\theta 22} \]

\[ K_p_{\theta} = -10.5 \]

\[ K_p_h = 0.5 \]

\[ K_i_h = 0.45 \]

\[ K_d_{\theta} = -0.5 \]

\[ 1 \]

\[ s \]

Disturbance

Altitude_cmd

Figure 3-11 Altitude PI Controller

Altitude Controller response to step disturbance

Figure 3-12 Altitude Controller Response to Step Disturbance
3.2.6 Airspeed PI

Airspeed PI Controller (using throttle)

![Airspeed PI Controller Diagram]

Figure 3-13 Airspeed PI Controller (using throttle)

![Airspeed Controller Step Response Graph]

Figure 3-14 Airspeed Controller Step Response
3.3 6-DOF overall linear controller

In order to develop the overall controller, non-linear model of airplane is divided into two sub-models: the dynamic model, which is the same rigid body 6 DOF model as shown in Figure 3-15, and the aerodynamics model which is shown in Figure 3-17:

**Figure 3-15 Six DOF rigid body dynamic model**

```matlab
function vdot = Transl(V,V,Mass,v)
    vdot = [1/Mass]*F-cross(v,V);

function vdot = Rotl[K, Inertia,R]
    vdot = inv(Inertia)*[X-cross(R, Inertia)*R];

function Eulerdot = StolRot(v,Euler)
    phi=Euler(1);
    theta=Euler(2);
    psi=Euler(3);
    StolRotmat=[1 0 0	sin(phi)*tan(theta) cos(phi)*tan(theta) 0
                0 cos(phi) -sin(phi)
                0 sin(phi)*sec(theta) cos(phi)*sec(theta)];
    Eulerdot=StolRotmat*v;

function Ve = Stol[Ve,Euler]
    phi=Euler(1);
    theta=Euler(2);
    psi=Euler(3);
    tsi=Euler(3);
    Rtsi = [ cos(tsi) sin(tsi) 0
             -sin(tsi) cos(tsi) 0
             0 0 1 ];
    Rtheta = [ cos(theta) 0 -sin(theta)
              0 1 0
              sin(theta) 0 cos(theta) ];
    Rphi = [ 1 0 0
             0 cos(phi) sin(phi)
             0 -sin(phi) cos(phi) ];
    Stolmat=[Rphi*Rtheta*Rtsi];
    Ve=Stolmat*Vb;
```

**Figure 3-16 Functions used in 6-DOF rigid body model**
Aerodynamic Forces and Moments Model

\[ \text{function} \ [\text{Force}, \text{Moment}] = \text{aero1}(\text{Va}, \alpha, \beta, \delta_a, \delta_r, \delta_e, \delta_t, p, q, r, \phi, \theta, C_{\alpha}) \]

\[ \text{vec1} = (-\cos(\alpha), \sin(\alpha)) \]
\[ \text{vec2} = (-\sin(\alpha), -\cos(\alpha)) \]

\[ C_{\alpha} = 0.5 \times \frac{\rho v \text{ Va}^2}{\text{ Va}} \]

\[ F = m \frac{\rho v}{2} \frac{\text{ Va}^2}{\text{ Va}^2} + \left(\frac{C_{\alpha} \rho v}{\text{ Va}^2} \right) \]

\[ M = \frac{m \rho v}{2} \frac{\text{ Va}^2}{\text{ Va}^2} + \left(\frac{C_{\alpha} \rho v}{\text{ Va}^2} \right) \]

Figure 3-17 Aerodynamics model

Figure 3-18 Aerodynamics functions
Another block in the aerodynamics model is $C(\alpha)$ block which gets angle of attack as input and has two outputs, drag and lift coefficients based on the equations (2-16) and (2-17). Function of this block shows in Figure 3-19, parameters for this function obtained from [1] is shown in Figure 3-20.

Another block which is used in overall control model is showing animation of aircraft in real time similar to Figure 3-21.
Complete overall PID controller model for non-linear 6-DOF aircraft is shown in Figure 3-22.

