ON WEAKLY 2-ABSORBING IDEALS OF COMMUTATIVE RINGS

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Abstract. Let \( R \) be a commutative ring with identity \( 1 \neq 0 \). Various generalizations of prime ideals have been studied. For example, a proper ideal \( I \) of \( R \) is weakly prime if \( a, b \in R \) with \( 0 \neq ab \in I \), then either \( a \in I \) or \( b \in I \). Also a proper ideal \( I \) of \( R \) is said to be 2-absorbing if whenever \( a, b, c \in R \) and \( abc \in I \), then either \( ab \in I \) or \( ac \in I \) or \( bc \in I \). In this paper, we introduce the concept of a weakly 2-absorbing ideal. A proper ideal \( I \) of \( R \) is called a weakly 2-absorbing ideal of \( R \) if whenever \( a, b, c \in R \) and \( 0 \neq abc \in I \), then either \( ab \in I \) or \( ac \in I \) or \( bc \in I \). For example, every proper ideal of a quasi-local ring \((R, M)\) with \( M^3 = \{0\} \) is a weakly 2-absorbing ideal of \( R \). We show that a weakly 2-absorbing ideal \( I \) of \( R \) with \( I^3 \neq 0 \) is a 2-absorbing ideal of \( R \). We show that every proper ideal of a commutative ring \( R \) is a weakly 2-absorbing ideal if and only if either \( R \) is a quasi-local ring with maximal ideal \( M \) such that \( M^3 = \{0\} \) or \( R \) is ring-isomorphic to \( R_1 \times F \) where \( R_1 \) is a quasi-local ring with maximal ideal \( M \) such that \( M^2 = \{0\} \) and \( F \) is a field or \( R \) is ring-isomorphic to \( F_1 \times F_2 \times F_3 \) for some fields \( F_1, F_2, F_3 \).

1. Introduction

In this paper, we study weakly 2-absorbing ideals in commutative rings with identity, which are a generalization of weakly prime ideals. Recall that 2-absorbing ideals, which are a generalization of prime ideals, were introduced and investigated in [4] and most recently in [3]. Recall from [2] that a proper ideal \( I \) of a commutative ring \( R \) is said to be a weakly prime ideal of \( R \) if whenever \( a, b \in R \) and \( 0 \neq ab \in I \), then either \( a \in I \) or \( b \in I \). Also recall from [4] that a proper ideal \( I \)

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of a commutative ring $R$ is called a 2-absorbing ideal of $R$ if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. We define a proper ideal of a commutative ring $R$ to be a weakly 2-absorbing ideal of $R$ if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. In the second section of this paper, many basic properties of weakly 2-absorbing ideals are studied, and in the third section, we characterize all commutative rings with the property that all proper ideals are weakly 2-absorbing ideals.

We assume throughout that all rings are commutative with $1 \neq 0$. Let $R$ be a ring. Then $\text{Nil}(R)$ denotes the ideal of nilpotent elements of $R$. An ideal $I$ of $R$ is said to be a proper ideal of $R$ if $I \neq R$. As usual, $\mathbb{Z}$, and $\mathbb{Z}/n\mathbb{Z}$ will denote integers, and integers modulo $n$, respectively. Some of our examples use the $R(+)M$ construction as in [5]. Let $R$ be a ring and $M$ an $R$-module. Then $R(+)M = R \times M$ is a ring with identity $(1, 0)$ under addition defined by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$. Note that $(0(+)M)^2 = 0$; so $0(+)M \subseteq \text{Nil}(R(+)M)$.

2. Basic properties of weakly 2-absorbing ideals

It is clear that every 2-absorbing ideal of a ring $R$ is a weakly 2-absorbing ideal of $R$. If $R$ is any commutative ring, then $I = \{0\}$ is a weakly 2-absorbing ideal of $R$ by definition. If $I = \{0\}$, then $I$ is a 2-absorbing ideal of $\mathbb{Z}_4$, but $I$ is a weakly 2-absorbing ideal of $\mathbb{Z}_8$ that is not a 2-absorbing ideal of $\mathbb{Z}_8$. The following is an example of a nonzero weakly 2-absorbing ideal that is not a 2-absorbing ideal (also see Theorem 2.9 and Theorem 2.13).

**Example 2.1.** Let $M = \{0, 4\}$. Then $M$ is an ideal of $\mathbb{Z}_8$. Let $R = \mathbb{Z}_8(+)M$ and let $I = \{(0, 0), (0, 4)\}$. Since $abc \in I$ for some $a, b, c \in R \setminus I$ if and only if $abc = (0, 0)$, we conclude that $I$ is a weakly 2-absorbing ideal of $R$. Since $(2, 0)(2, 0)(2, 0) \in I$ and $(4, 0) \notin I$, $I$ is not a 2-absorbing ideal of $R$. For an infinite weakly 2-absorbing ideal that is not a 2-absorbing ideal, let $M$ be as above and $K = M[\mathbb{Z}]$. Then $K$ is an infinite ideal of $\mathbb{Z}_8[\mathbb{X}]$. Let $R = \mathbb{Z}_8(+)K$ and let $I = \{0\}(+)K$. Then $I$ is an infinite ideal of $R$. Again, since $abc \in I$ for some $a, b, c \in R \setminus I$ if and only if $abc = (0, 0)$, $I$ is a weakly 2-absorbing ideal of $R$.

We start with the following trivial lemma that we omit its proof.

**Lemma 2.2.** If $P_1$ and $P_2$ are two distinct weakly prime ideals of a commutative ring $R$, then $P_1 \cap P_2$ is a weakly 2-absorbing ideal of $R$.