Figure 3-22 Non-linear aircraft PID controller

The details of linear controller block is shown in Figure 3-23

Figure 3-23 Linear Controller Block

Controllers in this block are the same that developed for transfer functions in the section 3.2 of course the system here in non-linear aircraft model rather than linear transfer function. Figure 3-24 to Figure 3-27 are showing the detail of each controller block.
Roll PID

\[ \begin{align*} \frac{1}{s} & \rightarrow 0.05 \\ K_l_{\phi} & \rightarrow 2.12 \\ K_p_{\phi} & \rightarrow 0.24 \\ p & \rightarrow 1 \\ \delta_a & \rightarrow \Delta_a \end{align*} \]

Figure 3-24 Roll PID

Side-Slip PI

\[ \begin{align*} \frac{1}{s} & \rightarrow 10 \\ K_l_{\beta} & \rightarrow 3 \\ K_p_{\beta} & \rightarrow 1 \\ \beta_c & \rightarrow \Delta_r \\ \delta_r & \rightarrow \Delta_r \\ \frac{1}{s} & \rightarrow 3 \\ v & \rightarrow \Delta_r \\ \text{Beta}_c & \rightarrow \Delta_r \end{align*} \]

Figure 3-25 Side-slip PI

Altitude PI and Pitch PD

\[ \begin{align*} \frac{1}{s} & \rightarrow 0.45 \\ K_l_{h} & \rightarrow 0.5 \\ K_p_{h} & \rightarrow 1 \\ \text{Z}_e & \rightarrow \text{Alt}_c \\ \text{Alt}_c & \rightarrow \text{Alt} \\ \frac{1}{s} & \rightarrow 0.5 \\ \text{q} & \rightarrow \text{Alt}_c \\ \text{Kd}_\theta & \rightarrow -0.5 \\ \theta_c & \rightarrow \text{Alt}_c \\ \text{Alt}_c & \rightarrow \text{Alt}_c \\ \text{Kp}_\theta & \rightarrow -10.5 \\ \theta_c & \rightarrow \text{Alt}_c \\ \text{Alt}_c & \rightarrow \text{Alt}_c \end{align*} \]

Figure 3-26 Altitude PI and Pitch PD

Airspeed PI

\[ \begin{align*} \frac{1}{s} & \rightarrow 50 \\ K_l_{V} & \rightarrow 7.35 \\ K_p_{V} & \rightarrow 1 \\ \text{Va}_{\text{cmd}} & \rightarrow \text{Va}_{\text{meas}} \\ \text{D_t}_{\text{trim}} & \rightarrow \text{delta}_{t} \\ \text{delta}_{t} & \rightarrow \text{delta}_{t} \\ \text{delta}_{t} & \rightarrow \text{delta}_{t} \end{align*} \]

Figure 3-27 Airspeed PI
Results for simulating of acting PID controllers on the non-linear aircraft model to keep the trim conditions are shown in Figure 3-28 to Figure 3-31.

![Figure 3-28 Velocities in inertial frame (PID)]](image)

![Figure 3-29 Euler Angles (PID)]](image)
Figure 3-30 Position of aircraft in inertial frame (PID)

Figure 3-31 Position of aircraft in 3D inertial frame
4 Dynamic inversion controller

4.1 Introduction
As explained in [3] to understand the basic concept of dynamic inversion we assume aircraft dynamics represent by the following equations:

\[ \dot{x} = F(x, u) \]
\[ y = H(x) \]

Where \( x \) is the state vector, \( u \) is the control vector and \( y \) is the output, for small perturbation from trim condition we can assume the function \( F \) is linear in \( u \) so we can re-write the above equation as follows:

\[ \dot{x} = f(x) + g(x)u \quad (4-1) \]

If \( g(x) \) is invertible, and we will define the desired states rate we have:

\[ u = g^{-1}(x)[\dot{x}_{des} - f(x)] \quad (4-2) \]

In ideal case if we have system shown in Figure 4-1:

![Figure 4-1 Ideal DI system [3]](image)

The result of multiplication of first two blocks in the left will be 1; of course due to non-linear model of aircraft and approximation which we implemented to get the linearized model, in addition to error of model parameters, in the real application that result will not become 1 so we need to have a feedback loop in order to compensate those errors, feedback loop could be similar to system in Figure 4-2:

![Figure 4-2 DI system with feedback [3]](image)

To define the input, output and states for our aircraft, we combine lateral and longitudinal model and develop one matrix equation for whole control system; we define our vector as follows:
\[
x = \begin{bmatrix} \alpha & \beta & \phi & p & q & r \end{bmatrix}^T
\]

\[
u = \begin{bmatrix} \delta_a & \delta_e & \delta_r \end{bmatrix}^T
\]

\[
CV = [\alpha_{cmd} \beta_{cmd} \phi_{cmd}]
\]

To define the \( f(x) \) and \( g(x) \) in (4-1) we will use lateral and longitudinal state space model, here we recall the equations (2-31) and (2-33) and remove \( \psi \) from lateral state-space model and remove \( h \) from longitudinal state-space model because the other states are independent to them and they are independent to inputs, check that all elements of \( 5^{th} \) column in \( A \) matrix and \( 5^{th} \) row in \( B \) matrix are zero. In addition \( \delta_t \) assumed to be constant so it will be cancelled from equation. The final equations are (4-3) for lateral and (4-4) for longitudinal state-space model.

\[
\begin{bmatrix}
\dot{\beta} \\
\dot{\gamma} \\
\dot{\psi}
\end{bmatrix} =
\begin{bmatrix}
Y_{\beta} & V_{\beta}\cos\beta' & Y_{\gamma} & V_{\gamma}\cos\beta' & 0 & g\cos\theta'\cos\phi' \\
L_{\beta} & V_{\beta}\cos\beta' & L_{\gamma} & V_{\gamma}\cos\beta' & 0 & 0 \\
N_{\beta} & V_{\beta}\cos\beta' & N_{\gamma} & V_{\gamma}\cos\beta' & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\beta \\
\gamma \\
\psi
\end{bmatrix}
+ \begin{bmatrix}
Y_{\beta} \\
L_{\beta} \\
N_{\beta}
\end{bmatrix}\delta_t
\]

\[
\begin{bmatrix}
\dot{u} \\
\dot{q}
\end{bmatrix} =
\begin{bmatrix}
X_{u} & X_{q}\cos\alpha' & X_{\beta} & -g\cos\theta' & X_{\gamma} & -g\sin\theta' \\
Z_{u} & Z_{q}\cos\alpha' & Z_{\beta} & 0 & Z_{\gamma} & 0 \\
M_{u} & M_{q}\cos\alpha' & M_{\beta} & 0 & M_{\gamma} & 0
\end{bmatrix}
\begin{bmatrix}
u \\
\kappa \\
\sigma
\end{bmatrix}
+ \begin{bmatrix}
X_{\beta} \\
X_{\gamma} \\
M_{\beta}
\end{bmatrix}\delta_t
\]

For Zagi airframe and trim conditions for straight line wing level with airspeed 12 m/s, the numerical value of matrices in equations (4-3) and (4-4) is shown in

\[
\begin{align*}
A &= \begin{bmatrix}
-0.3101 & -0.8187 & -1.1461 & -9.7562 \\
-0.0705 & -0.6025 & 1.0000 & -0.0704 \\
0.4098 & -5.0202 & -6.7124 & 0 \\
0 & 0 & 1.0000 & 0
\end{bmatrix} \\
B &= \begin{bmatrix}
-0.5211 \\
0.4548 \\
-67.7058 \\
0
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
A_{\text{bar}} &= \begin{bmatrix}
-1.2376 & 0.0555 & -0.9554 & 0.8129 \\
-3.5415 & -4.1466 & 2.3795 & 0 \\
48.0007 & 0.2165 & -4.0534 & 0 \\
0 & 1.0000 & 0 & 0
\end{bmatrix} \\
B_{\text{bar}} &= \begin{bmatrix}
0 & -0.2147 \\
23.6099 & 30.7085 \\
11.9915 & -6.0162 \\
0 & 0
\end{bmatrix}
\end{align*}
\]

Figure 4-3 Numerical value for lateral and longitudinal state-space model
4.2 Applying dynamic inversion

Following the method in [3], in order to bypass the singularity problem in the inversion we implement the DI in two steps for that divide our states into two parts, angle of attack $\alpha$, side-slip angle $\beta$ and bank angle $\phi$ will be the slow motion variables on the other hand $p$, $q$ and $r$ will be the fast variables, the point here is once a control surface has deflection Euler angles rates $p$, $q$ and $r$ are changing faster than $\alpha$, $\beta$ and $\phi$ so in the first step we calculate DI for slow variables and in second step we will find the DI for fast variables, note that we ignore the direct effect of elevator deflection on the angle of attack, that is very small value and cause the singularity problem. The simplified equation will be as follows:

$$
\begin{align*}
\begin{bmatrix}
\dot{\alpha} \\
\dot{\beta} \\
\dot{\phi}
\end{bmatrix} &=
\begin{bmatrix}
A_{21} & 0 & 0 \\
0 & A_{11} & A_{14} \\
0 & A_{41} & A_{44}
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta \\
\phi
\end{bmatrix} +
\begin{bmatrix}
0 & A_{23} & 0 \\
A_{12} & 0 & A_{13} \\
A_{42} & 0 & A_{43}
\end{bmatrix}
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix} \\

\begin{bmatrix}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{bmatrix}_{cmd} &= 
\begin{bmatrix}
0 & A_{23} & 0 \\
0 & A_{14} & A_{13} \\
0 & A_{43} & A_{44}
\end{bmatrix}
\begin{bmatrix}
\dot{\alpha} \\
\dot{\beta} \\
\dot{\phi}
\end{bmatrix}_{cmd}

\begin{bmatrix}
\delta_a \\
\delta_b \\
\delta_c_{cmd}
\end{bmatrix} &=
\begin{bmatrix}
B_{21} & 0 & B_{22} \\
0 & B_{31} & 0 \\
B_{41} & 0 & B_{42}
\end{bmatrix}
\begin{bmatrix}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{bmatrix}_{cmd} -
\begin{bmatrix}
0 & A_{21} & A_{24} & A_{22} \\
A_{12} & 0 & 0 & A_{13} \\
A_{42} & 0 & 0 & A_{43}
\end{bmatrix}
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\end{align*}
$$

(4-5)  
(4-6)  

The DI equations are (4-7) and (4-8):

$$
\begin{align*}
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix} &= 
\begin{bmatrix}
0 & A_{23} & 0 \\
0 & A_{14} & A_{13} \\
0 & A_{43} & A_{44}
\end{bmatrix}^{-1}
\begin{bmatrix}
\dot{\alpha} \\
\dot{\beta} \\
\dot{\phi}
\end{bmatrix}_{cmd} -
\begin{bmatrix}
A_{11} & A_{14} & 0 \\
A_{41} & 0 & A_{44}
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta \\
\phi
\end{bmatrix}

\begin{bmatrix}
\delta_a \\
\delta_b \\
\delta_c_{cmd}
\end{bmatrix} &=
\begin{bmatrix}
B_{21} & 0 & B_{22} \\
0 & B_{31} & 0 \\
B_{41} & 0 & B_{42}
\end{bmatrix}
\begin{bmatrix}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{bmatrix}_{cmd} -
\begin{bmatrix}
0 & A_{21} & A_{24} & A_{22} \\
A_{12} & 0 & 0 & A_{13} \\
A_{42} & 0 & 0 & A_{43}
\end{bmatrix}
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\end{align*}
$$

(4-7)  
(4-8)  

Block diagram for DI controller with fast and slow inversion shown in Figure 4-4:
4.2.1 Slow inversion

The block for slow inversion equation (4-7) is shown in Figure 4-5:

![Slow Dynamic Inversion Block Diagram]

\[ \hat{\alpha} = K_{unc} \frac{\alpha_{cmd} - \alpha_{meas}}{\Delta T} \] (4-9)

4.2.2 Fast inversion

The block for fast inversion equation (4-8) is shown in Figure 4-6:

![Fast Dynamic Inversion Block Diagram]

4.2.3 Desired dynamics

A proportional system is used for both fast and slow desired dynamics. In this block to find the desired rate of a certain state, for example angle of attack, \(\alpha\), the measurement value, \(\alpha_{meas}\), is deducted from the desired value, \(\alpha_{cmd}\), and multiplied by constant. The result will be feed to inversion block as desired rate. Since the sample time is constant the desired dynamics block calculates the approximate time derivative of the state as shown in equation (4-9):

\[ \hat{\alpha} = K_{unc} \frac{\alpha_{cmd} - \alpha_{meas}}{\Delta T} \] (4-9)
Desired dynamic block and proportional value for slow states, i.e. $\alpha, \beta$ and $\varphi$ is shown in Figure 4-7:

**Figure 4-7 Slow inversion desired dynamic block**

Block and values for fast states, $p, q$ and $r$ is shown in Figure 4-8:

**Figure 4-8 Fast inversion desired dynamic block**

### 4.2.4 Actuator dynamics

In order to have more realistic simulation actuator dynamics added to non-linear aircraft model, Figure 4-9 shows the block for actuator dynamics, second order model with 10 Hz natural frequency and 0.707 damping ratio is used as actuator model for elevator, aileron, rudder and throttle of the aircraft.