Let $I$ be a weakly 2-absorbing ideal of a ring $R$ and $a, b, c \in R$. We say $(a, b, c)$ is a triple-zero of $I$ if $abc = 0$, $ab \notin I$, $bc \notin I$, and $ac \notin I$. 


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Theorem 2.3. Let $I$ be a weakly 2-absorbing ideal of a ring $R$ and suppose that $(a, b, c)$ is a triple-zero of $I$ for some $a, b, c \in R$. Then

1. $abI = bcI = acI = \{0\}$.
2. $aI^2 = bI^2 = cI^2 = \{0\}$.

Proof. (1). Suppose that $abi \neq 0$ for some $i \in I$. Then $ab(c+i) \neq 0$. Since $ab \not\in I$, we conclude that either $a(c+i) \in I$ or $b(c+i) \in I$, and hence $ac \in I$ or $bc \in I$, a contradiction. Thus $abI = \{0\}$. Similarly, one can show that $bcI = acI = \{0\}$.

(2). Suppose that $ai_1i_2 \neq 0$ for some $i_1, i_2 \in I$. Since $abI = acI = bcI = \{0\}$ by (1), we conclude that $a(b+i_1)(c+i_2) = ai_1i_2 \neq 0$. Hence either $a(b+i_1) \in I$ or $a(c+i_2) \in I$ or $(b+i_1)(c+i_2) \in I$, and thus either $ab \in I$ or $ac \in I$ or $bc \in I$, a contradiction. Thus $aI^2 = \{0\}$. Similarly, $bI^2 = cI^2 = \{0\}$.

Theorem 2.4. Let $I$ be a weakly 2-absorbing ideal of $R$ that is not a 2-absorbing ideal. Then $I^3 = \{0\}$.

Proof. Since $I$ is not a 2-absorbing ideal of $R$, $I$ has a triple-zero $(a, b, c)$ for some $a, b, c \in R$. Suppose that $i_1i_2i_3 \neq 0$ for some $i_1, i_2, i_3 \in I$. Then by Theorem 2.3 we have $(a+i_1)(b+i_2)(c+i_3) = i_1i_2i_3 \neq 0$. Hence either $(a+i_1)(b+i_2) \in I$ or $(a+i_1)(c+i_3) \in I$ or $(b+i_2)(c+i_3) \in I$, and thus either $ab \in I$ or $ac \in I$ or $bc \in I$, a contradiction. Hence $I^3 = \{0\}$.

Corollary 2.5. Let $I$ be a weakly 2-absorbing ideal of $R$. If $I$ is not a 2-absorbing ideal of $R$, then $I \subseteq \text{Nil}(R)$.

It should be noted that a proper ideal $I$ of $R$ with $I^3 = 0$ need not be a weakly 2-absorbing ideal of $R$. We have the following example.

Example 2.6. $R = \mathbb{Z}_{16}$. Then $I = \{0, 8\}$ is an ideal of $\mathbb{Z}_{16}$ and $I^3 = 0$, but $2.2.2 = 8 \in I$ and $4 \not\in I$.

Theorem 2.7. Let $I$ be a weakly 2-absorbing ideal of $R$ that is not a 2-absorbing ideal. Then

1. If $w \in \text{Nil}(R)$, then either $w^2 \in I$ or $w^2I = wI^2 = \{0\}$.
2. $\text{Nil}(R)^2I^2 = \{0\}$.

Proof. (1). Let $w \in \text{Nil}(R)$. First, we show that if $w^2I \neq \{0\}$, then $w^2 \in I$. Hence assume that $w^2I \neq \{0\}$. Let $n$ be the least positive integer such that $w^n = 0$. Then $n \geq 3$ and for some $i \in I$ we have $w^2(i + w^{n-2}) = w^2i \neq 0$. Hence either $w^2 \in I$ or $(wi + w^{n-1}) \in I$. If $w^2 \in I$, then we are done. Thus assume $(wi + w^{n-1}) \in I$. Hence $w^{n-1} \in I$ and $w^{n-1} \neq 0$, and thus $w^2 \in I$. Hence for each
w ∈ \text{Nil}(R), we have either \(w^2 \in I\) or \(w^2I = \{0\}\). Now assume that \(v^2 \not\in I\) for some \(v \in \text{Nil}(R)\). Then \(v^2I = \{0\}\). We will show that \(vI^2 = \{0\}\). Assume that \(vi_1i_2 \neq 0\) for some \(i_1, i_2 \in I\). Let \(m\) be the least positive integer such that \(v^m = 0\). Since \(v^2 \not\in I\), \(m \geq 3\) and \(v^2I = 0\). Hence \(v(v + i_1)(v^{m-2} + i_2) = vi_1i_2 \neq 0\). Since \(0 \neq v(v + i_1)(v^{m-2} + i_2) \in I\), one can conclude that either \(v^2 \in I\) or \(v^{m-1} \neq 0\) and \(v^{m-1} \in I\). Hence in both cases, we have \(v^2 \in I\), a contradiction. Thus \(vI^2 = \{0\}\).

(2). Let \(a, b \in \text{Nil}(R)\). If either \(a^2 \not\in I\) or \(b^2 \not\in I\), then \(abI^2 = \{0\}\) by (1).