**Figure 4-9 Actuators dynamics**
4.2.5 Sensors noise and dynamics
Sensors noise and dynamics also added as a block to system, first order system with natural frequency of 30 Hz considered as dynamic model of sensor and white Gaussian noise added and feed to the controller. (Figure 4-10)

Sensor transfer function and measurement noise model

![Sensor transfer function and measurement noise model](image)

**Figure 4-10 Sensors noise and dynamic model**

4.2.6 Wind gust
To simulate wind gust disturbance pulse shown in Figure 4-11 is added to airspeed component in z direction (w).

![Wind Gust](image)

**Figure 4-11 Wind Gust**
4.3 6-DOF overall DI controller

Figure 4-12 shows the overall DI controller:

![DI_Controller](image)

DI controller block which include desired dynamics, fast and slow inversion is shown in Figure 4-13.

![Dynamic_Inversion_Controller](image)

Results of simulating whole system for time period of 20 seconds are shown in Figure 4-14 to Figure 4-17.

Velocities in inertial frame are shown in Figure 4-14.

Euler angles are shown in Figure 4-15.

Figure 4-16 and Figure 4-17 are showing the position of aircraft in inertial frame.
Figure 4-14 Velocities in inertial frame (DI)

Figure 4-15 Euler angles (DI)
Figure 4-16 Position of aircraft in inertial frame (DI)

Figure 4-17 Aircraft 3D position in inertial frame (DI)
5 Robustness Analysis

5.1 Introduction to μ-synthesis

As per the details in [3], the most commonly used methodology to analyze robustness of linear systems using DI controllers is based on the structured singular value (μ) and a technique now widely described in the literatures as μ-analysis. The bounded uncertainty in the parameters should be separated from the nominal value using the linear fraction transformers. Suppose we have a linear system that is described by the following equations:

\[
\dot{x} = ax + b \\
y = x
\]  \hspace{1cm} (5-1)

Assume that the value of \(a\) varies between \(a^-\) and \(a^+\).

\[
a^- \leq a \leq a^+
\]  \hspace{1cm} (5-2)

This relation can be rewritten in terms of nominal value as

\[
a = a^{\text{nom}} + \frac{k_1 \delta_a}{1 - k_2 \delta_a}
\]  \hspace{1cm} (5-3)

Where,

\[
a^{\text{nom}} = \text{nominal value of } a
\]

\[
k_1 = \frac{2(a^+ - a^-)(a^+ - a^-)}{a^+ - a^-}
\]

\[
k_2 = \frac{(a^+ + a^-) - 2a}{a^+ - a^-}
\]

\[-1 \leq \delta_a \leq 1\]

The perturbation in \(a\) described in (5-1) can be integrated into (5-3) and expressed in state-space form by introducing the fictitious terms \(z_a\) and \(w_a\) as

\[
\begin{bmatrix}
\dot{x} \\
z_a \\
y
\end{bmatrix} =
\begin{bmatrix}
\bar{a} & k_1 & b \\
1 & k_2 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
w_a \\
u
\end{bmatrix}
\]  \hspace{1cm} (5-4)

Block diagram in Figure 5-1 corresponds to (5-4).

![Figure 5-1 LFT block diagram [3]]
Here, what is known, matrix G, is separated from what is uncertain, \( \Delta \). Although the \( \Delta \) is uncertain, the range is known: \( -1 \leq \Delta \leq 1 \). The next case to be considered is uncertainty in \( b \). The variation in \( b \), is expressed similar to the previous one as:

\[
\begin{bmatrix}
\dot{x} \\
z_p \\
y
\end{bmatrix} =
\begin{bmatrix}
\bar{a} & k' & b' \\
0 & k_2' & 1 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
w_p \\
u
\end{bmatrix}
\]

(5-5)

Where,

\[
k_1' = \frac{2(b^+ - \bar{b})(\bar{b} - b^-)}{b^+ - b^-}
\]

\[
k_2' = \frac{(b^+ + b^-) - 2\bar{b}}{b^+ - b^-}
\]

Finally, combining the expressions for uncertainty in \( a \) and \( b \), (5-6) is the state-space results and block diagram shown in Figure 5-2.

\[
\begin{bmatrix}
\dot{x} \\
z_a \\
z_b \\
y
\end{bmatrix} =
\begin{bmatrix}
\bar{a} & k_1 & k'_1 & \bar{b}' \\
1 & k_2 & 0 & 0 \\
0 & 0 & k'_2 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
w_a \\
w_b \\
u
\end{bmatrix}
\]

(5-6)

\[\begin{array}{c}
\Delta \\
G
\end{array}\]

Figure 5-2 Companion to LFT block diagram [3]

We have separated what is known from what is uncertain but bounded. Since \( \Delta \) is no longer a scalar, we have to make another choice for a norm: the maximum singular value. It is not difficult to see that \( \sigma(\Delta) \leq 1 \). A very important observation is that the uncertain element \( \Delta \) has fixed structure: a diagonal matrix consisting of the individual uncertainties in \( a \) and \( b \). Thus, unstructured uncertainty at the component level has become a structured uncertainty as the system level. LFTs are mathematical tools that allow us to provide this systemic structure for the uncertainty.

After representing uncertainties in the system using LFTs, attention turns to analyzing the robustness of systems modeled in this way. Using structured singular value, \( \mu \), the technique is based on following theorem:

Robust Stability \( \Leftrightarrow \mu_{\Delta}(M_{11}(j\omega)) < 1, \forall \omega \)  

(5-7)

Where \( M_{11} \) is the left upper corner block of \( M \); i.e.,
\begin{equation}
M(j\omega) = \begin{bmatrix}
M_{11}(j\omega) & M_{12}(j\omega) \\
M_{21}(j\omega) & M_{22}(j\omega)
\end{bmatrix}
\end{equation}

and the function $\mu_\Delta$ is defined as
\begin{equation}
\mu_\Delta(M) = \frac{1}{\min\{\sigma(\Delta) : \Delta \in \Delta, \det(I - M\Delta) = 0\}}
\end{equation}

where $\Delta = \{\text{diag}(\Delta_1, \Delta_2, \ldots, \Delta_n)\}$.

According to this theorem, $\mu_\Delta$ is a function of $M$ that depends on the structure of $\Delta$. $\mu_\Delta$ is the reciprocal of the smallest $\Delta$ (where we use $\sigma$ as the norm) we can find for the set of $\Delta$ that makes the matrix $I-M\Delta$ singular. If no such $\Delta$ exists, $\mu_\Delta$ is taken to be zero.

![Figure 5-3 μ-analysis block diagram [3]](image)

Even though the function $\mu_\Delta$ is defined, we still should calculate it and since there is no exact calculation algorithm exists we calculate upper and lower bounds. Normally, the upper bound is used since these values of $\mu$ are safer (as mentioned in [3]). According to [3] steps needed to test the robust stability using $\mu$-analysis are as follows:

1. Construct the interconnection structure, $M$, which is a known linear system.
2. Define a structured perturbation set, $\Delta$.
3. Combine $M$ and $\Delta$ to form the feedback system shown in Figure 5-3.
4. Calculate a frequency response of $M$.
5. Calculate the upper and lower bounds for $\mu$.
6. Find the upper bound peak value.
7. If $\mu_{\text{peak}} < 1$ : pass; $\mu_{\text{peak}} > 1$: fail.

By the way to evaluate the $\mu$-analysis of DI controller MATLAB® software is used.

### 5.2 Robustness of DI controller

As per the technique explained in [5], robustness of DI controller for uncertainty in three major parameter of lateral model is tested using $\mu$-analysis tools of MATLAB®. The uncertainty of 20% in $L_p$, 40% in $N_r$ and 30% in $N_{\text{delta}_r}$ system remains stable.
Figure 5-4 Uncertain DI control system

Figure 5-4 shows the block diagram for uncertain DI control system.

Figure 5-5 $\mu$ upper and lower bounds for parameter uncertainty

As shown in Figure 5-5, the upper limit for whole range of frequency remains less than one so for the defined parameter uncertainty system remains stable. Report generated by MATLAB® in Figure 5-6 confirms the same.