Hence suppose that \(a^2 \in I\) and \(b^2 \in I\). Then \(ab(a + b) \in I\). If \((a, b, a + b)\) is a triple-zero of \(I\), then \(abI = \{0\}\) by Theorem 2.3(1), and hence \(abI^2 = \{0\}\). If \((a, b, a + b)\) is not a triple-zero of \(I\), then one can easily see that \(ab \in I\), and hence \(abI^2 = \{0\}\) by Theorem 2.4.

\[\text{Corollary 2.8. Suppose that } A, B, C \text{ are weakly 2-absorbing ideals of a ring } R \text{ such that none of them is a 2-absorbing ideal of } R. \text{ Then } A^2BC = AB^2C = ABC^2 = A^2B^2 = A^2C^2 = B^2C^2 = \{0\}\]

If \(I\) is a 2-absorbing ideal of a ring \(R\), then there are at most two prime ideals of \(R\) that are minimal over \(I\) (see [4, Theorem 2.3] and [3, Theorem 2.5]). In the following result, we show that for every \(n \geq 2\), there is a ring \(R\) and a nonzero weakly 2-absorbing ideal \(I\) of \(R\) such that there are exactly \(n\) prime ideals of \(R\) that are minimal over \(I\).

\[\text{Theorem 2.9. Let } n \geq 2. \text{ Then there is a ring } R \text{ and a nonzero weakly 2-absorbing } I \text{ ideal of } R \text{ such that there are exactly } n \text{ prime ideals of } R \text{ that are minimal over } I.\]

\[\text{Proof. Let } n \geq 2 \text{ and } D = \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n \text{ (n times). Let } M = \{0, 4\} \text{ an ideal of } \mathbb{Z}_n. \text{ For every } x = (a_1, \ldots, a_n) \in D, \text{ define } xM = a_1M. \text{ Then } M \text{ is a } D-\text{module.} \text{ Now consider the idealization ring } R = D(+M) \text{ and } I = \{(0, \ldots, 0)\}(+)M. \text{ We note that if } a, b, c \in R \backslash I \text{ and } abc \in I, \text{ then } abc = ((0, \ldots, 0), 0). \text{ Hence } I \text{ is a nonzero weakly 2-absorbing ideal of } R. \text{ Since every prime ideal of } R \text{ is of the form } P(+)M \text{ for some prime ideal } P \text{ of } D \text{ by [5, Theorem 25.1 (3)]}, \text{ we conclude that there are exactly } n \text{ prime ideals of } R \text{ that are minimal over } I. \]

\[\text{Theorem 2.10. Let } R = R_1 \times R_2 \text{ be a decomposable commutative ring and } I \text{ be a proper ideal of } R_1. \text{ The following statements are equivalent:}\]

\[\begin{align*}
(1) & \ I \times R_2 \text{ is a weakly 2-absorbing ideal of } R. \\
(2) & \ I \times R_2 \text{ is a 2-absorbing ideal of } R. \\
(3) & \ I \text{ is a 2-absorbing ideal of } R_1.
\end{align*}\]
Proof. (1) $\Rightarrow$ (2). Since $I \times R_2 \not\subseteq \text{Nil}(R)$, $I \times R_2$ must be a 2-absorbing ideal of $R$ by Corollary 2.5. (2) $\Rightarrow$ (3). The claim is clear. (3) $\Rightarrow$ (1). If $I$ is a 2-absorbing ideal of $R_1$, then it is easily verified that $I \times R_2$ is a 2-absorbing ideal of $R$, and thus $I \times R_2$ is a weakly 2-absorbing ideal of $R$. □

Theorem 2.11. Let $R = R_1 \times R_2$ where $R_1$ and $R_2$ are commutative rings with identity. Let $I_1$ be a nonzero proper ideal of $R_1$ and $J$ be a nonzero ideal of $R_2$. The following statements are equivalent:

1. $I \times J$ is a weakly 2-absorbing ideal of $R$.
2. $J = R_2$ and $I$ is a 2-absorbing ideal of $R_1$ or $J$ is a prime ideal of $R_2$ and $I$ is a prime ideal of $R_1$.
3. $I \times J$ is a 2-absorbing ideal of $R$.

Proof. (1) $\Rightarrow$ (2). Suppose that $I \times J$ is a weakly 2-absorbing ideal of $R$. If $J = R_2$, then $I$ is a 2-absorbing ideal of $R_1$ by Theorem 2.10. Suppose that $J \neq R_2$. We show that $J$ is a prime ideal of $R_2$ and $I$ is a prime ideal of $R_1$. Let $a, b \in R_2$ such that $ab \in J$, and let $0 \neq i \in I$. Then $(i, 1)(1, a)(1, b) = (i, ab) \in I \times J \setminus \{(0, 0)\}$. Since $(1, a)(1, b) = (1, ab) \notin I_1 \times J$, we conclude that either $(i, 1)(1, a) = (i, a) \in I \times J$ or $(i, 1)(1, b) = (i, b) \in I \times J$, and hence either $a \in J$ or $b \in J$. Thus $J$ is a prime ideal of $R_2$. Similarly, let $c, d \in R_1$ such that $cd \in I$, and let $0 \neq j \in J$. Then $(c, 1)(d, 1)(1, j) = (cd, j) \in I \times J \setminus \{(0, 0)\}$. Since $(c, 1)(d, 1) = (cd, 1) \notin I \times J$, we conclude that either $(c, 1)(1, j) = (c, j) \in I \times J$ or $(d, 1)(1, j) = (a, j) \in I \times J$, and thus either $c \in I$ or $d \in I$. Hence $I$ is a prime ideal of $R_1$. (2) $\Rightarrow$ (3). If $J = R_2$ and $I$ is a 2-absorbing ideal of $R_1$, then $I \times R_2$ is a 2-absorbing ideal of $R$ by Theorem 2.10. Suppose that $I$ is a prime ideal of $R_1$ and $J$ is a prime ideal of $R_2$. Suppose that $(a_1, b_1)(a_2, b_2)(a_3, b_3) \in I \times J$ for some $a_1, a_2, a_3 \in R_1$ and for some $b_1, b_2, b_3 \in R_2$. Then at least one of the $a_i's$ is in $I$, say $a_1$, and at least one of the $b_i's$ is in $J$, say $b_2$. Thus $(a_1, b_1)(a_2, b_2) \in I \times J$. Hence $I \times J$ is a 2-absorbing ideal of $R$. (3) $\Rightarrow$ (1). No comments. □