REPORT

Uncertain System is robustly stable to modeled uncertainty.
-- It can tolerate up to 58% of the modeled uncertainty.
-- A destabilizing combination of 61% of the modeled uncertainty exists, causing an instability at 1.30m×1009 rad/s.
-- Sensitivity with respect to uncertain element ...
  'U_Lp' is 14%. Increasing 'U_Lp' by 25% leads to a 4% decrease in the margin.
  'U_Ndr' is 51%. Increasing 'U_Ndr' by 25% leads to a 5% decrease in the margin.
  'U_Nk' is 58%. Increasing 'U_Nk' by 25% leads to a 14% decrease in the margin.

Figure 5-6 Stability report for uncertain system


% Make uncertain model for Aircraft
% 

echo off;

clear all;
close all;
clos;

% Calculate trim states
UAV_trim_states;

% Calculate lateral state space model A_bar and B_bar
UAV_lateral;

% Calculate longitudinal state space model A and B
UAV_long;

% Uncertainty for Lp, Nr and N_delta_r

% Lp=A_bar(2,2)
U_Lp=ureal('U_Lp',A_bar(2,2),{'Percentage',20});

% Nr=A_bar(3,3)
U_Nr=ureal('U_Nr',A_bar(3,3),{'Percentage',40});

% B=B_bar(2,1)
U_B=ureal('U_B',B_bar(2,1),{'Percentage',30});

Figure 5-7 Define uncertain parameters in MATLAB®

U_A_bar=umat(A_bar);
U_A_bar(2,2)=U_Lp;
U_A_bar(3,3)=U_Nr;

U_B_bar=umat(B_bar);
U_B_bar(3,2)=U_B;

C_airplane=[1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0];
B_airplane=[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0]

U_airplane=usample(U_A_bar,U_B_bar,C_airplane,B_airplane);

K1=inv([B_bar(2,2);B_bar(2,2);B_bar(3,1);B_bar(3,2)])
K1=A_bar(2,1,1:3)
K2=[7.3 0.0 0.0];
K3=[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0];

systemnames = 'U_airplane K1 B1 K2 B3';
inputvar = '([p_cmd r_cmd] rcmd)';
outputvar = '([U_airplane])';
input_to_U_airplane = '([K1] rcmd
input_to_R1 = '([U_airplane])';
input_to_R2 = '([p_cmd r_cmd(1):r_cmd(2)] rcmd_B3)';
input_to_R3 = '([U_airplane] rcmd_B3)';
CLOOP = sysic;

options侬=alphapos(\'n'侬 tol=0.01); % Change tol to tol for more accurate result

REPORT1

Figure 5-8 Evaluate μ analysis using MATLAB®
Further to the stability information step responses of roll rate to p_cmd for ten random values of uncertain parameters are shown in Figure 5-9.

![Figure 5-9 Roll rate (p) step response to p_cmd](image)

Furthermore, the step responses of yaw rate to r_cmd for ten random values of uncertain parameters are shown in Figure 5-10.

![Figure 5-10 Yaw rate (r) step response to r_cmd](image)
5.3 Compare PID and DI controller (Uncertain)

In this section, the performance of PID controller is compared to the one of DI controller while uncertain parameters are taken into consideration. The aircraft lateral state space nominal model is used and saturation of the control surfaces is also implemented. Both PID and DI controllers are designed for roll command with an uncertainty of +/- 20% for Lp.

Figure 5-11 shows the block diagram of the PID controller system.

![Roll PID Controller Nominal Model](image)

The step responses of roll angle to the 0.08 [rad] (=4.58 [deg]) roll command for nominal Lp, Lp-20% and Lp+20% are shown in Figure 5-15.

DI controller for the same state (roll angle) and same aircraft model is shown in Figure 5-12; the corresponding subsystem blocks for slow and fast dynamic inversion are shown in Figure 5-13 and Figure 5-14. In order to overcome the actuator saturation, a second order filter is implemented between step command and roll command input of DI controller. This filter has 7 Hz natural frequency and 0.9 damping ratio.

The step response for DI controller with the same condition is also shown in Figure 5-15.