The following example shows that the hypothesis that $J$ is a nonzero ideal of $R_2$ in Theorem 2.11 is crucial.

Example 2.12. Let $R_1 = \mathbb{Z}_8(+)M$ and $I = \{0\}(+)M$ as in example 2.1. Let $R_2$ be a field. Then $I \times \{0\}$ is a weakly 2-absorbing ideal of $R_1 \times R_2$ that is not a 2-absorbing ideal of $R_1 \times R_2$. Observe that $I$ is not a prime ideal of $R_1$.

Theorem 2.13. Let $R = R_1 \times R_2$ be a commutative ring. Let $I$ be a nonzero proper ideal of $R_1$ and $J$ be an ideal of $R_2$. The following statements are equivalent:

...
(1) $I \times J$ is a weakly 2-absorbing ideal of $R$ that is not a 2-absorbing ideal.
(2) $I$ is a weakly prime ideal of $R_1$ that is not a prime ideal and $J = \{0\}$ is a prime ideal of $R_2$.

Proof. (1) $\Rightarrow$ (2). Assume that $I \times J$ is a weakly 2-absorbing ideal of $R$ that is not a 2-absorbing ideal. Suppose that $J \neq \{0\}$. Then $I \times J$ is a 2-absorbing ideal of $R$ by Theorem 2.11, which contradicts the hypothesis. Thus $J = \{0\}$. We show that $J = \{0\}$ is a prime ideal of $R_2$ (and hence $R_3$ is an integral domain). Suppose that $ab \in J = \{0\}$ for some $a, b \in R_2$. Let $0 \neq i \in I$. Since $(i, 1)(1, a) = (i, ab) \in I \times J \setminus \{(0, 0)\}$ and $(1, a)(1, b) \neq (1, ab) \notin I \times J$, we conclude that either $(i, 1)(1, a) = (i, a) \in I \times J$ or $(i, 1)(1, b) = (i, b) \in I \times J$, and thus $a \in J$ or $b \in J$. Hence $J = \{0\}$ is a prime ideal of $R_2$. We show that $I$ is a weakly prime ideal of $R_1$. Suppose that $ab \in I \setminus \{0\}$ for some $a, b \in R_1$. Since $(a, 1)(b, 1)(1, 0) = (ab, 0) \in I \times \{0\} \setminus \{(0, 0)\}$ and $(a, 1)(b, 1) = (ab, 1) \notin I \times \{0\}$, we conclude that either $(a, 1)(1, 0) = (a, 0) \in I \times \{0\}$ or $(b, 1)(1, 0) = (b, 0) \in I \times \{0\}$, and thus either $a \in I$ or $b \in I$. Hence $I$ is a weakly prime ideal of $R_1$. If $I$ is a prime ideal of $R_1$, then it is easily verified that $I \times \{0\}$ is a 2-absorbing ideal of $R$, which is a contradiction. (2) $\Rightarrow$ (1). Suppose that $I$ is a weakly prime ideal of $R_1$ that is not a prime ideal and $J = \{0\}$ is a prime ideal of $R_2$. We show that $I \times \{0\}$ is a weakly 2-absorbing ideal of $R$. Suppose that $(a, b)(c, d)(e, f) = (ace, bdf) \in I \times \{0\} \setminus \{(0, 0)\}$. Since $I$ is a weakly prime of $R_1$, we may assume $a \in I$. Since $R_2$ is an integral domain, we may assume $d = 0$. Hence $(a, b)(c, d) = (a, b)(c, 0) = (ac, 0) \in I \times \{0\}$. Thus $I \times \{0\}$ is a weakly 2-absorbing ideal of $R$. We show that $I \times \{0\}$ is not a 2-absorbing ideal of $R$. Since $I$ is a weakly prime ideal of $R_1$ that is not a prime ideal, there are $a, b \in R_1$ such that $ab = 0$ but neither $a \in I$ nor $b \in I$. Since $(a, 1)(b, 1)(1, 0) = (0, 0)$ and neither $(a, 1)(b, 1) = (ab, 1) \in I \times \{0\}$ nor $(a, 1)(1, 0) = (a, 0) \in I \times \{0\}$ nor $(b, 1)(1, 0) = (b, 0) \in I \times \{0\}$, we conclude that $I \times \{0\}$ is not a 2-absorbing ideal of $R$.

Let $R_1, R_2$ and $R_3$ be commutative rings with identity and set $R = R_1 \times R_2 \times R_3$. An ideal $I$ of $R$ will have the form $I_1 \times I_2 \times I_3$ where $I_1, I_2$ and $I_3$ are ideals of $R_1, R_2$ and $R_3$, respectively. The next two theorems show that weakly 2-absorbing ideals are really of interest in rings of this form.