![Roll DI Controller Nominal Model](image)

Figure 5-12 DI Roll Controller
Lateral slow inversion block

\[
\begin{bmatrix}
A_{\text{bar}}(1,1) & A_{\text{bar}}(1,4) \\
A_{\text{bar}}(4,1) & A_{\text{bar}}(4,4)
\end{bmatrix}
\]

Matrix Multiply

\[
\begin{bmatrix}
A_{\text{bar}}(2,1) & A_{\text{bar}}(2,2) \\
A_{\text{bar}}(3,1) & A_{\text{bar}}(3,2)
\end{bmatrix}
\]

Divide

\[
\begin{bmatrix}
A_{\text{bar}}(4,1) & A_{\text{bar}}(4,4)
\end{bmatrix}
\]

Figure 5-13 Slow inversion block (DI Roll Controller)

Lateral fast inversion block

\[
\begin{bmatrix}
A_{\text{bar}}(2,1) & A_{\text{bar}}(2,2) \\
A_{\text{bar}}(3,1) & A_{\text{bar}}(3,2)
\end{bmatrix}
\]

Matrix Multiply

\[
\begin{bmatrix}
B_{\text{bar}}(2,1) & B_{\text{bar}}(2,2) \\
B_{\text{bar}}(3,1) & B_{\text{bar}}(3,2)
\end{bmatrix}
\]

Divide

\[
\begin{bmatrix}
A_{\text{bar}}(2,1) & A_{\text{bar}}(2,2) & A_{\text{bar}}(2,3) \\
A_{\text{bar}}(3,1) & A_{\text{bar}}(3,2) & A_{\text{bar}}(3,3)
\end{bmatrix}
\]

Figure 5-14 Fast inversion block (DI Roll Controller)

Roll Angle Step Response

![Roll Angle Step Response Graph](image)

Figure 5-15 Compare roll command step response of PID and DI with uncertain Lp
Comparing the response of PID and DI controller as shown in Figure 5-15, we can observe for the same plant, DI controller has obvious better performance, faster rise time, lower overshoot and faster settling time. In addition, if uncertainty in a parameter exists such as +/-20% in Lp, we can see the performance of PID controller is changing due to a change in the parameter value. On the other hand, for the same condition, the DI controller has less change in the performance. While change in the performance of PID controller is clear in Figure 5-15, it is very difficult to see the performance change of DI controller in the same Figure 5-15.

By magnifying a section of the DI controller step response, a tiny performance change can be observed in Figure 5-16.

![Figure 5-16 Magnified DI step response in Figure 5-15](image-url)
6 Conclusion

In this thesis, the rigid body nonlinear equations of motion of a Zagi fixed wing aircraft are derived. Aerodynamics forces and moments are taken into consideration. Actuator dynamics, including actuator saturation, have been appropriately modeled so as to make the simulation more realistic. Sensor dynamics are accounted for by deriving simple dynamic models that include sensor noise and time delays. The derived full non-linear model of the aircraft is used to test and validated all control laws designed for this aircraft.

To ensure good performance of the aircraft, a Dynamic Inversion controller has been designed, implemented, and simulated. Implementation is performed in a MATLAB® SIMULINK® environment. To avoid singularities in implementing the Dynamic Inversion control law, fast and slow dynamic of the system are considered. Instead of using the decoupled longitudinal and lateral linear models traditionally used in PID design, the control action is taken directly at the Attitude command level of the coupled equations of motion. Stability and robustness of the DI controller is evaluated using the $\mu$-analysis tools. Subject to several types of modeling uncertainties, simulation has proven the stability and robustness of the DI controller. The DI controller showed several advantages compare to the conventional PID controller. The DI controller has faster time response, less overshoot, tighter tracking, and performed better under uncertainties in plant parameters.

In conclusion, though the DI controller is more complex and time consuming during the design phase, its implementation resulted in a less complex code and smaller memory requirement compare to a PID controller. Using simpler and smaller software translates into using lighter and more cost effective hardware. This intern results in better system reliability. This make the DI controller a good choice for smaller and light weight unmanned aircrafts where weight and reliability are of great concern.
References


Vita

Roozbeh Falah Ramezani was born on September 17, 1975, in Abadan, city in the southwest of Iran. He was graduated from Dr. Sharyati High School (Tehran) in 1993. Same year he joined to the Iran University of Science and Technology, Tehran, from which he graduated in 1997. His degree was a Bachelor of Science in Electrical Engineering – Power.

Mr. Ramezani worked as product developer and technical manager for seven years at Parto Sanat Co and Parnyan Electronics Co in Iran. He moved to United Arab Emirates in 2004 and established RDA Electronics in Dubai. In 2007, Mr. Ramezani began a Master’s program in Mechatronics Engineering at the American University of Sharjah. He was awarded the Master of Science degree in Mechatronics Engineering in 2012.