Theorem 2.14. Let $R = R_1 \times R_2 \times R_3$ where $R_1, R_2$ and $R_3$ are commutative rings with identity. If $I$ is a weakly 2-absorbing ideal of $R$, then either $I = \{(0, 0, 0)\}$, or $I$ is a 2-absorbing ideal of $R$. 


Proof. Since \( \{0\} \) is a weakly 2-absorbing ideal in any ring, we may assume that \( I = I_1 \times I_2 \times I_3 \neq \{(0,0,0)\} \). Since \( I \neq \{(0,0,0)\} \), there is an element \((0,0,0) \neq (a,b,c) \in I \). Then \((a,1,1)(1,b,1)(1,1,c) = (a,b,c) \), and hence either \((a,b,1) \in I \) or \((a,1,c) \in I \) or \((1,b,c) \in I \). If \((a,b,1) \in I \), then \( I_3 = R_3 \). Likewise if \((a,1,c) \in I \) or \((1,b,c) \in I \), then \( I_2 = R_2 \) or \( I_1 = R_1 \), respectively. So \( I = I_1 \times I_2 \times R_3 \) or \( I = I_1 \times R_2 \times R_3 \) or \( I = R_1 \times I_2 \times R_3 \). Hence \( I \not\subseteq \text{Nil}(R) \). Since \( I \) is a weakly 2-absorbing ideal of \( R \) and \( I \not\subseteq \text{Nil}(R) \), \( I \) is a 2-absorbing ideal of \( R \) by Corollary 2.5. \( \square \)

Theorem 2.15. Let \( R = R_1 \times R_2 \times R_3 \) where \( R_1, R_2 \) and \( R_3 \) are commutative rings with identity. Let \( I_1 \) be a proper ideal of \( R_1 \), \( I_2 \) be an ideal of \( R_2 \), and \( I_3 \) be an ideal of \( R_3 \) such that \( L = I_1 \times I_2 \times I_3 \neq \{(0,0,0)\} \). The following statements are equivalent:

1. \( L = I_1 \times I_2 \times I_3 \) is a weakly 2-absorbing ideal of \( R \).
2. \( L = I_1 \times I_2 \times I_3 \) is a 2-absorbing ideal of \( R \).
3. \( L = I_1 \times R_2 \times R_3 \) and \( I_1 \) is a 2-absorbing ideal of \( R_1 \) or \( L = I_1 \times I_2 \times R_3 \) such that \( I_1 \) is a prime ideal of \( R_1 \) and \( I_2 \) is a prime ideal of \( R_2 \) or \( L = I_1 \times R_2 \times I_3 \) such that \( I_1 \) is a prime ideal of \( R_1 \) and \( I_3 \) is a prime ideal of \( R_3 \).

Proof. (1) \( \Rightarrow \) (2). Since \( L \) is a nonzero weakly 2-absorbing ideal, \( L \) is a 2-absorbing ideal of \( R \) by Theorem 2.14. (2) \( \Rightarrow \) (3). Since \( L \) is a 2-absorbing ideal of \( R \), \( I_1 \) is a 2-absorbing ideal of \( R_1 \). Since \( I_2 \) is a proper ideal of \( R_1 \), by the proof of Theorem 2.14 either \( I_2 \neq R_2 \) or \( I_3 = R_3 \). Assume that \( I_2 \neq R_2 \) and \( I_3 = R_3 \). We show that \( I_1 \) is a prime ideal of \( R_1 \) and \( I_2 \) is a prime of \( R_2 \). Let \( a, b \in R_1 \) such that \( ab \in I_1 \), and let \( c, d \in R_2 \) such that \( cd \in I_2 \). Then \((a,1,1)(1,cd,1)(b,1,1) = (ab,cd,1) \in L \) \( \setminus \{(0,0,0)\} \). Since \((a,1,1)(b,1,1) \notin L \), we have \((a,1,1)(1,cd,1) = (a,cd,1) \in L \) or \((1,cd,1)(b,1,1) = (b,cd,1) \in L \), and hence \( a \in I_1 \) or \( b \in I_1 \). Thus \( I_1 \) is a prime ideal of \( R_1 \). Similarly, since \((ab,1,1)(1,c,1)(1,d,1) = (ab,cd,1) \in L \) \( \setminus \{(0,0,0)\} \) and \((1,c,1)(1,d,1) = (1,cd,1) \notin L \), we conclude that either \((ab,1,1)(1,c,1) \in L \) or \((ab,1,1)(1,d,1) = (ab,d,1) \in L \), and hence either \( c \in I_2 \) or \( d \in I_2 \). Thus \( I_2 \) is a prime ideal of \( R_2 \). Finally, assume \( I_2 = R_2 \) and \( I_3 \neq R_3 \). By an argument similar to that we applied on the ideal \( I_1 \times I_2 \times R_3 \), we conclude that \( I_1 \) is a prime ideal of \( R_1 \) and \( I_3 \) is a prime ideal of \( R_3 \). (3) \( \Rightarrow \) (1). If \( L \) is one of the given three forms, then it is easily verified that \( L \) is a 2-absorbing ideal of \( R \), and hence \( L \) is a weakly 2-absorbing ideal of \( R \). \( \square \)

Theorem 2.16. Let \( A \) be a weakly 2-absorbing ideal of a commutative ring \( R \). Then:
(1) If $I$ is an ideal of $R$ with $I \subseteq A$, then $A/I$ is a weakly 2-absorbing ideal of $R/I$.

(2) If $R_0$ is a subring of $R$, then $A \cap R_0$ is a weakly 2-absorbing ideal of $R_0$.

(3) If $S$ is a multiplicatively closed subset of $R$ with $A \cap S = \emptyset$, then $A_S$ is a weakly 2-absorbing ideal of $R_S$.

**Proof.** (1). Let $\bar{R} = R/I$, $\bar{A} = A/I$, and pick $\bar{a}, \bar{b}, \bar{c} \in \bar{R}$ such that $0 \neq \bar{a}\bar{b}\bar{c} \in \bar{A}$. Since $\bar{a}\bar{b}\bar{c} \neq 0$, we have $abc \in R - I$. Hence $0 \neq abc \in A$. Since $A$ is weakly 2-absorbing, we have $ab \in A$ or $ac \in A$ or $bc \in A$. Consequently, $\bar{a}\bar{b} \in \bar{A}$ or $\bar{a}\bar{c} \in \bar{A}$ or $\bar{b}\bar{c} \in \bar{A}$. (2). The proof is straightforward. (3). Suppose that $0 \neq (x/r)(y/s)(z/t) \in A_S$ where $x, y, z \in R$ and $r, s, t \in S$ but $(x/r)(y/s) \notin A_S$ and $(x/r)(z/t) \notin A_S$. Then $(xyz)/(rst) = a/u$ for some $a \in A$ and $u \in S$. So there exists $v \in S$ with $vuxyz = vrsta \in A$. Thus we have $0 \neq (vux)yz \in A$ but $(vux)y \notin A$ and $(vux)z \notin A$. Since $A$ is weakly 2-absorbing, it follows that $yz \in A$, that is $(y/s)(z/t) \in A_S$. □

3. RINGS WITH THE PROPERTY THAT ALL PROPER IDEALS ARE WEAKLY 2-ABSORBING

For a commutative ring $R$, let $J(R)$ denotes the intersection of all maximal ideals of $R$.

**Lemma 3.1.** Let $R$ be a commutative ring and $a, b, c \in J(R)$. Then the ideal $abcR$ is a weakly 2-absorbing ideal of $R$ if and only if $abc = 0$.

**Proof.** Let $a, b, c \in J(R)$. If $abc = 0$, then $abcR$ is a weakly 2-absorbing ideal of $R$. Now suppose that $abc \neq 0$ and $abcR$ is a weakly 2-absorbing ideal of $R$. Since $abcR$ is a weakly 2-absorbing ideal of $R$ and $0 \neq abc \in abcR$, we conclude that either $ab \in abcR$ or $ac \in abcR$ or $bc \in abcR$. Without lost of generality, we may assume that $ab \in abcR$. Thus $ab = abk$ for some $k \in R$, and hence $ab(1 - ck) = 0$. Since $ck \in J(R)$, $1 - ck$ is a unit of $R$. Thus $ab(1 - ck) = 0$ implies that $ab = 0$, and thus $abc = 0$ which is a contradiction. Hence $abc = 0$. □

**Theorem 3.2.** Let $(R, M)$ be a quasi-local ring. Then every proper ideal of $R$ is weakly 2-absorbing if and only if $M^3 = \{0\}$.

**Proof.** Assume that every proper ideal of $R$ is weakly 2-absorbing. Let $a, b, c \in M$. Since $abcR$ is a weakly 2-absorbing ideal of $R$, $abc = 0$ by Lemma 3.1. Thus $M^3 = \{0\}$. Conversely, assume that $M^3 = \{0\}$, and let $I$ be a proper ideal of $R$ such that $I \neq \{0\}$. Suppose that $abc \in I$ and $abc \neq 0$. Since $M^3 = \{0\}$ and $abc \neq 0$, $a$ is a unit of $R$ or $b$ is a unit of $R$ or $c$ is a unit of $R$, and thus either $ab \in I$ or $ac \in I$ or $bc \in I$. Hence $I$ is a weakly 2-absorbing ideal of $R$. □
Corollary 3.3. Let \((R, M)\) be a quasi-local ideal of \(R\) such that \(M^2 = \{0\}\). Then every proper ideal of \(R\) is a 2-absorbing ideal of \(R\).

Proof. Let \(I\) be a proper ideal of \(R\) and suppose that \(abc \in I\) for some \(a, b, c \in R\). Since \(M^3 = \{0\}\), \(I\) is a weakly 2-absorbing ideal of \(R\) by Theorem 3.2. Hence if \(abc \in I \setminus \{0\}\), then there is nothing to prove. Thus assume that \(abc = 0\). Since \(M^2 = \{0\}\) and \(abc = 0\), either \(ab = 0 \in I\) or \(ac \in I\) or \(bc = 0 \in I\). Thus \(I\) is a 2-absorbing ideal of \(R\).  

Theorem 3.4. Let \((R_1, M_1)\) and \((R_2, M_2)\) be quasi-local commutative rings with maximal ideals \(M_1\) and \(M_2\) respectively, and let \(R = R_1 \times R_2\). Then every proper ideal of \(R\) is a weakly 2-absorbing ideal of \(R\) if and only if \(M_1^2 = M_2^2 = \{0\}\) and either \(R_1\) or \(R_2\) is a field.

Proof. Suppose that every proper ideal of \(R\) is a weakly 2-absorbing ideal of \(R\). Let \(a, b \in M_1\) and suppose that \(ab \neq 0\). Then \(I = abR_1 \times \{0\}\) is a weakly 2-absorbing ideal of \(R\). Since \((a, 1)(b, 1)(1, 0) = (ab, 0) \in I \setminus \{(0, 0)\}\) and \((a, 1)(b, 1) \notin I\), either \((a, 1)(1, 0) = (a, 0) \in I\) or \((b, 1)(1, 0) = (b, 0) \in I\). Assume that \((a, 0) \in I\). Then \(a = abk\) for some \(k \in R_1\). Hence \(a(1 - bk) = 0\). Since \(1 - bk\) is a unit of \(R_1\), \(a = 0\) which is a contradiction. Also, if \((b, 0) \in I\), then one can conclude that \(b = 0\) which is a contradiction again. Thus \(M_1^2 = \{0\}\). Now assume \(a, b \in M_2\) such that \(ab \neq 0\). Then \(I = \{0\} \times abR_2\) is a weakly 2-absorbing ideal of \(R\). Since \((1, a)(1, b)(0, 1) = (0, ab) \in I\), by an argument similar to that we applied on \(M_1\) we conclude that either \(a = 0\) or \(b = 0\) which is a contradiction. Thus \(M_2^2 = \{0\}\). Suppose that \(R_1\) is not a field. We show that \(R_2\) is a field. Since \(R_1\) is not a field, \(M_1 \neq \{0\}\) and \(J = M_1 \times \{0\}\) is a weakly 2-absorbing ideal of \(R\). Suppose that \(R_2\) is not a field. Since \(M_2^2 = \{0\}\) and \(R_2\) is not a field, there is a \(c \in M_2\) such that \(c \neq 0\) and \(c^2 = 0\). Let \(m \in M_1\) such that \(m \neq 0\). Then \((m, 1)(1, c)(1, c) = (m, c^2) = (m, 0) \in J = M_1 \times \{0\}\ \setminus \{(0, 0)\}\), but neither \((m, 1)(1, c) = (m, c) \in J\) nor \((1, c)(1, c) = (1, c) \in J\), which is a contradiction. Hence \(M_2 = \{0\}\), and thus \(R_2\) is a field. Conversely, suppose that \(M_1^2 = \{0\}\) and \(R_2\) is a field. Since \(M_1^2 = \{0\}\), every proper ideal of \(R_1\) is a 2-absorbing ideal of \(R_1\) by Corollary 3.3. Since \(M_1^2 = \{0\}\) and \(R_2\) is a field, the ideal \(\{0\} \times R_2\) is a weakly 2-absorbing ideal of \(R\). Since \(R_2\) is a field, the ideal \(R_1 \times \{0\}\) is a weakly 2-absorbing ideal of \(R\). Let \(J\) be a proper ideal of \(R_1\) such that \(J \neq \{0\}\). Since \(J\) is a 2-absorbing ideal of \(R_1\), \(J \times R_2\) is a weakly 2-absorbing ideal of \(R\) by Theorem 2.10. Finally, we show that \(I = J \times \{0\}\) is a weakly 2-absorbing ideal of \(R\). Suppose that \((a_1, b_1)(a_2, b_2)(a_3, b_3) \in R \setminus \{(0, 0)\}\) for some \(a_1, a_2, a_3 \in R_1\) and for some \(b_1, b_2, b_3 \in R_2\). Since \(M_1^2 = \{0\}\), only one of the \(a_i\)’s is in \(M_1\), say \(a_1 \in M_1\) and \(a_2, a_3\) are units of \(R_1\). Since \(a_1a_2a_3 \in J\) and \(a_2, a_3\) are units of \(R_1\),
$a_1 \in J$. Since $R_2$ is a field and $b_1b_2b_3 = 0$, at least one of the $b_i$'s is equal to 0, say $b_2 = 0$. Hence $(a_1, b_1)(a_2, 0) = (a_1a_2, 0) \in I$. Thus $I$ is a weakly 2-absorbing ideal of $R$. \hfill \Box

**Theorem 3.5.** Let $R_1$, $R_2$, and $R_3$ be commutative rings, and let $R = R_1 \times R_2 \times R_3$. Then every proper ideal of $R$ is a weakly 2-absorbing ideal of $R$ if and only if $R_1$, $R_2$, and $R_3$ are fields.

**Proof.** If $R_1$, $R_2$, and $R_3$ are fields, then by [4, Theorem 3.4(3)] every nonzero proper ideal of $R$ is a 2-absorbing ideal of $R$, and hence every proper ideal of $R$ is a weakly 2-absorbing ideal of $R$. Conversely, suppose that every proper ideal of $R$ is a weakly 2-absorbing ideal of $R$ and one of the $R_i$'s, $1 \leq i \leq 3$, is not a field. Without loss of generality, we may assume $R_1$ is not a field. Hence $R_1$ has a proper ideal $J$ such that $J \neq \{0\}$. Let $I = J \times \{0\} \times \{0\}$. Then $I$ is a weakly 2-absorbing ideal of $R$. Let $m \in J$ such that $m \neq 0$. Then $(m, 1,1)(1,0,1)(1,1,0) = (m, 0, 0) \in I \setminus \{(0,0,0)\}$ but neither $(m,1,1)(1,0,1) = (m,0,1) \in I$ nor $(m,1,1)(1,1,0) = (m,1,0) \in I$ nor $(1,0,1)(1,1,0) = (1,0,0) \in I$, which is a contradiction. Thus $R_1$, $R_2$, and $R_3$ are fields. \hfill \Box

**Lemma 3.6.** Suppose that every proper ideal of $R$ is a weakly 2-absorbing ideal. Then $R$ has at most three maximal ideals.

**Proof.** Suppose that $M_1, M_2, M_3, M_4$ are distinct maximal ideals of $R$. Let $I = M_1 \cap M_2 \cap M_3$. Since there are three prime ideals of $R$ that are minimal over $I$, $I$ is not a 2-absorbing ideal of $R$ by [3, Theorem 2.5]. Hence $I$ is a weakly 2-absorbing ideal of $R$ that is not a 2-absorbing ideal of $R$. Thus $I^3 = \{0\}$ by Theorem 2.4. Hence $I^3 = M_1^3M_2^3M_3^3 = \{0\} \subset M_4$, and thus one of the $M_i$'s, $1 \leq i \leq 3$, is contained in $M_4$, which is a contradiction. Hence $R$ has at most three distinct maximal ideals. \hfill \Box

**Theorem 3.7.** A commutative ring $R$ has the property that every proper ideal is a weakly 2-absorbing ideal of $R$ if and only if one of the following statements hold:

1. $(R, M)$ is a quasi-local ring with $M^2 = 0$.
2. $R$ is ring-isomorphic to $R_1 \times F$, where $R_1$ is a quasi-local ring with maximal ideal $M$ such that $M^2 = \{0\}$ and $F$ is a field.
3. $R$ is ring-isomorphic to $F_1 \times F_2 \times F_3$, where $F_1, F_2, F_3$ are fields.

**Proof.** If $R$ satisfies condition (1), then every proper ideal of $R$ is a weakly 2-absorbing ideal of $R$ by Theorem 3.2. If $R$ satisfies condition (2), then every proper ideal of $R$ is a weakly 2-absorbing ideal of $R$ by Theorem 3.4. If $R$ satisfies...
condition (3), then every proper ideal of \( R \) is a weakly 2-absorbing ideal of \( R \) by Theorem 3.5. Conversely, suppose that every proper ideal of \( R \) is a weakly 2-absorbing ideal. Then \( R \) has at most three maximal ideals by Lemma 3.6. Hence we consider the following three cases: \textbf{Case 1.} Suppose that \( R \) has exactly one maximal ideal, call it \( M \). Then \( M^3 = \{0\} \) by Theorem 3.2. \textbf{Case 2.} Suppose that \( R \) has exactly two maximal ideals, say \( M_1 \) and \( M_2 \) are the maximal ideals of \( R \). Then \( J(R) = M_1 \cap M_2 \) is a weakly 2-absorbing ideal of \( R \) (in fact, \( J(R) \) is a 2-absorbing ideal of \( R \)). We show \( J(R)^3 = \{0\} \). Let \( a, b, c \in J(R) \). Since \( abcR \) is a weakly 2-absorbing ideal of \( R \), we conclude that \( abc = 0 \) by Lemma 3.1. Thus \( J(R)^3 = M_1^2 \cap M_2^2 = \{0\} \). Hence \( R \) is ring-isomorphic to \( R/M_1^2 \times R/M_2^2 \). Since \( R/M_1^2 \) and \( R/M_2^2 \) are quasi-local commutative rings, we conclude that \( R \) is ring-isomorphic to \( R_1 \times F \), where \( R_1 \) is quasi-local ring with maximal ideal \( M \) such that \( M^2 = \{0\} \) and \( F \) is a field by Theorem 3.4. \textbf{Case 3.} Suppose that \( R \) has exactly three maximal ideals, say \( M_1, M_2, M_3 \) are the maximal ideals of \( R \). Hence \( J(R) = M_1 \cap M_2 \cap M_3 \) is a weakly 2-absorbing ideal of \( R \). Since \( J(R) \) is the intersection of three prime ideals of \( R \), \( J(R) \) is not a 2-absorbing ideal of \( R \) by [4]. Hence \( J(R)^3 = \{0\} \) by Theorem 2.4. Since \( J(R)^3 = M_1^3 \cap M_2^3 \cap M_3^3 = \{0\} \), we conclude that \( R \) is ring-isomorphic to \( R/M_1^3 \times R/M_2^3 \times R/M_3^3 \). Thus \( R \) is ring-isomorphic to \( F_1 \times F_2 \times F_3 \), where \( F_1, F_2, F_3 \) are fields by Theorem 3.5. \( \square \)

\textbf{Corollary 3.8.} Let \( n \) be a positive integer. Then every proper ideal of \( R = \mathbb{Z}_n \) is a weakly 2-absorbing ideal of \( R \) if and only if either \( n = q^3 \) for some prime positive integer \( q \) or \( n = q^2 p \) for some distinct prime positive integers \( q, p \) or \( n = q_1 q_2 q_3 \) for some distinct prime positive integers \( q_1, q_2, q_3 \).

\begin{itemize}
\item Let \( I \) be a 2-absorbing ideal of a commutative ring \( R \) and suppose that \( I_1 I_2 I_3 \subseteq I \) for some ideals \( I_1, I_2, \) and \( I_3 \) of \( R \). Then by [4] either \( I_1 I_2 \subseteq I \) or \( I_1 I_3 \subseteq I \) or \( I_2 I_3 \subseteq I \). We are unable to answer the following question:
\item \textbf{Question.} Suppose that \( I \) is a weakly 2-absorbing ideal of a commutative ring \( R \) that is not a 2-absorbing ideal and \( 0 \neq I_1 I_2 I_3 \subseteq I \) for some ideals \( I_1, I_2, \) and \( I_3 \) of \( R \). Does it imply that \( I_1 I_2 \subseteq I \) or \( I_1 I_3 \subseteq I \) or \( I_2 I_3 \subseteq I \)?
\end{itemize}

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